On the CR transversality of holomorphic maps into hyperquadrics

Dedicated to Professor Yum-Tong Siu on the occasion of his 70th birthday

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Abstract Let M_ℓ be a smooth Levi-nondegenerate hypersurface of signature ℓ in $\mathbf{C}^n$ with $n \geq 3$, and write H_ℓ^N for the standard hyperquadric of the same signature in $\mathbf{C}^N$ with $N - n < \frac{n-1}{2}$. Let F be a holomorphic map sending M_ℓ into H_ℓ^N. Assume F does not send a neighborhood of M_ℓ in $\mathbf{C}^n$ into H_ℓ^N. We show that F is necessarily CR transversal to M_ℓ at any point. Equivalently, we show that F is a local CR embedding from M_ℓ into H_ℓ^N.

1 Introduction and the main theorems

Let M_1 and M_2 be two connected smooth CR hypersurfaces in $\mathbf{C}^n$ and $\mathbf{C}^N$, respectively, with $3 \leq n \leq N$. Let F be a holomorphic map from some small neighborhood $U \subset \mathbf{C}^n$ of M into $\mathbf{C}^N$ with $F(M_1) \subset M_2$. Given a point $p \in M$, denote by $T_p^{(1,0)}M$ the holomorphic tangent vector space of M at p. Assume F does not send a neighborhood of p in $\mathbf{C}^n$ into M_2. An important question in the study of the geometric structure of F is to understand the geometric conditions for the manifolds in which F is CR transversal to M_1 at p. Recall that F is said to be CR transversal at p if

$$T_{F(p)}^{(1,0)}M_2 + dF(T_p^{(1,0)}\mathbf{C}^n) = T_{F(p)}^{(1,0)}\mathbf{C}^N.$$

Roughly speaking, the CR transversality property can be interpreted as an non-vanishing property of the normal derivative of the normal components for the map.

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The problem has been extensively investigated in the literature. When both the target and the source manifolds are strongly pseudoconvex, CR transversality always holds due to the classical Hopf lemma. In the equal dimensional case ($n = N$), work has been done by Pinchuk [Pi], Fornaess [Fo], Baouendi-Rothschild [BR], Ebenfelt-Rothschild [ER], Huang [Hu2] and the references therein. The study of the higher codimensional case starts with the work of Baouendi-Huang in [BH] where it is proved that the CR transversality always holds when the manifolds are hyperquadrics of the same signature. Baouendi-Ebenfelt-Rothschild [BER2] proved, under rather general setting, that the CR transversality holds in an open dense subset. See also a recent paper of Ebenfelt-Son [ES] and the references therein.

While there exist examples where CR transversality fails on certain thin sets (see, for instance [BER2]), as mentioned above, the rigidity theorem due to Baouendi-Huang [BH] indicates that the CR transversality holds everywhere when both M_1 and M_2 are hyperquadrics of the same signature ℓ. Enlightened by this result, the following conjecture concerning the CR transversality was asked by Baouendi and the first author in the year of 2005:

Conjecture (Baounedi-Huang, 2005): *Let $M_1 \subset \mathbf{C}^n$ and $M_2 \subset \mathbf{C}^N$ be two (connected) Levi non-degenerate real analytic hypersurfaces with the same signature $\ell > 0$. Here $3 \leq n < N$. Let F be a holomorphic map defined in a neighborhood U of M_1, sending M_1 into M_2. Then either F is a local CR embedding from M_1 into M_2 or F is totally degenerate in the sense that it maps a neighborhood U of M_1 in $\mathbf{C}^n$ into M_2.*

We point out that, for the M_1 and M_2 given in the conjecture, the fact that F is CR transversal at p is equivalent to the fact that F is a CR embedding from a neighborhood of p in M_1 into M_2. Along these lines, in a recent paper of the authors [HZ2], by developing a new technique, we showed the CR transversality holds when $M_2 = H_\ell^{n+1}$ and the point under study is not CR umbilical in the sense of Chern-Moser.

In this paper, combining a quantitative version of a very useful lemma due to the first author with the tools developed in [HZ2], we are able to drop the geometric assumption of the umbilicality and relax the codimension-one restriction in [HZ2]. The generalization of the above mentioned lemma in [Hu] will be addressed in detail in section 3.

We next state our main theorems:

Theorem 1. *Let M_ℓ be a smooth Levi non-degenerate hypersurface of signature ℓ in $\mathbf{C}^n$ with $n \geq 3$ and $0 \in M_\ell$. Suppose that F is a holomorphic map in a small neighborhood U of $0 \in \mathbf{C}^n$ such that*

$$F(M_\ell \cap U) \subset H_\ell^N$$

with $N - n < \frac{n-1}{2}$. If $F(U) \not\subset H_\ell^N$, then F is CR transversal to M_ℓ at 0, or equivalently, F is a CR embedding from a small neighborhood of $0 \in M_\ell$ into H_ℓ^N.

Theorem 2. *Let M_ℓ be a germ of a smooth Levi non-degenerate hypersurface at 0 of signature ℓ in $\mathbf{C}^n$, $n \geq 3$. Suppose that there exists a holomorphic map F in a neighborhood U of 0 in $\mathbf{C}^n$ sending M_ℓ into H_ℓ^N but $F(U) \not\subset H_\ell^N$, $N < 2n-1$. Then M_ℓ is CR embeddable into H_ℓ^N near 0. Equivalently, there exists a holomorphic map $\tilde{F} : M_\ell \to H_\ell^N$ near 0, which is CR transversal to M_ℓ at 0.*

The idea of the proof is based on a re-scaling technique that was initially introduced in [HZ2]. With the aid of a quantitative lemma of the first author in [Hu], we generate a formal CR transversal map which, by a result of Meylan-Mir-Zaitsev proved in [MMZ], is necessarily convergent. Finally, using a rigidity result in [EHZ], F differs from the CR transversal map only by an automorphisms of the target and hence it is CR transversal as well.

The outline of the paper is as follows. In section 2, the notations and a normalization procedure of Baouendi-Huang is revisited. A modified lemma in [Hu] is discussed and proved in section 3. Section 4 is devoted to the proof of the main theorem.

