Optimal Sobolev regularity of $\bar\partial$ on the Hartogs triangle

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Abstract

In this paper, we show that for each $k \in \mathbb{Z}^+, p > 4$, there exists a solution operator $\mathcal{T}_k$ to the $\bar\partial$ problem on the Hartogs triangle that maintains the same $W^{k,p}$ regularity as that of the data. According to a Kerzman-type example, this operator provides solutions with the optimal Sobolev regularity.

1 Introduction

The Hartogs triangle

$$\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}$$

is a pseudoconvex domain with non-Lipschitz boundary. It serves as a model counterexample for many questions in several complex variables. For instance, it does not admit a Stein neighborhood basis or a bounded plurisubharmonic exhaustion function. Meanwhile, Chaumat and Chollet showed in [2] that the corresponding $\bar\partial$ problem on $\mathbb{H}$ is not globally regular in the sense that there is a smooth $\bar\partial$-closed $(0,1)$-form $\mathbf{f}$ on $\overline{\mathbb{H}}$, such that $\bar\partial u = \mathbf{f}$ has no smooth solution on $\overline{\mathbb{H}}$. Interestingly, at each Hölder level the $\bar\partial$ equation does admit Hölder solutions with the same Hölder regularity as that of the data. For more properties on $\mathbb{H}$ please refer to a survey [15] of Shaw. On the other hand, the study of Sobolev regularity was initiated by Chakrabarti and Shaw in [5], where they carried out a weighted L^2-Sobolev estimate for the canonical solution on $\mathbb{H}$. See also a recent work [18] of Yuan and the second author on weighted L^p-Sobolev estimates of $\bar\partial$ on general quotient domains.

The goal of this paper is to study the optimal $\bar\partial$ regularity on $\mathbb{H}$ at each (unweighted) Sobolev level. Recently, the optimal L^p regularity of $\bar\partial$ on $\mathbb{H}$ was obtained by the second author in [19]. The following is our main theorem concerning the $W^{k,p}$ regularity, $k \ge 1$. As demonstrated by a Kerzman-type Example 2 (in Section 4), it gives the optimal $W^{k,p}$ regularity in the sense that for any $\epsilon > 0$, there exists a $W^{k,p}$ datum which has no $W^{k,p+\epsilon}$ solution to $\bar\partial$ on $\mathbb{H}$.

Theorem 1.1. *For each $k \in \mathbb{Z}^+, 4 < p < \infty$, there exists a solution operator $\mathcal{T}_k$ such that for any $\bar\partial$-closed $(0,1)$ form $\mathbf{f} \in W^{k,p}(\mathbb{H})$, $\mathcal{T}_k\mathbf{f} \in W^{k,p}(\mathbb{H})$ and solves $\bar\partial u = \mathbf{f}$ on $\mathbb{H}$. Moreover, there exists a constant C dependent only on k and p such that*

$$\|\mathcal{T}_k\mathbf{f}\|_{W^{k,p}(\mathbb{H})} \le C\|\mathbf{f}\|_{W^{k,p}(\mathbb{H})}.$$

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The general idea of the proof is as follows. According to a heuristic procedure to treat the $\bar{\partial}$ problem on the Hartogs triangle $\mathbb{H}$, one first uses the biholomorphism between the punctured bidisc and $\mathbb{H}$ to pull back the data and solve $\bar{\partial}$ on the punctured bidisc, and then pushes the solutions forward onto the Hartogs triangle. As a consequence of this, the corresponding Sobolev regularity of the $\bar{\partial}$ problem requires a weighted Sobolev regularity on product domains due to the presence of the nontrivial Jacobian of the biholomorphism. Based upon our recent weighted Sobolev result [14] about Cauchy-type integrals, we first obtain the following Sobolev regularity for $\bar{\partial}$ on product domains with respect to weights in some refined Muckenhoupt space A_p^* (see Definition 2.1).

Theorem 1.2. *Let $\Omega = D_1 \times \cdots D_n, n \geq 2$, where each D_j is a bounded domain in $\mathbb{C}$ with $C^{k,1}$ boundary. There exists a solution operator T such that for any $\bar{\partial}$-closed $(0,q)$ form $\mathbf{f} \in W^{k+n-2,p}(\Omega, \mu), k \in \mathbb{Z}^+, 1 < p < \infty, \mu \in A_p^*$, $T\mathbf{f} \in W^{k,p}(\Omega, \mu)$ and solves $\bar{\partial}u = \mathbf{f}$ on Ω. Moreover, there exists a constant C dependent only on Ω, k, p and the A_p^* constant of μ such that*

$$\|T\mathbf{f}\|_{W^{k,p}(\Omega,\mu)} \leq C\|\mathbf{f}\|_{W^{k+n-2,p}(\Omega,\mu)}.$$

As shown by Example 1 (in Section 3), Theorem 1.2 gives the optimal Sobolev regularity of solutions on product domains with dimension $n = 2$. Jin and Yuan obtained in [8] a similar Sobolev estimate for polydiscs in the case when $\mu \equiv 1$ and $q = 1$. It is also worth pointing out that the operator T considered in Theorem 1.2 fails to maintain the L^p (where $k = 0$) regularity in general. See [3] of Chen and McNeal for a $\bar{\partial}$-closed (0,1) form $\mathbf{f}$ in $L^p(\triangle^2)$ such that that $T\mathbf{f}$ fails to lie in $L^p(\triangle^2), p < 2$. Instead, [19] made use of the canonical solution operator to provide an optimal weighted L^p regularity for $\bar{\partial}$ on product domains in $\mathbb{C}^n$.

Theorem 1.2 readily gives a semi-weighted L^p-Sobolev estimate below for a (fixed) solution operator to $\bar{\partial}$ on $\mathbb{H}, p > 2$.

Corollary 1.3. *There exists a solution operator $\mathcal{T}$ such that for any $\bar{\partial}$-closed $(0,1)$ form $\mathbf{f} \in W^{k,p}(\mathbb{H}), k \in \mathbb{Z}^+, 2 < p < \infty$, $\mathcal{T}\mathbf{f} \in W^{k,p}(\mathbb{H}, |z_2|^{kp})$ and solves $\bar{\partial}u = \mathbf{f}$ on $\mathbb{H}$. Moreover, there exists a constant C dependent only on k and p such that*

$$\|\mathcal{T}\mathbf{f}\|_{W^{k,p}(\mathbb{H},|z_2|^{kp})} \leq C\|\mathbf{f}\|_{W^{k,p}(\mathbb{H})}.$$

The estimate in Corollary 1.3 maintains the Sobolev index (k, p), and in particular improves a result in [18]. We note that the $p > 2$ assumption in the corollary is due to the fact that the weight after pulling the data on $\mathbb{H}$ back to the bidisc lies in A_p^* only when $p > 2$, where Theorem 1.2 can be applied. Unfortunately, the solution operator $\mathcal{T}$ here subjects to some quantified loss in the exponent of the weight at each Sobolev level. Although this weight loss is not unexpected due to the global irregularity of $\bar{\partial}$ on $\mathbb{H}$, $\mathcal{T}$ does not provide an optimal Sobolev regularity.