2 Notations and a normalization procedure

Let M_ℓ be a germ at 0 of a smooth Levi non-degenerate hypersurface of signature ℓ in $\mathbf{C}^n$. After a holomorphic change of coordinates, M_ℓ near the origin can be expressed as follows.

$$M_\ell = \left\{(z,w) \in \mathbf{C}^{n-1} \times \mathbf{C} : \Im w = |z|_\ell^2 - \frac{1}{4}\mathscr{S}(z) + o(4)\right\}. \tag{1}$$

Here for any n-tuples a and b, $\langle a, \bar{b}\rangle_\ell := -\sum_{j=1}^{\ell} a_j \bar{b}_j + \sum_{j=\ell+1}^{n} a_j \bar{b}_j$ and $|a|_\ell^2 = \langle a, \bar{a}\rangle_\ell$, $\mathscr{S}(z) := \sum_{1\leq \alpha,\beta,\gamma,\delta\leq n} s_{\alpha\bar{\beta}\gamma\bar{\delta}} z_\alpha \bar{z}_\beta z_\gamma \bar{z}_\delta$ is a homogeneous polynomial of bi-degree (2,2), called the Chern-Moser-Weyl curvature function of M_ℓ at 0. See [CM] for more details. Without loss of generality, we always assume that $\ell \leq (n-1)/2$ so ℓ becomes an invariant.

As in [CM], assign the weighted degree 1 to variable z and 2 to variable w. Given a holomorphic function h, denote by $h^{(k)}$ the terms of weighted degree k, and by $h^{(\mu,\nu)}$ the terms of degree μ in z variable and of degree ν in w variable in the power series expansion of h at 0. For each integer $k \geq 0$, we write $o(k)$ for terms of degree larger than k, and $o_{wt}(k)$ for terms of weighted degree larger than k. To simplify our notation, we also preassign the coefficient of h with negative degrees to be 0.

Let $\tilde{M}_\ell$ be a germ at 0 of another smooth Levi-nondegenerate hypersurface in $\mathbf{C}^N$ of signature ℓ given by

$$\tilde{M}_\ell = \left\{(\tilde{z},\tilde{w}) \in \mathbf{C}^{N-1} \times \mathbf{C} : \Im \tilde{w} = |\tilde{z}|_\ell^2 - \frac{1}{4}\tilde{\mathscr{S}}(\tilde{z}) + o(4)\right\}. \tag{2}$$

Here $\tilde{\mathscr{S}}$ is the corresponding Chern-Moser curvature tensor function of $\tilde{M}_\ell$ at 0.

Let F be a smooth CR map sending $(M_\ell,0)$ into $(\tilde{M}_\ell,0)$. Write

$$F := (\tilde{f}, g) = (f, \phi, g) \tag{3}$$

with $f=(f_1,\ldots,f_{n-1})$ and $\phi=(\phi_1,\ldots,\phi_{N-n})$ being components of F. Assume that F is CR transversal at 0. Then, following a normalization procedure as in [§2, BH], we have

$$\begin{aligned}\tilde{z} = (f_1(z,w),\ldots,f_{n-1}(z,w),\phi_1(z,w),\ldots,\phi_{N-n}(z,w)) = \lambda zU + \mathbf{a}w + O(|(z,w)|^2)\\ \tilde{w} = g(z,w) = \sigma\lambda^2 w + O(|(z,w)|^2).\end{aligned} \tag{4}$$

Here U can be extended to an $(N-1)\times(N-1)$ matrix $\tilde{U}\in SU(N-1,\ell)$ (namely $\langle X\tilde{U}, Y\overline{\tilde{U}}\rangle_\ell = \langle X,Y\rangle_\ell$ for any $X,Y\in\mathbf{C}^{N-1}$), $\mathbf{a}\in\mathbf{C}^{N-1}$ and $\lambda>0$, $\sigma=\pm1$ with $\sigma=1$ for $\ell<\frac{n-1}{2}$. When $\sigma=-1$, by considering $F\circ\tau_{n-1/2}$ instead of F, where $\tau_{\frac{n-1}{2}}(z_1,\ldots,z_{\frac{n-1}{2}},z_{\frac{n-1}{2}+1},\ldots,z_{n-1},w) = (z_{\frac{n-1}{2}+1},\ldots,z_{n-1},z_1,\ldots,z_{\frac{n-1}{2}},-w)$, we can make $\sigma=1$. Hence, we will assume in what follows that $\sigma=1$. Moreover, as in [HZ], F can be normalized as follows:

Proposition 1. *([HZ]) Let M_ℓ and $\tilde{M}_\ell$ be defined by (1) and (2), respectively, and let F be a smooth CR map sending M_ℓ into $\tilde{M}_\ell$ given by (3) and (4) with $\sigma=1$. Then after composing F from the left by some automorphism $T\in Aut_0(H^N_\ell)$ preserving the origin, the following holds:*

$$F^\sharp = (f^\sharp, \phi^\sharp, g^\sharp) := T\circ F,$$

with

$$\begin{aligned} f^\sharp(z,w) &= z + \tfrac{i}{2}a^{(1,0)}(z)w + o_{wt}(3),\\ \phi^\sharp(z,w) &= \phi^{(2,0)}(z) + o_{wt}(2),\\ g^\sharp(z,w) &= w + o_{wt}(4),\end{aligned}$$

and

$$\langle a^{(1,0)}(z),\bar{z}\rangle_\ell |z|^2_\ell = |\phi^{(2,0)}(z)|^2 + \frac{1}{4}(\mathscr{S}(z) - \lambda^{-2}\tilde{\mathscr{S}}(\lambda(z,0)\widetilde{U})).$$

In particular, the automorphism T is given by

$$T(\tilde{z},\tilde{w}) = \frac{(\lambda^{-1}(\tilde{z}-\lambda^{-2}\mathbf{a}\tilde{w})\tilde{U}^{-1}, \lambda^{-2}\tilde{w})}{q(\tilde{z},\tilde{w})}$$

with $r_0=\frac{1}{2}\Re\{g''_{ww}(0)\}$, $q(\tilde{z},\tilde{w}) = 1+2i\langle\tilde{z},\lambda^{-2}\overline{\mathbf{a}}\rangle_\ell + \lambda^{-4}(r_0 - i|\mathbf{a}|^2_\ell)\tilde{w}$. Moreover, $F^\sharp$ sends M_ℓ into $\tilde{M}^\sharp := T(\tilde{M}_\ell)$ given by

$$\tilde{M}^\sharp = \left\{(\tilde{z}^\sharp,\tilde{w}^\sharp)\in\mathbf{C}^{N+1} : \Im\tilde{w}^\sharp = |\tilde{z}^\sharp|^2_\ell + \frac{1}{4}\tilde{\mathscr{S}}^\sharp(\tilde{z}^\sharp) + o(4)\right\}$$

with $\tilde{\mathscr{S}}^\sharp(z^\sharp) = \lambda^{-2}\tilde{\mathscr{S}}(\lambda z^\sharp \tilde{U})$.

3 A quantitative version of a basic lemma

In this section, some simple preparation facts will be given without proof at first. In the second part of the section, we will discuss a quantitative version of a lemma obtained in [Hu], which played crucial role for us to get the convergence in our rescaling argument.

Given a polynomial ϕ, define $\|\phi\|$ to be the maximum modulus of all the coefficients in ϕ. For a given vector-valued polynomial $\phi = (\phi_1, \ldots, \phi_s)$, $\|\phi\| := \max_{1\leq j\leq s}\|\phi_j\|$. We first refer to a lemma in [HZ2] without proof.

Lemma 1. *[HZ2] (1). Let* $X(z,\bar{z})$ *and* $Y(z,\bar{z})$ *be two polynomials such that* $X(z,\bar{z}) = Y(z,\bar{z})|z|_\ell^2$. *Then* $\|Y\|$ *is bounded by a constant depending only on* $\|X\|$ *and the degree of* X.

(2). Let $h(z)$ *be a homogeneous holomorphic polynomial of degree* d *in* $z \in \mathbf{C}^n$. *If* $|h(z)| \leq c|z|^d$ *on* $\{|z|_\ell^2 = 0\}$, *then* $\|h\| \leq C$ *for some* C *depending only on* c *and* d.

In various rigidity problems concerning CR immersions, the following lemma in [Hu] plays an essential role in deriving key identities to eventually conclude uniqueness:

Lemma 2. *[Hu] Let* $\{\phi_j\}_{j=1}^{n-1}$ *and* $\{\psi_j\}_{j=1}^{n-1}$ *be two families of holomorphic functions in* $\mathbf{C}^n$. *Let* $B(z,\xi)$ *be a real-analytic function in* (z,ξ). *Suppose that*

$$\sum_{j=1}^{n-1}\phi_j(z)\psi_j(\xi) = B(z,\xi)\langle z,\xi\rangle_\ell.$$

Then $B(z,\xi) = \sum_{j=1}^{n-1}\phi_j(z)\psi_j(\xi) = 0$.

We find a quantitative version of the above lemma serves our purpose under this context perfectly well.

Lemma 3. *Let* $\{\phi_j\}_{j=1}^{n-1}$ *and* $\{\psi_j\}_{j=1}^{n-1}$ *be two families of holomorphic polynomials of degree* k *and* m *in* $\mathbf{C}^n$, *respectively. Let* $H(z,\xi), B(z,\xi)$ *be two polynomials in* (z,ξ). *Suppose that*

$$\sum_{j=1}^{n-1}\phi_j(z)\psi_j(\xi) = H(z,\xi) + B(z,\xi)\langle z,\xi\rangle_\ell$$

and $\|H\| \leq C$. *Then* $\|B\| \leq \tilde{C}$ *and* $\|\sum_{j=1}^{n-1}\phi_j(z)\psi_j(\xi)\| \leq \tilde{C}$ *with* $\tilde{C}$ *dependent only on* (C,k,m,n).

The proof of the lemma is based on the following algorithm together with Lemma 2. First, let us formulate the algorithm procedure so as to re-adjust two families $\{\phi_j\}_{j=1}^{n-1}$ and $\{\psi_j\}_{j=1}^{n-1}$ in Lemma 3.

Lemma 4. *Let $\phi := \{\phi_j\}_{j=1}^{s}$ and $\psi := \{\psi_j\}_{j=1}^{s}$ be two families of holomorphic polynomials of degree k and m in $\mathbf{C}^n$, respectively. There exist two families $\tilde{\phi} := \{\tilde{\phi}_j\}_{j=1}^{s}$ and $\tilde{\psi} := \{\tilde{\psi}_j\}_{j=1}^{s}$ of holomorphic polynomials of degree k and m in $\mathbf{C}^n$, respectively, such that*

$$\sum_{j=1}^{s} \phi_j(z)\psi_j(\xi) = \sum_{j=1}^{s} \tilde{\phi}_j(z)\tilde{\psi}_j(\xi) \tag{5}$$

and

$$\|\tilde{\phi}\| \leq 1, \;\; C\|\tilde{\psi}\| \leq \| \sum_{j=1}^{s} \tilde{\phi}_j(z)\tilde{\psi}_j(\xi)\| \leq s\|\tilde{\psi}\| \tag{6}$$

for some positive constant C dependent only on s.

Proof of Lemma 4: Without loss of generality, assume $\|\phi_j\| \neq 0$ for all $1 \leq j \leq s$ and $\{\phi_j\}_{j=1}^{s}$ are linearly independent. Moreover, by replacing ϕ_j and ψ_j by $\frac{\phi_j}{\|\phi_j\|}$ and $\|\phi_j\|\psi_j$, respectively, one can assume that $\|\phi_j\| = 1$ for all $1 \leq j \leq s$. Denote by $\{e_l\}_{l=1}^{d(k)}$ a basis of unit monomials to span the polynomial spaces of degree k and write $\phi_j = \sum_{1\leq l\leq d(k)} D_j^l e_l, 1 \leq j \leq s$. Here $d(k)$ is the dimension of polynomial spaces of degree k. Hence $\|\phi_j\| = \max_{1\leq l\leq d(k)} D_j^l$ for each $1 \leq j \leq s$. Arranging the order of $\{e_l\}$ if necessary, we can make $D_1^1 = 1$ and $|D_1^l| \leq 1$.