In order to obtain the optimal Sobolev regularity for $\bar{\partial}$ on $\mathbb{H}$, one needs to further adjust the solution operator $\mathcal{T}$ in Corollary 1.3 accordingly at different Sobolev levels. In fact, we apply to $\mathcal{T}$ a surgical procedure – truncation by Taylor polynormials: one on the data, and another on the $\bar{\partial}$ solution on the punctured bidisc. The idea was initially introduced by Ma and Michel in [11] to treat the Hölder regularity. In the Sobolev category when $p > 4$, this procedure at order $k-1$ is meaningful and in the strong (continuous) sense due to the Sobolev embedding theorem. Note that the top k-th order derivatives are still in the weak (distributional) sense where we need to use discretion. After a careful inspection of the post-surgical regularity on the pull-back of the data and push-forward of the solutions on the punctured bidisc, we utilize a weighted Hardy-type inequality to obtain a sequence of refined Sobolev estimates. These estimates eventually allow the weight loss from the singularity at $(0,0)$ to be precisely (and fortunately) compensated by the weight gain from the truncation, so that the truncated solution enjoys the (unweighted) Sobolev regularity in Theorem 1.1. Throughout our proof, the assumptions $k \geq 1, p > 4$ are crucial and repeatedly used. It is not clear whether the theorem still holds if $p \leq 4$.

The organization of the paper is as follows. In Section 2, we give notations and preliminaries that are needed in the paper. In Section 3, we prove Theorem 1.2 for the weighted Sobolev estimate on product domains, from which Corollary 1.3 follows. Section 4 is devoted to the proof of the main Theorem 1.1 for the Sobolev estimate on the Hartogs triangle.

2 Notations and preliminaries

2.1 Weighted Sobolev spaces

Denote by $|S|$ the Lebesgue measure of a subset S in $\mathbb{C}^n$, and dV_{z_j} the volume integral element in the complex z_j variable. For $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, let $\hat{z}_j = (z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_n) \in \mathbb{C}^{n-1}$, where the j-th component of z is skipped. Our weight space under consideration is as follows.

Definition 2.1. *Given $1 < p < \infty$, a weight $\mu : \mathbb{C}^n \to [0, \infty)$ is said to be in A_p^* if the A_p^* constant*

$$A_p^*(\mu) := \sup \left(\frac{1}{|D|} \int_D \mu(z) dV_{z_j} \right) \left(\frac{1}{|D|} \int_D \mu(z)^{\frac{1}{1-p}} dV_{z_j} \right)^{p-1} < \infty,$$

where the supremum is taken over a.e. $\hat{z}_j \in \mathbb{C}^{n-1}, j = 1, \ldots, n$, and all discs $D \subset \mathbb{C}$.

When $n = 1$, the A_p^* space coincides with the standard Muckenhoupt's class A_p, the collection of all weights $\mu : \mathbb{C}^n \to [0, \infty)$ satisfying

$$A_p(\mu) := \sup \left(\frac{1}{|B|} \int_B \mu(z) dV_z \right) \left(\frac{1}{|B|} \int_B \mu(z)^{\frac{1}{1-p}} dV_z \right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{C}^n$. Clearly, $A_q \subset A_p$ if $1 < q < p < \infty$. A_p spaces also satisfy an open-end property: if $\mu \in A_p$ for some $p > 1$, then $\mu \in A_{\tilde{p}}$ for some $\tilde{p} < p$. See [16, Chapter V] for more details of the A_p class.

When $n \geq 2$, Definition 2.1 essentially says that $\mu \in A_p^*$ if and only if the restriction of μ on any complex one-dimensional slice $\hat{z}_j$ belongs to A_p, with a uniform A_p bound independent of $\hat{z}_j$. On the other hand, $\mu \in A_p^*$ if and only if the δ-dilation $\mu_\delta(z) := \mu(\delta_1 z_1, \ldots, \delta_n z_n) \in A_p$ with a uniform A_p constant for all $\delta = (\delta_1, \ldots, \delta_n) \in (\mathbb{R}^+)^n$ (see [6, pp. 454]). This in particular implies $A_p^* \subset A_p$. As will be seen in the rest of the paper, the setting of A_p^* weights allows us to apply the slicing property of product domains rather effectively.

Let Ω be a bounded domain in $\mathbb{C}^n$. Denote by $\mathbb{Z}^+$ the set of all positive integers. Given $k \in \mathbb{Z}^+ \cup \{0\}, p \geq 1$, the weighted Sobolev space $W^{k,p}(\Omega, \mu)$ with respect to a weight $\mu \geq 0$ is the set of functions on Ω whose weak derivatives up to order k exist and belong to $L^p(\Omega, \mu)$. The corresponding weighted $W^{k,p}$ norm of a function $h \in W^{k,p}(\Omega, \mu)$ is

$$\|h\|_{W^{k,p}(\Omega,\mu)} := \left(\sum_{l=0}^{k} \int_\Omega |\nabla_z^l h(z)|^p \mu(z) dV_z \right)^{\frac{1}{p}} < \infty.$$

Here $\nabla_z^l h$ represents all l-th order weak derivatives of h. When $\mu \equiv 1$, $W^{k,p}(\Omega,\mu)$ is reduced to the (unweighted) Sobolev space $W^{k,p}(\Omega)$. As a direct consequence of the open-end property for A_p and Hölder inequality, if $\mu \in A_p, p>1$, there exists some $q>1$ such that $W^{k,p}(\Omega,\mu) \subset W^{k,q}(\Omega)$.

In the rest of the paper, for each $j=1,\dots,n$, we use $\nabla_{z_j}^{\alpha_j} h$ to specify all α_j-th order weak derivatives of h in the complex z_j-th direction. For a multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$, denote $\nabla_{z_1}^{\alpha_1}\cdots\nabla_{z_n}^{\alpha_n}$ by ∇_z^α. Then for $l \in \mathbb{Z}^+$, $\nabla_z^l = \sum_{|\alpha|=l}\nabla_z^\alpha$. We also represent the α_j-th order derivative of h with respect to the holomorphic z_j and anti-holomorphic $\bar z_j$ variable by $\partial_{z_j}^{\alpha_j} h$ and $\bar\partial_{z_j}^{\alpha_j} h$, respectively. When the context is clear, the letter z may be dropped from those differential operators and we write instead $\nabla^l, \nabla_j^{\alpha_j}, \nabla^\alpha, \partial_j^{\alpha_j}$ and $\bar\partial_j^{\alpha_j}$ etc.