Step 1: Let ${}^1\phi_1 := \phi_1, {}^1\phi_j := \phi_j - D_j^1 \cdot \phi_1, 2 \leq j \leq s$. Then in terms of the basis representation ${}^1\phi_j := {}^1D_j^l \cdot e_l$, one has

$$\begin{gathered} {}^1D_1^1 = 1, \; |{}^1D_1^l| \leq 1, \; 2 \leq l \leq d(k); \\ {}^1D_j^1 = 0, \; |{}^1D_j^l| \leq 2, \; 2 \leq j \leq s, \;\; 2 \leq l \leq d(k). \end{gathered}$$

Moreover, letting ${}^1\psi_1 := \psi_1 + \sum_{j=2}^{s} D_j^1 \cdot \psi_j, {}^1\psi_j := \psi_j, 2 \leq j \leq s$, then

$$\sum_{j=1}^{s} \phi_j(z)\psi_j(\xi) = \sum_{j=1}^{s} {}^1\phi_j(z) \cdot {}^1\psi_j(\xi). \tag{7}$$

Step 2: Normalize ${}^1\phi_j, 2 \leq j \leq s$ by replacing ${}^1\phi_j, {}^1\psi_j$ by $\frac{{}^1\phi_j}{\|{}^1\phi_j\|}$ and $\|{}^1\phi_j\| \cdot {}^1\psi_j$, respectively. By abuse of notation, we still denote them by ${}^1\phi_j, {}^1\psi_j$ and the representation matrix under the basis $\{e_l\}$ by $\{{}^1D_j^l\}$. Moreover, since $\{\phi_j\}_{j=1}^{s}$ are linearly independent, by rearranging the order of $\{e_l\}_{l=2}^{d(k)}$ if necessary, we have (7) holds with

$$
\begin{gathered}
{}^1D_1^1 = 1,\ |{}^1D_1^l| \le 1,\ 2 \le l \le d(k);\\
{}^1D_2^1 = 0,\ {}^1D_2^2 = 1,\ |{}^1D_2^l| \le 1,\ 3 \le l \le d(k);\\
{}^1D_j^1 = 0,\ |{}^1D_j^l| \le 1,\ 3 \le j \le s,\ \ 2 \le l \le d(k)
\end{gathered}
$$

and for each $1 \le j \le s$,

$$
\max_{1 \le l \le d(k)} {}^1D_j^l = 1.
$$

Step 3: Let ${}^2\phi_2 = {}^1\phi_2, {}^2\phi_j := {}^1\phi_j - {}^1D_j^2 \cdot {}^1\phi_2$ for $1 \le j \le s, j \ne 2$. Then in terms of the basis representation ${}^2\phi_j := {}^2D_j^l \cdot e_l$, we deduce

$$
\begin{gathered}
{}^2D_1^1 = 1,\ {}^2D_1^2 = 0,\ |{}^2D_1^l| \le 2,\ 3 \le l \le d(k);\\
{}^2D_2^1 = 0,\ {}^2D_2^2 = 1,\ |{}^2D_2^l| \le 1,\ 3 \le l \le d(k);\\
{}^2D_j^1 = 0,\ {}^2D_j^2 = 0,\ |{}^2D_j^l| \le 2,\ 3 \le j \le s,\ \ 3 \le l \le d(k).
\end{gathered}
$$

Moreover, letting ${}^2\psi_2 := {}^1\psi_2 + \sum_{j \ne 2} {}^1D_j^2 \cdot {}^1\psi_j, {}^2\psi_j := {}^1\psi_j, 1 \le j \le s$ with $j \ne 2$, then

$$
\sum_{j=1}^{s} \phi_j(z)\psi_j(\xi) = \sum_{j=1}^{s} {}^2\phi_j(z) \cdot {}^2\psi_j(\xi). \tag{8}
$$

Step 4: Normalize ${}^2\phi_j, 1 \le j \le s, j \ne 2$ by replacing ${}^2\phi_j, {}^2\psi_j$ by $\frac{{}^2\phi_j}{\|{}^2\phi_j\|}$ and $\|{}^2\phi_j\| \cdot {}^2\psi_j$, respectively. As before, we still denote them by ${}^2\phi_j, {}^2\psi_j$ and the representation matrix under the basis $\{e_l\}$ by $\{{}^2D_j^l\}$. Furthermore, (8) holds with

$$
\begin{gathered}
1 \ge {}^2D_1^1 \ge \tfrac{1}{2},\ {}^2D_1^2 = 0,\ |{}^2D_1^l| \le 1,\ 3 \le l \le d(k);\\
{}^2D_2^1 = 0,\ {}^2D_2^2 = 1,\ |{}^2D_2^l| \le 1,\ 3 \le l \le d(k);\\
{}^2D_j^1 = 0,\ {}^2D_j^2 = 0,\ |{}^2D_j^l| \le 1,\ 3 \le j \le s,\ \ 3 \le l \le d(k)
\end{gathered}
$$

and for each $1 \le j \le s$,

$$
\max_{1 \le l \le d(k)} {}^2D_j^l = 1.
$$

Step 5: Continue the above process until we get new families $\{{}^s\phi_j\}_{j=1}^s, \{{}^s\psi_j\}_{j=1}^s$ such that under the basis representation, ${}^s\phi_j := {}^sD_j^l \cdot e_l$ with

$$
{}^sD = \begin{bmatrix}
{}^sD_1^1 & 0 & 0 & \cdots & 0 & {}^sD_1^{s+1} & \cdots & {}^sD_1^{d(k)} \\
0 & {}^sD_2^2 & 0 & \cdots & 0 & {}^sD_2^{s+1} & \cdots & {}^sD_2^{d(k)} \\
0 & 0 & {}^sD_3^3 & \cdots & 0 & {}^sD_2^{s+1} & \cdots & {}^sD_3^{d(k)} \\
 & & & & \cdots & & & \\
 & & & & \cdots & & & \\
 & & & & \cdots & & & \\
0 & 0 & 0 & \cdots & {}^sD_s^s & {}^sD_s^{s+1} & \cdots & {}^sD_s^{d(k)}
\end{bmatrix},
$$

where

$$1 \geq {}^sD_j^j \geq \frac{1}{2^{s-j}},\ 1 \leq j \leq s-1;\ \ {}^sD_s^s = 1;$$

and for each $1 \leq j \leq s$,

$$\max_{1 \leq l \leq d(k)} {}^sD_j^l = 1.$$

Moreover,

$$\sum_{j=1}^{s} \phi_j(z)\psi_j(\xi) = \sum_{j=1}^{s} {}^s\phi_j(z) \cdot {}^s\psi_j(\xi).$$

Let $\tilde{\phi}_j := {}^s\phi_j, \tilde{\psi}_j := {}^s\psi_j, 1 \leq j \leq s$. Then from the construction, for $1 \leq j \leq s$, $\|\tilde{\phi}_j\| = 1$ with $\sum_{j=1}^{s} \phi_j(z)\psi_j(\xi) = \sum_{j=1}^{s} \tilde{\phi}_j(z)\tilde{\psi}_j(\xi)$. Hence

$$\|\sum_{j=1}^{s} \phi_j(z)\psi_j(\xi)\| \leq \sum_{j=1}^{s} \|\tilde{\phi}_j\|\|\tilde{\psi}_j\| \leq s\|\tilde{\psi}\|.$$

Furthermore, since ${}^sD_j^j \geq \frac{1}{2^{s-j}}$ when $1 \leq j \leq s$,

$$\|\sum_{j=1}^{s} \phi_j(z)\psi_j(\xi)\| \geq \max_{1 \leq j \leq s} {}^sD_j^j \cdot \|\tilde{\psi}_j\| \geq \frac{1}{2^{s-1}}\|\tilde{\psi}\|.$$

The proof of Lemma 4 is therefore complete. $\square$

Proof of Lemma 3: Assume by contradiction that there exist families of $\{\phi^\lambda\}$ and $\{\psi^\lambda\}$, such that

$$\sum_{j=1}^{n-1} \phi_j^\lambda(z)\psi_j^\lambda(\xi) = H^\lambda(z,\xi) + B^\lambda(z,\xi)\langle z,\xi\rangle_\ell \tag{9}$$

with $\|H^\lambda\| \leq C$ while $\|\sum_{j=1}^{n-1} \phi_j^\lambda(z)\psi_j^\lambda(\xi)\| = \lambda \to \infty$. Applying Lemma 4 to ϕ^λ and ψ^λ if necessary, we can further assume that ϕ^λ and ψ^λ satisfy

$$\|\phi^\lambda\| \leq 1, \quad C\|\psi^\lambda\| \leq \|\sum_{j=1}^{n-1} \phi_j^\lambda(z)\psi_j^\lambda(\xi)\| = \lambda \leq (n-1)\|\psi^\lambda\|.$$

In special, for each $1 \leq j \leq n-1$,

$$\|\phi_j^\lambda\| \leq 1, \ \ \frac{1}{n-1} \leq \|\frac{\psi_j^\lambda}{\lambda}\| \leq \frac{1}{C}. \tag{10}$$

Dividing both sides of (9) by λ, then one obtains for some polynomial $\tilde{B}^\lambda$ that

$$\sum_{j=1}^{n-1}\phi_j^\lambda(z)\frac{\psi_j^\lambda(\xi)}{\lambda}=\frac{H^\lambda(z,\xi)}{\lambda}+\tilde{B}^\lambda(z,\xi)\langle z,\xi\rangle_\ell. \tag{11}$$

Since ϕ^λ and ψ^λ satisfy (10), we deduce after passing to a subsequence that ϕ^λ and $\frac{\psi^\lambda}{\lambda}$ converges, say, to polynomials ϕ^∞ and ψ^∞. Moreover, the same inequalities in (10) pass onto ϕ^∞ and ψ^∞ without change, i.e., $\|\phi^\infty\|\leq 1$, $\frac{1}{n-1}\leq\|\psi^\infty\|\leq C$ and $C\|\psi^\infty\|\leq\|\sum_{j=1}^{n-1}\phi_j^\infty(z)\psi_j^\infty(\xi)\|\leq(n-1)\|\psi^\infty\|$.

On the other hand, from (11) and Lemma 1, after passing $\lambda\to\infty$, there exists some polynomial B^∞ such that

$$\sum_{j=1}^{n-1}\phi_j^\infty(z)\psi_j^\infty(\xi)=B^\infty(z,\xi)\langle z,\xi\rangle_\ell.$$

According to Lemma 2, it immediately gives that

$$\sum_{j=1}^{n-1}\phi_j^\infty(z)\psi_j^\infty(\xi)=0.$$

This however contradicts with the fact that $\|\sum_{j=1}^{n-1}\phi_j^\infty(z)\psi_j^\infty(\xi)\|\geq C\|\psi^\infty\|\geq\frac{C}{n-1}$. Therefore, there exists some $\tilde{C}$ dependent only on (C,k,m,n) such that $\|\sum_{j=1}^{n-1}\phi_j(z)\psi_j(\xi)\|\leq\tilde{C}$ and hence $\|B\|\leq\tilde{C}$ because of Lemma 1. $\square$

With a routine induction process, Lemmas 1 and 3 combined together can be used to show the following:

Lemma 5. *Let $\{\phi_{jr}\}_{j=1}^{n-1}$ and $\{\psi_{jr}\}_{j=1}^{n-1}$ be two families of holomorphic polynomials in $\mathbf{C}^n$, $1\leq r\leq m$. Let $H(z,\xi),B(z,\xi)$ be two polynomials in (z,ξ). Suppose that*

$$\sum_{r=1}^{m}\left(\sum_{j=1}^{n-1}\phi_{jr}(z)\psi_{jr}(\xi)\right)\langle z,\xi\rangle_\ell^r=H(z,\xi)+B(z,\xi)\langle z,\xi\rangle_\ell^{m+1}$$

and $\|H\|\leq C$. Then $\|B\|\leq\tilde{C}$ and $\|\sum_{j=1}^{n-1}\phi_{jr}(z)\psi_{jr}(\xi)\|\leq\tilde{C}$ for all $1\leq r\leq m$ with $\tilde{C}$ dependent only on (C,n,m) and the degrees of ϕ_{jr},ψ_{jr} for all $1\leq r\leq m$.

4 Proof of the main theorems

The proof of the main theorems is motivated by the ideas in [HZ2] and [EHZ]. Assume F is not CR transversal to M_ℓ at 0 and $F(U)\not\subset H_\ell^N$. Assume also $N-n<n-1$ for the moment.

By a result of [BER2], the set of points where the CR transversality holds for such an F forms an open dense subset in M_ℓ. Choose a sequence $\{p_j\}\in M_\ell$ such that $p_j\to 0$ and F is CR transversal at each p_j with $j\geq 1$. Write $q_j:=F(p_j)$. Now for each j, applying the normalization process to F at p_j as in section 2, we obtain $F^\sharp_{p_j}$ in the following form:

$$F^\sharp_{p_j}=(f^\sharp_{p_j},\phi^\sharp_{p_j},g^\sharp_{p_j})=({f_1}^\sharp_{p_j},\ldots,{f_n}^\sharp_{p_j},\phi^\sharp_{p_j},g^\sharp_{p_j}):=T_{p_j}\circ\tau_{F(p_j)}\circ F\circ\sigma_{p_j}, \qquad (12)$$

where

$$\begin{aligned} f^\sharp_{p_j}(z,w)&=z+\tfrac{i}{2}a^{(1,0)}_{p_j}(z)w+o_{wt}(3),\\ \phi^\sharp_{p_j}(z,w)&=\phi^{(2,0)}_{p_j}(z)+o_{wt}(2),\\ g^\sharp_{p_j}(z,w)&=w+o_{wt}(4),\end{aligned}$$

with the following CR Gauss-Codazzi equation

$$\langle a^{(1,0)}_{p_j}(z),\bar z\rangle_\ell|z|^2_\ell=|\phi^{(2,0)}_{p_j}(z)|^2+\frac{1}{4}\mathscr{S}_{p_j}(z). \qquad (13)$$