2.2 Weighted Sobolev estimates on planar domains

Let D be a bounded domain in $\mathbb{C}$ with Lipschitz boundary. For $p>1, z\in D$, define

$$\begin{aligned} Th(z) &:= \frac{-1}{2\pi i}\int_D \frac{h(\zeta)}{\zeta - z} d\bar\zeta \wedge d\zeta, \quad \text{for} \quad h \in L^p(D);\\ Sh(z) &:= \frac{1}{2\pi i}\int_{bD} \frac{h(\zeta)}{\zeta - z} d\zeta, \quad \text{for} \quad h \in L^p(bD). \end{aligned}$$

Clearly, $d\bar\zeta\wedge d\zeta = 2i dV_\zeta$ in the above. T and S satisfy the Cauchy-Green formula below: for any $h \in W^{1,p}(D), p>1$,

$$h = Sh + T\bar\partial h \quad \text{on} \quad D$$

in the sense of distributions.

The following weighted Sobolev regularity of T and S is essential in order to carry out the weighted Sobolev regularity of $\bar\partial$ on product domains. It is worthwhile to note that (2.2) below fails if $k=0$, where S is not even well-defined.

Theorem 2.2. *[14] Let $D \subset \mathbb{C}$ be a bounded domain with $C^{k,1}$ boundary and $\mu \in A_p, 1<p<\infty$. For $k \in \mathbb{Z}^+ \cup \{0\}$, there exists a constant C dependent only on D, k, p and $A_p(\mu)$, such that for all $h \in W^{k,p}(D,\mu)$,*

$$\|Th\|_{W^{k+1,p}(D,\mu)} \le C\|h\|_{W^{k,p}(D,\mu)}. \tag{2.1}$$

If in addition $k \in \mathbb{Z}^+$, then

$$\|Sh\|_{W^{k,p}(D,\mu)} \le C\|h\|_{W^{k,p}(D,\mu)}. \tag{2.2}$$

2.3 Product domains and the Hartogs triangle

A subset $\Omega \subset \mathbb{C}^n$ is said to be a product domain, if $\Omega = D_1 \times \cdots \times D_n$, where each $D_j \subset \mathbb{C}, j=1,\dots,n$, is a bounded domain in $\mathbb{C}$ such that its boundary bD_j consists of a finite number of rectifiable Jordan curves which do not intersect one another. A product domain Ω is always pseudoconvex, and has Lipschitz boundary if in addition each bD_j is Lipschitz, $j=1,\dots,n$.

Denote by $\triangle$ the unit disc in $\mathbb{C}$, and by $\triangle^* := \triangle \setminus \{0\}$ the punctured disc on $\mathbb{C}$. Then the punctured bidisc $\triangle \times \triangle^*$ is biholomorphic to the Hartogs triangle $\mathbb{H}$ through the map $\psi : \triangle \times \triangle^* \to \mathbb{H}$, where

$$(w_1, w_2) \in \triangle \times \triangle^* \mapsto (z_1, z_2) = \psi(w) = (w_1 w_2, w_2) \in \mathbb{H}. \tag{2.3}$$

The inverse $\phi : \mathbb{H} \to \triangle \times \triangle^*$ is given by

$$(z_1, z_2) \in \mathbb{H} \mapsto (w_1, w_2) = \phi(z) = \left(\frac{z_1}{z_2}, z_2\right) \in \triangle \times \triangle^*. \tag{2.4}$$

Note that $\mathbb{H}$ is not Lipschtiz near $(0, 0)$.

It is well-known that any domain with Lipschtiz boundary is a uniform domain (see [7] for the definition). Recently, it was shown in [1, Theorem 2.12] that the Hartogs triangle is also a uniform domain. Thus according to [9] [4, Theorem 1.1], both Lipschitz product domains and the Hartogs triangle satisfy a weighted Sobolev extension property. Namely, let Ω be either a Lipschitz product domain or the Hartogs triangle. Then for any weight $\mu \in A_p, 1 < p < \infty, k \in \mathbb{Z}^+$, any $h \in W^{k,p}(\Omega, \mu)$ can be extended as an element $\tilde{h}$ in $W^{k,p}(\mathbb{C}^n, \mu)$ such that

$$\|\tilde{h}\|_{W^{k,p}(\mathbb{C}^n,\mu)} \leq C\|h\|_{W^{k,p}(\Omega,\mu)}$$

for some constant C dependent only on k, p and the A_p constant of μ.

For simplicity of notations, throughout the rest of the paper, we shall say the two quantities a and b satisfy $a \lesssim b$, if $a \leq Cb$ for some constant $C > 0$ dependent only possibly on Ω, k, p and the A_p^* constant $A_p^*(\mu)$ (or $A_p(\mu)$).

3 Weighted Sobolev estimates on product domains

Let $D_j \subset \mathbb{C}$, $j = 1, \ldots, n$, be bounded domains with $C^{k,1}$ boundary, $n \geq 2, k \in \mathbb{Z}^+ \cup \{0\}$, and let $\Omega := D_1 \times \cdots \times D_n$. Denote by T_j and S_j the solid and boundary Cauchy integral operators T and S acting on functions along the j-th slice of Ω, respectively. Namely, for $p > 1, z \in \Omega$,

$$\begin{aligned} T_j h(z) &:= \frac{-1}{2\pi i} \int_{D_j} \frac{h(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n)}{\zeta - z_j} d\bar{\zeta} \wedge d\zeta, \quad \text{for} \quad h \in L^p(\Omega); \\ S_j h(z) &:= \frac{1}{2\pi i} \int_{bD_j} \frac{h(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n)}{\zeta - z_j} d\zeta, \quad \text{for} \quad h \in L^p(b\Omega). \end{aligned} \tag{3.1}$$

Proposition 3.1. *Let $\Omega = D_1 \times \cdots \times D_n$, where each D_j is a bounded domain in $\mathbb{C}$ with $C^{k,1}$ boundary, $k \in \mathbb{Z}^+ \cup \{0\}$. Assume $\mu \in A_p^*, 1 < p < \infty$. Then for any $h \in W^{k,p}(\Omega, \mu)$,*

$$\|T_j h\|_{W^{k,p}(\Omega,\mu)} \lesssim \|h\|_{W^{k,p}(\Omega,\mu)}. \tag{3.2}$$

If in addition $k \in \mathbb{Z}^+$, then

$$\|S_j h\|_{W^{k-1,p}(\Omega,\mu)} \lesssim \|h\|_{W^{k,p}(\Omega,\mu)}. \tag{3.3}$$

Proof. Without loss of generality, assume $j = 1$ and $n = 2$. For any multi-index $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| \leq k$, since $\bar{\partial}_1 T_1 = id$, we can further assume $\nabla^\alpha T_1 h = \partial_1^{\alpha_1} T_1 (\nabla_2^{\alpha_2} h)$. For a.e. fixed $z_2 \in D_2$, $\mu(\cdot, z_2) \in A_p$ and $\nabla_2^{\alpha_2} h(\cdot, z_2) \in W^{\alpha_1,p}(D_1, \mu(\cdot, z_2))$. Making use of (2.1), we have

$$\int_{D_1} |\partial_1^{\alpha_1} T_1 (\nabla_2^{\alpha_2} h)(z_1, z_2)|^p \mu(z_1, z_2) dV_{z_1} \lesssim \sum_{l=0}^{\alpha_1} \int_{D_1} |\nabla_1^l \nabla_2^{\alpha_2} h(z_1, z_2)|^p \mu(z_1, z_2) dV_{z_1}.$$

Thus

$$\|\nabla^\alpha T_1 h\|_{L^p(\Omega,\mu)}^p = \int_{D_2} \int_{D_1} |\partial_1^{\alpha_1} T_1 (\nabla_2^{\alpha_2} h)(z_1, z_2)|^p \mu(z_1, z_2) dV_{z_1} dV_{z_2} \lesssim \|h\|_{W^{k,p}(\Omega,\mu)}^p.$$

The boundedness of S_1 is proved similarly. Since $S_1 h$ is holomorphic with respect to the z_1 variable, we only consider $\nabla^\alpha S_1 h(z) = \partial_1^{\alpha_1} S_1(\nabla_2^{\alpha_2} h)$ with $|\alpha| \le k-1$. Then $\nabla_2^{\alpha_2} h(\cdot, z_2) \in W^{k-\alpha_2,p}(D_1)$ for a.e. $z_2 \in D_2$. Noting that $k - \alpha_2 \ge 1$, by (2.2),

$$\int_{D_1} |\partial_1^{\alpha_1} S_1\left(\nabla_2^{\alpha_2} h\right)(z_1,z_2)|^p \mu(z_1,z_2) dV_{z_1} \lesssim \sum_{l=0}^{\alpha_1+1} \int_{D_1} |\nabla_1^l \nabla_2^{\alpha_2} h(z_1,z_2)|^p \mu(z_1,z_2) dV_{z_1}.$$

Here the sum for l up to $\alpha_1 + 1$ is necessary in the case when $\alpha = (0, k-1)$, due to the absence of (2.2) at $k = 0$ there. Hence $\|\nabla^\alpha S_1 h\|_{L^p(\Omega,\mu)} \lesssim \|h\|_{W^{k,p}(\Omega,\mu)}$.

$\square$

Remark 3.2. *a). The estimate (3.2) is optimal. Indeed, consider $h(z_1,z_2) = |z_2|^{k-\frac{2}{p}}$ on $\triangle \times \triangle$. Then $h \in W^{k,s}(\triangle \times \triangle)$ for all $s < p$. However, $T_1 h(z_1,z_2) = \bar{z}_1 |z_2|^{k-\frac{2}{p}} \notin W^{k,p}(\triangle \times \triangle)$.*
b). As a consequence of Theorem 2.2, one also has when $k \in \mathbb{Z}^+, 1 < p < \infty, j = 1, \ldots, n$,

$$\sum_{l=0}^{k} \|\nabla_j^l T_j h\|_{L^p(\Omega,\mu)} \lesssim \sum_{l=0}^{k-1} \|\nabla_j^l h\|_{L^p(\Omega,\mu)} \lesssim \|h\|_{W^{k-1,p}(\Omega,\mu)}, \tag{3.4}$$

$$\sum_{l=0}^{k} \|\nabla_j^l T_j h\|_{W^{1,p}(\Omega,\mu)} \lesssim \|h\|_{W^{k,p}(\Omega,\mu)}, \tag{3.5}$$

and

$$\sum_{l=0}^{k} \|\nabla_j^l S_j h\|_{L^p(\Omega,\mu)} \lesssim \sum_{l=0}^{k} \|\nabla_j^l h\|_{L^p(\Omega,\mu)} \lesssim \|h\|_{W^{k,p}(\Omega,\mu)}. \tag{3.6}$$

In the case when $\mu \equiv 1$ and $k = 0$, an application of the classical complex analysis theory (see [17] etc.) and Fubini theorem gives for $1 \le p < \infty$,

$$\|T_j h\|_{L^p(\Omega)} \lesssim \|h\|_{L^p(\Omega)}. \tag{3.7}$$

These inequalities will be used later.

Given a $(0,q)$ form

$$\mathbf{f} = \sum_{j_1 < \cdots < j_q} f_{\bar{j}_1 \cdots \bar{j}_q} d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \in C^1(\bar{\Omega}),$$

define $T_j \mathbf{f}$ and $S_j \mathbf{f}$ to be the action on the corresponding component functions. Namely,

$$T_j \mathbf{f} := \sum_{1 \le j_1 < \cdots < j_q \le n} T_j f_{\bar{j}_1 \cdots \bar{j}_q} d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q};$$
$$S_j \mathbf{f} := \sum_{1 \le j_1 < \cdots < j_q \le n} S_j f_{\bar{j}_1 \cdots \bar{j}_q} d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.$$

Furthermore, define a projection $\pi_k \mathbf{f}$ to be a $(0, q-1)$ form with

$$\pi_k \mathbf{f} := \sum_{1 \le k < j_2 < \cdots < j_q \le n} f_{\bar{k}\bar{j}_2 \cdots \bar{j}_q} d\bar{z}_{j_2} \wedge \cdots \wedge d\bar{z}_{j_q}.$$

In their celebrated work [12, pp. 430], Nijenhuis and Woolf constructed a solution operator of the $\bar{\partial}$ equation for $(0,q)$ forms on product domains.

Theorem 3.3. *[12] Let $\Omega = D_1 \times \cdots \times D_n$, where each D_j is a bounded domain in $\mathbb{C}$ with $C^{k,1}$ boundary, $k \in \mathbb{Z}^+$. If $\mathbf{f} \in C^1(\bar{\Omega})$ is a $\bar{\partial}$-closed $(0,q)$ form on Ω, then*

$$T\mathbf{f} := T_1\pi_1\mathbf{f} + T_2S_1\pi_2\mathbf{f} + \cdots + T_nS_1\cdots S_{n-1}\pi_n\mathbf{f} \tag{3.8}$$

is a solution to $\bar{\partial}u = \mathbf{f}$ on Ω.