Here $\tau_{F(p_j)}$ is the translation map of H_ℓ^N sending $F(p_j)$ to 0, σ_{p_j} is a biholomorphic map sending 0 to p_j such that $\sigma^{-1}_{p_j}(M_\ell)$ is normalized up to the 4th order, and $\mathscr{S}_{p_j}$ is the resulting Chern-Moser-Weyl curvature function of M_ℓ at p_j. Note σ_{p_j} depends smoothly on p_j. Since F is not CR transversal at 0, $\lim_{j\to\infty}\lambda_{p_j}=0$ with λ_{p_j} defined in (4) for the map $\tau_{F(p_j)}\circ F\circ\sigma_{p_j}$. By construction, at each point p_j, $F^\sharp_{p_j}$ sends $\sigma^{-1}_{p_j}(M_\ell)$ into H_ℓ^N. We then have for $(z,u)\approx 0$,

$$\begin{aligned} -\Im g^\sharp_{p_j}(z,u+i(|z|^2_\ell+o_{wt}(3)))+|f^\sharp_{p_j}(z,u+i(|z|^2_\ell+o_{wt}(3)))|^2_\ell+\\ +|\phi^\sharp_{p_j}(z,u+i(|z|^2_\ell+o_{wt}(3)))|^2=0, \end{aligned} \qquad (14)$$

Here $(z,u+i(|z|^2_\ell+o_{wt}(3)))$ is a local parametrization of $\sigma^{-1}_{p_j}(M_\ell)$ near 0. Due to the smooth dependence of σ_{p_j} with respect to p_j, the error term $o_{wt}(3)$ depends smoothly on p_j. With an abuse of notation, we shall suppress $\sharp$ and the subindex j of p for the map in (14).

Given any positive integer $k\geq 2$, collect terms of weighted degree k in the power series expansion of (14). We have:

$$\begin{aligned} &\Im g^{(k)}_p(z,w)-2\Re\langle f^{(k-1)}_p(z,w),\bar z\rangle_\ell=(|\phi_p(z,w)|^2)^{(k)}\\ +\quad &H(g^{(r)}_p|_{0\leq r\leq k-1},f^{(r)}_p|_{0\leq r\leq k-2},(|\phi_p|^2)^{(r)}|_{0\leq r\leq k-1}) \end{aligned} \qquad (15)$$

on $w = u + i|z|^2_\ell$. Here H is a certain bounded polynomial on all its variables. From now on and in what follows, we use C in general to represent constants independent of p, and use $H(\cdot,\cdot)$ in general to represent polynomials whose norm is bounded by C. C and H may be different in different contexts.

Lemma 6. *Assume that $N-n<n-1$. For F_p constructed as above and for each k, $\|F_p^{(k)}\| \le C$ with C independent of p.*

Proof of Lemma 6: According to the normalization procedure conducted in Section 2, $\|g_p^{(k)}\| \le C, \|f_p^{(k-1)}\| \le C, \|(|\phi_p|^2)^{(k)}\| \le C$ automatically hold when $k \le 4$. Indeed, $\|g_p^{(k)}\| \le 1, \|f_p^{(k-2)}\| \le 1, \|(|\phi_p|^2)^{(k-1)}\| = 0, k \le 4$ by (12). Moreover, since $\|\mathscr{S}_p\| \le C$, applying Lemma 3 to (13), one has $\|f_p^{(3)}\| \le C, \|(|\phi_p|^2)^{(4)}\| \le C$.

Assuming by induction that $(\|g_p^{(j)}\|, \|f_p^{(j-1)}\|, \|(|\phi_p|^2)^{(j)}\|)$ are all uniformly bounded by some constant independent of p for $j \le k$, we shall show the unform boundedness of $(\|g_p^{(k+1)}\|, \|f_p^{(k)}\|, \|(|\phi_p|^2)^{(k+1)}\|)$. Complexifying (15) at level $k+1$, we obtain

$$\begin{aligned} &g_p^{(k+1)}(z,w) - \bar{g}_p^{(k+1)}(\xi,\eta) - 2i\langle f_p^{(k)}(z,w),\xi\rangle_\ell - 2i\langle \bar{f}_p^{(k)}(\xi,\eta),z\rangle_\ell \\ &\qquad = 2i\langle \phi_p(z,w),\bar{\phi}_p(\xi,\eta)\rangle^{(k+1)} + H(z,\xi,w,\eta) \end{aligned} \tag{16}$$

which holds on $w-\eta = 2i\langle z,\xi\rangle_\ell$.

Let $L_j = \frac{\partial}{\partial z_j} + 2i\delta_j\xi_j\frac{\partial}{\partial w}, 1 \le j \le n-1$ with $\delta_j = -1$ when $j \le \ell$ and $\delta_j = 1$ with $j \ge \ell+1$. Then L_j is a holomorphic tangent vector field on $w-\eta = 2i\langle z,\xi\rangle_\ell$ for each j. Applying L_j onto (16), we get

$$\begin{aligned} &L_j g_p^{(k+1)}(z,w) - 2i\langle L_j f_p^{(k)}(z,w),\xi\rangle_\ell - 2i\bar{f}_{p,j}^{(k)}(\xi,\eta) \\ &\qquad = 2iL_j\langle \phi_p(z,w),\bar{\phi}_p(\xi,\eta)\rangle^{(k+1)} + H(z,\xi,w,\eta) \end{aligned} \tag{17}$$

on $w-\eta = 2i\langle z,\xi\rangle_\ell$.