Proof of Theorem 1.2: Given a $\bar{\partial}$-closed $(0,q)$ form $\mathbf{f} \in W^{n-1,p}(\Omega,\mu), p > 1$ (the $k = 1$ case in the theorem), we first verify that $T\mathbf{f}$ in (3.8) is a weak solution to $\bar{\partial}u = \mathbf{f}$ on Ω. Since $W^{n-1,p}(\Omega,\mu) \subset W^{n-1,q}(\Omega)$ for some $q > 1$, for simplicity we directly assume $\mathbf{f} \in W^{n-1,p}(\Omega), p > 1$. Following an idea in [13], for each $j = 1,\dots,n$, let $\{D_j^{(m)}\}_{m=1}^{\infty}$ be a family of strictly increasing open subsets of D_j such that
a). for $m \geq N_0 \in \mathbb{N}$, $bD_j^{(m)}$ is $C^{k,1}$, $\frac{1}{m+1} < dist(D_j^{(m)}, D_j^c) < \frac{1}{m}$;
b). $H_j^{(m)} : \bar{D}_j \to \bar{D}_j^{(m)}$ is a C^1 diffeomorphism with $\lim_{m\to\infty} \|H_j^{(m)} - id\|_{C^1(D_j)} = 0$.

Let $\Omega^{(m)} = D_1^{(m)} \times \cdots \times D_n^{(m)}$ be the product of those approximating planar domains. Denote by $T_j^{(m)}, S_j^{(m)}$ and $T^{(m)}$ the operators defined in (3.1) and (3.8) accordingly, with Ω replaced by $\Omega^{(m)}$. Then $T^{(m)}\mathbf{f} \in W^{1,p}(\Omega^{(m)})$. Adopting the mollifier argument to $\mathbf{f} \in W^{n-1,p}(\Omega)$, we obtain $\mathbf{f}^\epsilon \in C^1(\overline{\Omega^{(m)}}) \cap W^{n-1,p}(\Omega^{(m)})$ such that

$$\|\mathbf{f}^\epsilon - \mathbf{f}\|_{W^{n-1,p}(\Omega^{(m)})} \to 0$$

as $\epsilon \to 0$ and $\bar{\partial}\mathbf{f}^\epsilon = 0$ on $\Omega^{(m)}$.

For each fixed m, $T^{(m)}\mathbf{f}^\epsilon \in W^{n-1,p}(\Omega^{(m)})$ when ϵ is small and

$$\bar{\partial}T^{(m)}\mathbf{f}^\epsilon = \mathbf{f}^\epsilon \quad \text{in} \quad \Omega^{(m)}$$

by Theorem 3.3. Furthermore,

$$\|T^{(m)}\mathbf{f}^\epsilon - T^{(m)}\mathbf{f}\|_{W^{1,p}(\Omega^{(m)})} \lesssim \|\mathbf{f}^\epsilon - \mathbf{f}\|_{W^{n-1,p}(\Omega^{(m)})} \to 0$$

as $\epsilon \to 0$. In particular, $\lim_{\epsilon\to0} T^{(m)}\mathbf{f}^\epsilon$ exists a.e. in $\Omega^{(m)}$ and is equal to $T^{(m)}\mathbf{f} \in W^{n-1,p}(\Omega^{(m)})$ pointwisely.

Given a testing form ϕ with a compact support K, let $m_0 \geq N_0$ be such that $K \subset \Omega^{(m_0-2)}$. Denote by $\langle\cdot,\cdot\rangle_\Omega$ (and $\langle\cdot,\cdot\rangle_{\Omega^{(m_0)}}$) the inner product(s) in $L^2(\Omega)$ (and in $L^2(\Omega^{(m_0)})$, respectively), and $\bar{\partial}^*$ the formal adjoint of $\bar{\partial}$. For all $m \geq m_0$, one has

$$\langle T^{(m)}\mathbf{f}, \bar{\partial}^*\phi\rangle_{\Omega^{(m_0)}} = \lim_{\epsilon\to0}\langle T^{(m)}\mathbf{f}^\epsilon, \bar{\partial}^*\phi\rangle_{\Omega^{(m_0)}} = \lim_{\epsilon\to0}\langle\bar{\partial}T^{(m)}\mathbf{f}^\epsilon, \phi\rangle_{\Omega^{(m_0)}} = \lim_{\epsilon\to0}\langle\mathbf{f}^\epsilon, \phi\rangle_{\Omega^{(m_0)}} = \langle\mathbf{f}, \phi\rangle_\Omega. \tag{3.9}$$

We further show that

$$\langle T\mathbf{f}, \bar{\partial}^*\phi\rangle_\Omega = \lim_{m\to\infty}\langle T^{(m)}\mathbf{f}, \bar{\partial}^*\phi\rangle_{\Omega^{(m_0)}}. \tag{3.10}$$

For simplicity, assume $\pi_j\mathbf{f}$ contains only one component function f_j, so does ϕ. We will also drop various integral measures, which should be clear from the context. For each $j = 1,\dots,n$,

$$\begin{aligned}&\langle T_j^{(m)}S_1^{(m)}\cdots S_{j-1}^{(m)}\pi_j\mathbf{f}, \bar{\partial}^*\phi\rangle_{\Omega^{(m_0)}}\\ =&\frac{1}{(2\pi i)^{j-1}}\int_{z\in K} T_j\left(\int_{\zeta_1\in bD_1^{(m)}}\cdots\int_{\zeta_{j-1}\in bD_{j-1}^{(m)}}\frac{f_j(\zeta_1,\cdots,\zeta_j,z_{j+1},\cdots,z_n)\chi_{D_j^{(m)}}(\zeta_j)}{(\zeta_1-z_1)\cdots(\zeta_{j-1}-z_{j-1})}\right)\overline{\bar{\partial}^*\phi(z)}.\end{aligned}$$

Here $\chi_{D_j^{(m)}}$ is the characteristic function of $D_j^{(m)} \subset \mathbb{C}$.

For each $(z, \zeta_j) \in K \times D_j \setminus \{z_j = \zeta_j\}$, after a change of variables, there exists some function $h^{(m)} \in C(\bar{D}_1 \times \cdots \bar{D}_{j-1})$, such that $\|h^{(m)} - 1\|_{C(D_1 \times \cdots D_{j-1})} \to 0$ as $m \to \infty$ and

$$\begin{aligned}
&\int_{\zeta_1 \in bD_1^{(m)}} \cdots \int_{\zeta_{j-1} \in bD_{j-1}^{(m)}} \frac{f_j(\zeta_1, \cdots, \zeta_j, z_{j+1}, \cdots, z_n)\chi_{D_j^{(m)}}(\zeta_j)}{(\zeta_1 - z_1)\cdots(\zeta_{j-1} - z_{j-1})} \\
&= \int_{\zeta_1 \in bD_1} \cdots \int_{\zeta_{j-1} \in bD_{j-1}} \frac{f_j(\zeta_1, \cdots, \zeta_j, z_{j+1}, \cdots, z_n) h^{(m)}(\zeta_1, \cdots, \zeta_{j-1})\chi_{D_j^{(m)}}(\zeta_j)}{(\zeta_1 - z_1)\cdots(\zeta_{j-1} - z_{j-1})}.
\end{aligned}$$

Notice that when $z \in K (\subset \Omega^{(m_0-2)})$ and $\zeta_l \in bD_l^{(m)}, m \geq m_0, l = 1, \ldots, j-1$,

$$\frac{1}{|\zeta_l - z_l|} \leq \frac{1}{dist((\Omega^{(m)})^c, \Omega^{(m_0-2)})} \leq \frac{1}{dist((\Omega^{(m_0)})^c, \Omega^{(m_0-2)})} < m_0^2.$$

Hence

$$\begin{aligned}
&\left| \langle T_j^{(m)} S_1^{(m)} \cdots S_{j-1}^{(m)} \pi_j \mathbf{f}, \bar{\partial}^* \phi\rangle_{\Omega^{(m_0)}} - \langle T_j S_1 \cdots S_{j-1} \pi_j \mathbf{f}, \bar{\partial}^* \phi\rangle_{\Omega^{(m_0)}} \right| \\
&\lesssim \left\| T_j \left(\int_{bD_1 \times \cdots \times bD_{j-1}} \left| f_j h^{(m)} \chi_{D_j^{(m)}} - f_j \right| \right) \right\|_{L^1(\Omega)}.
\end{aligned} \tag{3.11}$$

On the other hand,

$$\begin{aligned}
&\left\| \int_{bD_1 \times \cdots \times bD_{j-1}} \left| f_j h^{(m)} \chi_{D_j^{(m)}} - f_j \right| \right\|_{L^1(\Omega)} \\
&\lesssim \left\| |f_j| \left| h^{(m)} \chi_{D_j^{(m)}} - 1 \right| \right\|_{L^1(bD_1 \times \cdots \times bD_{j-1} \times D_j \times \cdots \times D_n)} \\
&\lesssim \|f_j\|_{L^1(bD_1 \times \cdots \times bD_{j-1} \times D_j \times \cdots \times D_n)} \|h^{(m)} - 1\|_{C(D_1 \times \cdots \times D_{j-1})} \\
&\quad + \|f_j\|_{L^p(bD_1 \times \cdots \times bD_{j-1} \times D_j \times \cdots \times D_n)} \mathrm{vol}^{1-\frac{1}{p}}(D_j \setminus D_j^{(m)}) \\
&\lesssim \|f_j\|_{W^{n-1,p}(\Omega)} \left(\|h^{(m)} - 1\|_{C(D_1 \times \cdots \times D_{j-1})} + \mathrm{vol}^{1-\frac{1}{p}}(D_j \setminus D_j^{(m)}) \right) \to 0
\end{aligned} \tag{3.12}$$

as $m \to \infty$. Here we used the trace theorem in the third inequality. Combining (3.7), (3.11) and (3.12) we finally get

$$\left| \left\langle T_j^{(m)} S_1^{(m)} \cdots S_{j-1}^{(m)} \pi_j \mathbf{f} - T_j S_1 \cdots S_{j-1} \pi_j \mathbf{f}, \bar{\partial}^* \phi \right\rangle_{\Omega^{(m_0)}} \right| \to 0$$

as $m \to \infty$. (3.10) is thus proved. Combining (3.9) with (3.10), we deduce that

$$\langle T\mathbf{f}, \bar{\partial}^* \phi\rangle_\Omega = \lim_{m \to \infty} \langle T^{(m)} \mathbf{f}, \bar{\partial}^* \phi\rangle_{\Omega^{(m_0)}} = \langle \mathbf{f}, \phi\rangle_\Omega,$$

which verifies $T\mathbf{f}$ as a weak solution to $\bar{\partial}$ on Ω.

We next prove the weighted Sobolev estimate for the operator T defined in (3.8). Since $\bar{\partial} T\mathbf{f} = \mathbf{f}$, we can further assume $\nabla^k = \partial^\alpha$ for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \leq k$. In view of (3.8) and the fact that π_j being a projection is automatically bounded in $W^{k,p}(\Omega, \mu)$, we only need to estimate $\|\partial^\alpha T_n S_1 \cdots S_{n-1} h\|_{L^p(\Omega,\mu)}$ in terms of $\|h\|_{W^{k+n-2,p}(\Omega,\mu)}$. Write $\partial^\alpha T_n S_1 \cdots S_{n-1} h = (\partial_n^{\alpha_n} T_n)(\partial_1^{\alpha_1} S_1) \cdots (\partial_{n-1}^{\alpha_{n-1}} S_{n-1})h$. If $\alpha_n \geq 1$, we apply (3.4) and (3.3) inductively to have

$$
\begin{aligned}
\|\partial^\alpha T_n S_1 \cdots S_{n-1} h\|_{L^p(\Omega,\mu)} &\lesssim \|(\partial_1^{\alpha_1} S_1) \cdots (\partial_{n-1}^{\alpha_{n-1}} S_{n-1})h\|_{W^{\alpha_n-1,p}(\Omega,\mu)} \\
&\lesssim \|(\partial_2^{\alpha_2} S_2) \cdots (\partial_{n-1}^{\alpha_{n-1}} S_{n-1})h\|_{W^{\alpha_n-1+\alpha_1+1,p}(\Omega,\mu)} \\
&\lesssim \cdots \\
&\lesssim \|h\|_{W^{\sum_{j=1}^n \alpha_j+n-2,p}(\Omega,\mu)} \leq \|h\|_{W^{k+n-2,p}(\Omega,\mu)}.
\end{aligned}
$$

If $\alpha_n = 0$, then there exists some $1 \leq j \leq n-1$, such that $\alpha_j \geq 1$. Without loss of generality, assume $\alpha_1 \geq 1$. Then by (3.4), (3.6) and (3.3) inductively,

$$
\begin{aligned}
\|\partial^\alpha T_n S_1 \cdots S_{n-1} h\|_{L^p(\Omega,\mu)} &\lesssim \|(\partial_1^{\alpha_1} S_1) \cdots (\partial_{n-1}^{\alpha_{n-1}} S_{n-1})h\|_{L^p(\Omega,\mu)} \\
&\lesssim \|(\partial_2^{\alpha_2} S_2) \cdots (\partial_{n-1}^{\alpha_{n-1}} S_{n-1})h\|_{W^{\alpha_1,p}(\Omega,\mu)} \\
&\lesssim \|(\partial_3^{\alpha_3} S_3) \cdots (\partial_{n-1}^{\alpha_{n-1}} S_{n-1})h\|_{W^{\alpha_1+\alpha_2+1,p}(\Omega,\mu)} \\
&\lesssim \cdots \\
&\lesssim \|h\|_{W^{k+n-2,p}(\Omega,\mu)}.
\end{aligned}
$$

The theorem is thus proved.

□

Similar to an example in [19], the following one shows that the $\bar{\partial}$ problem does not improve regularity in weighted Sobolev spaces on product domains. As such the weighted Sobolev regularity obtained in Theorem 1.2 is optimal when $n = 2$.