Now we expand $g_p^{(k+1)}, f_p^{(k)}, \langle \phi_p(z,w),\bar{\phi}_p(\xi,\eta)\rangle^{(k+1)}$ in the following manner:

$$g_p^{(k+1)}(z,w) = \sum_{\mu+2\nu=k+1} (g_p)_{\mu\nu}(z)w^\nu;$$

$$f_p^{(k)}(z,w) = \sum_{\mu+2\nu=k} (f_p)_{\mu\nu}(z)w^\nu;$$

$$\langle \phi_p(z,w),\bar{\phi}_p(\xi,\eta)\rangle^{(k+1)} = \sum_{\mu+\gamma+2(\nu+\delta)=k+1} (A_p)_{\mu\gamma\nu\delta}(z,\xi)w^\nu\eta^\delta.$$

Here $(g_p)_{\mu\nu}$ and $(f_p)_{\mu\nu}$ are homogeneous polynomials of degree μ in z, $(A_p)_{\mu\gamma\nu\delta}$ is a homogeneous polynomial of bi-degree (μ,γ) in (z,ξ).

Let $w=0, \eta = -2i\langle z,\xi\rangle_\ell$ in (16). Then we have

$$(g_p)_{(k+1)0}(z) - \sum_{\mu+2\nu=k+1} (\bar{g}_p)_{\mu\nu}(\xi)\eta^\nu - 2i\langle (f_p)_{k0}(z),\xi\rangle_\ell$$

$$-2i\langle \sum_{\mu+2\nu=k} (\bar{f}_p)_{\mu\nu}(\xi)\eta^\nu, z\rangle_\ell = 2i \sum_{\mu+\gamma+2\delta=k+1} (A_p)_{\mu\gamma0\delta}(z,\xi)\eta^\delta + H(z,\xi,\eta) \quad (18)$$

on $\eta = -2i\langle z,\xi\rangle_\ell$.

Collect terms in (z,ξ) of bi-degree $(k+1,0)$ and $(k,1)$ in (18). By the fact that $\phi_p(0) = \frac{\partial\phi_p}{\partial z}(0) = \frac{\partial\phi_p}{\partial w}(0) = 0$ and the definition of $(A_p)_{\mu\gamma\nu\delta}$,

$$(A_p)_{k+1,0,0,0} = (A_p)_{k,1,0,0} = (A_p)_{k-1,0,0,1} = 0. \quad (19)$$

Then we have that

$$\|(f_p)_{k0}\| \le C, \quad \|(g_p)_{(k+1)0}\| \le C.$$

Hence for each $1 \le j \le n-1$,

$$\begin{aligned} L_j f_p^{(k)}(z,0) &= 2i\delta_j\xi_j \sum_{\mu=k-2} (f_p)_{\mu 1}(z) + H(z); \\ L_j g_p^{(k+1)}(z,0) &= 2i\delta_j\xi_j \sum_{\mu=k-1} (g_p)_{\mu 1}(z) + H(z). \end{aligned} \quad (20)$$

Collecting terms in (z,ξ) of bi-degree $(\alpha,\beta), \beta \ge 2$ with $\alpha+\beta = k+1$ in (18) gives

$$\begin{aligned} &-(\bar{g}_p)_{\beta-\alpha,\alpha}(\xi)\eta^\alpha - 2i\langle z, (\bar{f}_p)_{\beta-\alpha+1,\alpha-1}(\xi)\eta^{\alpha-1}\rangle_\ell \\ &\quad = 2i\sum_{\theta=0}^{\alpha-2} (A_p)_{\alpha-\theta,\beta-\theta,0,\theta}(z,\xi)\eta^\theta + H(z,\xi,\eta) \end{aligned} \quad (21)$$

with $\eta = -2i\langle z,\xi\rangle_\ell$. Here once again we used the fact that $\phi_p(0) = \frac{\partial\phi_p}{\partial z}(0) = 0$ which implies $(A_p)_{1,(\beta-\alpha-1),0,(\alpha-1)} = (A_p)_{0,\beta-\alpha,0,\alpha} = 0$, so the summation on the right hand side runs only till $\theta = \alpha - 2$. Recall from the definition of A_p, $(A_p)_{\mu\gamma\nu\delta}(z,\xi) = \sum_{j=1}^{N-n} \phi_{p,j}^{(\mu,\nu)}(z,1)\bar{\phi}_{p.j}^{(\gamma,\delta)}(\xi,1)$. Since $N-n < n-1$ by assumption, we immediately have, by applying Lemma 5 to (21) and by using (19), that

$$\|(\bar{g}_p)_{\beta-\alpha,\alpha}(\xi)\langle z,\xi\rangle_\ell - \langle z, (\bar{f}_p)_{\beta-\alpha+1,\alpha-1}(\xi)\rangle_\ell\| \le C \quad (22)$$

with $\beta \ge 2$, and

$$\|(A_p)_{\mu\gamma0\delta}\| \le C.$$

Hence from the above inequality,

$$L_j(A_p)(z,\xi,0,\eta) = 2i\delta_j\xi_j \sum_{\mu+\gamma+2\delta=k-1} (A_p)_{\mu\gamma1\delta}(z,\xi)\eta^\delta + H(z,\xi,\eta). \quad (23)$$

Letting $w = 0, \eta = -2i\langle z,\xi\rangle_\ell$ and then substituting (20) and (23) in (17), we have for each $1 \le j \le n-1$,

$$2i\delta_j\xi_j \sum_{\mu=k-1} (g_p)_{\mu 1}(z) - 2i\langle 2i\delta_j\xi_j \sum_{\mu=k-2} (f_p)_{\mu 1}(z), \xi\rangle_\ell - 2i \sum_{\mu+2\nu=k+1} (\bar{f}_{p,j})_{\mu\nu}(\xi)\eta^\nu$$

$$= 2i\delta_j\xi_j \sum_{\mu+\gamma+2\delta=k-1} (A_p)_{\mu\gamma 1\delta}(z,\xi)\eta^\delta + H(z,\xi,\eta) \tag{24}$$

on $\eta = -2i\langle z,\xi\rangle_\ell$. Collect terms in (z,ξ) of bi-degree $(k-1,1)$ and $(k-2,2)$ in (24). Since $(A_p)_{k-1,0,1,0} = (A_p)_{k-3,0,1,1} = 0$, one obtains that

$$\begin{gathered}\|(g_p)_{(k-1)1}\| \leq C,\\ \|2i\delta_j\langle \xi_j (f_p)_{(k-2)1}(z),\xi\rangle_\ell + (\bar f_{p,j})_{(4-k)(k-2)}(\xi)(-2i\langle z,\xi\rangle_\ell)^{k-2}\| \leq C.\end{gathered} \tag{25}$$

Here we have used the convention that $h_\mu = 0$ if μ is negative.