Example 1. *For each $k \in \mathbb{Z}^+, 1 < p < \infty, \epsilon > 0$ and any $s \in \left(\frac{2}{1+\epsilon}, 2\right) \setminus \{1\}$, consider $\mathbf{f} = (z_2 - 1)^{k-s} d\bar{z}_1$ on $\triangle \times \triangle$, $\frac{1}{2}\pi < \arg(z_2 - 1) < \frac{3}{2}\pi$ and $\mu = |z_2 - 1|^{s(p-1)}$. Then $\mu \in A_p^*$, $\mathbf{f} \in W^{k,p}(\triangle \times \triangle, \mu)$ and is $\bar{\partial}$-closed on $\triangle \times \triangle$. However, there does not exist a solution $u \in W^{k,p+\epsilon}(\triangle \times \triangle, \mu)$ to $\bar{\partial} u = \mathbf{f}$ on $\triangle \times \triangle$.*

Proof. One can directly verify that $\mathbf{f} \in W^{k,p}(\triangle \times \triangle, \mu)$ is $\bar{\partial}$-closed on $\triangle \times \triangle$ and $\mu \in A_p^*$.

Suppose there exists some $u \in W^{k,p+\epsilon}(\triangle \times \triangle, \mu)$ satisfying $\bar{\partial} u = \mathbf{f}$ on $\triangle \times \triangle$. Then there exists some holomorphic function h on $\triangle \times \triangle$, such that $u = (z_2 - 1)^{k-s} \bar{z}_1 + h \in W^{k,p+\epsilon}(\triangle \times \triangle, \mu)$. For each $(r, z_2) \in U := (0,1) \times \triangle \subset \mathbb{R}^3$, consider

$$
v(r, z_2) := \int_{|z_1|=r} u(z_1, z_2) dz_1.
$$

By Hölder inequality, Fubini theorem and the fact that $p > 1$,

$$
\begin{aligned}
\|\partial_{z_2}^k v\|_{L^{p+\epsilon}(U,\mu)}^{p+\epsilon} &= \int_U \left| \int_{|z_1|=r} \partial_{z_2}^k u(z_1, z_2) dz_1 \right|^{p+\epsilon} \mu(z_2) dV_{z_2} dr \\
&= \int_{|z_2|<1} \int_0^1 \left| r \int_0^{2\pi} |\partial_{z_2}^k u(re^{i\theta}, z_2)| d\theta \right|^{p+\epsilon} dr \mu(z_2) dV_{z_2} \\
&\lesssim \int_{|z_2|<1} \int_0^1 \int_0^{2\pi} |\partial_{z_2}^k u(re^{i\theta}, z_2)|^{p+\epsilon} d\theta r dr \mu(z_2) dV_{z_2} \\
&= \int_{|z_2|<1, |z_1|<1} |\partial_{z_2}^k u(z)|^{p+\epsilon} \mu(z_2) dV_z \leq \|u\|_{W^{k,p+\epsilon}(\triangle \times \triangle, \mu)}^{p+\epsilon} < \infty.
\end{aligned}
$$

Thus $\partial_{z_2}^k v \in L^{p+\epsilon}(U,\mu)$.

On the other hand, by Cauchy's theorem, for each $(r, z_2) \in U$,

$$\begin{aligned}\partial_{z_2}^k v(r,z_2) =&(k-s)\cdots(1-s)\int_{|z_1|=r}(z_2-1)^{-s}\bar{z}_1 dz_1\\ =&(k-s)\cdots(1-s)(z_2-1)^{-s}\int_{|z_1|=r}\frac{r^2}{z_1}dz_1 = 2(k-s)\cdots(1-s)\pi r^2 i(z_2-1)^{-s},\end{aligned}$$

which is not in $L^{p+\epsilon}(U,\mu)$ by the choice of $s > \frac{2}{1+\epsilon}$. This is a contradiction!

□

Making use of Theorem 1.2, one can immediately prove the weighted Sobolev estimate for the $\bar\partial$ problem on $\mathbb{H}$ in Corollary 1.3. In comparison to the statement of Theorem 1.1, the solution operator in Corollary 1.3 is the same for all Sobolev levels.

Proof of Corollary 1.3: For any $\mathbf{f} = \sum_{j=1}^2 f_j(z)d\bar{z}_j \in W^{k,p}(\mathbb{H})$, making use of the change of variables formula we have the pull-back

$$\psi^*\mathbf{f} = \bar{w}_2 f_1\circ\psi d\bar{w}_1 + (\bar{w}_1 f_1\circ\psi + f_2\circ\psi)\, d\bar{w}_2. \tag{3.13}$$

Moreover, noting by the chain rule

$$\partial_{w_1} = w_2\partial_{z_1}, \quad \partial_{w_2} = w_1\partial_{z_1} + \partial_{z_2},$$

we have $\psi^*\mathbf{f} \in W^{k,p}(\triangle\times\triangle, |w_2|^2)$ with

$$\begin{aligned}\|\psi^*\mathbf{f}\|^p_{W^{k,p}(\triangle\times\triangle,|w_2|^2)} &\lesssim \sum_{j=1}^2\sum_{l=0}^k\int_{\triangle\times\triangle}|\nabla^l_w(f_j\circ\psi)(w)|^p|w_2|^2dV_w\\ &\lesssim \sum_{j=1}^2\sum_{l=0}^k\int_{\mathbb{H}}|\nabla^l_z f_j(z)|^p dV_z = \|\mathbf{f}\|^p_{W^{k,p}(\mathbb{H})}.\end{aligned} \tag{3.14}$$

Since $k\in\mathbb{Z}^+, p>2$, by (3.14) one has $\psi^*\mathbf{f}$ to be $\bar\partial$-closed on $\triangle\times\triangle$ (see, for instance, [10, pp. 28]). Making use of Theorem 1.2, there exists a solution $\tilde{u}\in W^{k,p}(\triangle\times\triangle,|w_2|^2)$ solving $\bar\partial\tilde{u} = \psi^*\mathbf{f}$. Arguing in the same way as in the proof of [19, Theorem 1.2], we know that $\mathcal{T}\mathbf{f} := \tilde{u}\circ\phi$ solves $\bar\partial u = \mathbf{f}$. Moreover,

$$\begin{aligned}\|\mathcal{T}\mathbf{f}\|^p_{W^{k,p}(\mathbb{H},|z_2|^{kp})} &= \sum_{l=0}^k\int_{\mathbb{H}}|\nabla^l_z(\tilde{u}\circ\phi)(z)z_2^k|^p dV_z\\ &\lesssim \sum_{l=0}^k\int_{\triangle\times\triangle}|\nabla^l_w\tilde{u}(w)|^p|w_2|^2dV_w = \|\tilde{u}\|^p_{W^{k,p}(\triangle\times\triangle,|w_2|^2)}.\end{aligned} \tag{3.15}$$

Here we used the chain rule

$$\partial_{z_1} = \frac{1}{z_2}\partial_{w_1}, \quad \partial_{z_2} = -\frac{z_1}{z_2^2}\partial_{w_1} + \partial_{w_2}$$

and the fact that $|z_1| < |z_2|$ on $\mathbb{H}$.