Moreover, collecting terms of bi-degree (α,β) in (z,ξ) with $\beta \geq 3$ and $\alpha+\beta = k$ in (24), one gets for each $1 \leq j \leq n$,

$$\begin{aligned}&(\bar f_{p,j})_{(\beta-\alpha)\alpha}(\xi)(-2i\langle z,\xi\rangle_\ell)^\alpha\\ &= -\delta_j\xi_j \sum_{\theta=0}^{\alpha-1}(A_p)_{(\alpha-\theta)(\beta-\theta-1)1\theta}(z,\xi)(-2i\langle z,\xi\rangle_\ell)^\theta + H(z,\xi).\end{aligned}$$

Here we use the fact that $(A_p)_{0,\beta-\alpha-1,1,\alpha} = 0$, so the summation on the right hand sides runs only till $\alpha - 1$. Applying Lemma 5 onto the above identity as before, we obtain $\|(f_p)_{\mu\nu}\| \leq C$ for $\mu+2\nu = k$ with $\mu+\nu \geq 3$. When $\mu+2\nu = k \geq 4$ with $\mu+\nu \leq 2$, or equivalently, when $\mu = 0, \nu = 2$, one substitutes the fact that $\|(f_p)_{21}\| \leq C$ into (25) and gets $\|(f_p)_{02}\| \leq C$ and hence

$$\|(f_p)_{\mu\nu}\| \leq C \tag{26}$$

for $\mu+2\nu = k$. Substitute (26) into (22), then

$$\|(g_p)_{\mu\nu}\| \leq C \tag{27}$$

for $\mu+2\nu = k+1$ (with $\mu+\nu \geq 3$, which is always fulfilled when $\mu+2\nu = k+1 \geq 5$).

Using equation (16), we then have from (26) and (27),

$$\langle \phi_p(z,w), \bar\phi_p(\xi,\eta)\rangle^{(k+1)} = H(z,\xi,w,\eta) \tag{28}$$

on $w - \eta = 2i\langle z,\xi\rangle_\ell$.

We claim that, for arbitrary (z,w,ξ,η), we have

$$\langle \phi_p(z,w), \bar\phi_p(\xi,\eta)\rangle^{(k+1)} = H(z,\xi,w,\eta).$$

Indeed, by (28), we have

$$\sum_{\mu+\gamma+2(\nu+\delta)=k+1} (A_p)_{\mu\gamma\nu\delta}(z,\xi)\left(\eta + 2i\langle z,\xi\rangle_\ell\right)^\nu \eta^\delta = H(z,\xi,\eta) \tag{29}$$

near 0. If $\|(A_p)_{\mu\gamma\nu\delta}\| \leq C$ does not hold uniformly in p, then there exists a smallest integer δ_0 such that $\|(A_p)_{\mu\gamma\nu\delta_0}\| \to \infty$ as $p \to 0$ after passing to a subsequence if necessary. Moving the terms with $\delta < \delta_0$ to the right, we obtain

$$\sum_{\mu+\gamma+2(\nu+\delta_0)=k+1} (A_p)_{\mu\gamma\nu\delta_0}(z,\xi)\big(2i\langle z,\xi\rangle_\ell\big)^\nu = H(z,\xi).$$

Collecting terms in (z,ξ) of bi-degree (α,β) with $\alpha+\beta = k+1-2\delta_0$ in the above expression, we get

$$\sum_{\theta=0}^{\alpha} (A_p)_{(\alpha-\theta)(\beta-\theta)\theta\delta_0}(z,\xi)\big(2i\langle z,\xi\rangle_\ell\big)^\theta = H(z,\xi).$$

Recall $(A_p)_{\mu\gamma\nu\delta} = \sum_{j=1}^{N-n} \phi_{p,j}^{(\mu,\nu)}(z,1)\bar{\phi}_{p.j}^{(\gamma,\delta)}(\xi,1)$ by definition and $N-n < n-1$. Applying Lemma 5 to the above identity, one deduces $\|(A_p)_{\mu\gamma\nu\delta_0}\| \leq C$ for $\mu+\gamma+2(\nu+\delta_0) = k+1$. This is a contradiction! Hence the claim holds.

The induction is thus complete. Consequently, for each k, $\|\phi_p^{(k)}\| \leq C$ with C independent of p. We have shown for each fixed k, $\{\|F_{p_j}^{(k)}\|\}_{j=1}^\infty$ is bounded by some constant independent of j. $\square$

We are now in a position to prove Theorem 2 and Theorem 1, making use of the result in [MMZ].

Proof of Theorem 2: If F is CR transversal to M_ℓ at 0, then we are done. Assume F is not CR transversal at 0. Then there exists $p_j \to 0$ such that F_{p_j} as constructed at the beginning of the section satisfies (12). Moreover, for each k, $\|F_{p_j}^{(k)}\| \leq C$ with C independent of j by Lemma 6. Following the same trick as in [HZ2], for each k, $\{F_{p_j}^{(k)}\}_{j=1}^\infty$ converges as $j \to \infty$ after passing to subsequences, to a certain $F^{*(k)}$. By the way these maps were constructed, the nontrivial formal map $F^*(=(f^*,\phi^*,g^*)) := \sum_k F^{*(k)}$ sends M_ℓ into H_ℓ^N satisfying the following normalization:

$$\begin{aligned} f^*(z,w) &= z + o_{wt}(2), \\ \phi^*(z,w) &= o_{wt}(1), \\ g^*(z,w) &= w + o_{wt}(4). \end{aligned}$$

According to a result of Meylan-Mir-Zaitsev [MMZ], the formal map F^* is convergent. Hence, F^* is a CR immersion at 0 sending M_ℓ into H_ℓ^N. $\square$

Proof of Theorem 1: Assume by contradiction that F neither is CR transversal to M_ℓ at 0 nor sends U into H_ℓ^N. Then there exists a CR immersion F^* sending M_ℓ into H_ℓ^N by Theorem 2. On the other hand, since any two CR transversal maps between a Levi-nondegenerate hypersurface and a hyperquadric of the same signature differ only by an automorphism of the hyperquadric (see [EHZ]) when the codimension is less than $\frac{n-1}{2}$, there exists an automorphism T of H_ℓ^N such that near $p_j \approx 0$, and hence at all points in M_ℓ near the origin,

$$F = T \circ F^*.$$

Since T extends to an automorphism of the projective space $\mathbf{P}^N$ and $T(0) = 0$, F must be CR transversal at 0. This is a contradiction. $\square$

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