Finally, combining (3.14)-(3.15) and Theorem 1.2,

$$\|\mathcal{T}\mathbf{f}\|_{W^{k,p}(\mathbb{H},|z_2|^{kp})} \lesssim \|\tilde{u}\|_{W^{k,p}(\triangle\times\triangle,|w_2|^2)} \lesssim \|\psi^*\mathbf{f}\|^p_{W^{k,p}(\triangle\times\triangle,|w_2|^2)} \lesssim \|\mathbf{f}\|^p_{W^{k,p}(\mathbb{H})}.$$

□

4 Optimal Sobolev regularity on the Hartogs triangle

In this section, following an idea of Ma and Michel in [11], we shall adjust the solution operator provided by Corollary 1.3, so that the new operator cancels the loss in the exponent of the weight. In detail, given a $W^{k,p}$ datum on the Hartogs triangle $\mathbb{H}$, we truncate its $(k-1)$-th order Taylor polynomial at $(0,0)$ and then pull it back to the punctured bidisc $\triangle \times \triangle^*$. Upon extension and solving the $\bar{\partial}$ problem on the bidisc $\triangle \times \triangle$ using Theorem 1.2, we once again truncate the $(k-1)$-th order holomorphic Taylor polynomial in the w_2 variable at $w_2 = 0$ from the solution. Both Taylor polynomials are meaningful when $p > 4$ due to the Sobolev embedding theorem. Moreover, we can obtain a refined weighted Sobolev regularity at each operation (Proposition 4.1 and Proposition 4.5) as a consequence of the truncation. Finally, pushing forward this truncated solution to $\mathbb{H}$, we show it is a solution to $\bar{\partial}$ on $\mathbb{H}$ that maintains the same Sobolev regularity as that of the datum.

Throughout the rest of the paper, $z = (z_1, z_2)$ will serve as the variable on $\mathbb{H}$, and $w = (w_1, w_2)$ as the variable on $\triangle \times \triangle$.

4.1 Truncating data on the Hartogs triangle

Given a $\bar{\partial}$-closed $(0,1)$ form $\mathbf{f} \in W^{k,p}(\mathbb{H}), k \in \mathbb{Z}^+, p > 4$, recalling $\mathbb{H}$ satisfies the Sobolev extension property, it extends to an element, still denoted by $\mathbf{f}$, in $W^{k,p}(\mathbb{C}^2)$. In particular, by Sobolev embedding theorem, $\mathbf{f} \in C^{k-1,\alpha}(\mathbb{H})$ for some $\alpha > 0$. Denote by $\mathcal{P}_k$ the $(k-1)$-th order Taylor polynomial operator at $(0,0)$. Namely, if $h \in C^{k-1}$ near $(0,0)$, then

$$\mathcal{P}_k h(z) := \sum_{l_1+l_2+s_1+s_2=0}^{k-1} \frac{\partial_{z_1}^{l_1} \bar{\partial}_{z_1}^{l_2} \partial_{z_2}^{s_1} \bar{\partial}_{z_2}^{s_2} h(0)}{l_1! l_2! s_1! s_2!} z_1^{l_1} \bar{z}_1^{l_2} z_2^{s_1} \bar{z}_2^{s_2}.$$

Then $\mathcal{P}_k \mathbf{f}$ is $\bar{\partial}$-closed on $\mathbb{H}$ and thus on $\triangle \times \triangle$ (see [11, Lemma 3]). Applying the $W^{k,p}$ estimate of $\bar{\partial}$ on $\triangle \times \triangle$ (i.e., Theorem 1.2 with $n = 2$ and $\mu \equiv 1$), one obtains some $u_k \in W^{k,p}(\triangle \times \triangle)$ satisfying

$$\begin{aligned} \bar{\partial} u_k = \mathcal{P}_k \mathbf{f} \quad \text{on} \quad \triangle \times \triangle; \\ \|u_k\|_{W^{k,p}(\triangle \times \triangle)} \lesssim \|\mathcal{P}_k \mathbf{f}\|_{W^{k,p}(\triangle \times \triangle)} \lesssim \|\mathcal{P}_k \mathbf{f}\|_{C^{k-1}(\mathbb{H})} \lesssim \|\mathbf{f}\|_{W^{k,p}(\mathbb{H})}. \end{aligned} \tag{4.1}$$

Let ψ be defined in (2.3). We truncate $\mathbf{f}$ by $\mathcal{P}_k \mathbf{f}$, and then pull back the truncated datum by ψ to obtain $\psi^*(\mathbf{f} - \mathcal{P}_k \mathbf{f})$ on the punctured bidisc.

Denote by $\mathcal{P}_{2,k}$ the $(k-1)$-th order Taylor polynomial operator in the complex w_2 variable at $w_2 = 0$ of C^{k-1} functions on $\triangle \times \triangle$. Then for any $h \in W^{k,p}(\mathbb{H}), k \in \mathbb{Z}^+, p > 4$,

$$\psi^* \left(\mathcal{P}_k h\right) = \mathcal{P}_{2,k} \left(\psi^* h\right).$$

In particular,

$$\mathcal{P}_{2,k} \left(\psi^*(h - \mathcal{P}_k h)\right) = 0. \tag{4.2}$$

The following proposition states that the pull-back $\psi^*(\mathbf{f} - \mathcal{P}_k \mathbf{f})$ of the truncated datum satisfies a more refined Sobolev estimate than (3.14).

Proposition 4.1. *Let $\mathbf{f} \in W^{k,p}(\mathbb{H})$ be a $\bar{\partial}$-closed $(0,1)$ form on $\mathbb{H}, k \in \mathbb{Z}^+, p > 4$ and ψ be defined in (2.3). Let*

$$\tilde{\mathbf{f}} = \tilde{f}_1 d\bar{w}_1 + \tilde{f}_2 d\bar{w}_2 := \psi^*(\mathbf{f} - \mathcal{P}_k \mathbf{f}) \quad on \quad \triangle \times \triangle^*.$$

Then $\tilde{\mathbf{f}}$ *extends as a* $\bar\partial$*-closed* $(0,1)$ *form on* $\triangle\times\triangle$, *with* $\tilde{\mathbf{f}}\in W^{k,p}(\triangle\times\triangle,|w_2|^2)$ *and*

$$\mathcal{P}_{2,k}\tilde{\mathbf{f}}=0. \tag{4.3}$$

Moreover, for $t,s\in\mathbb{Z}^+\cup\{0\}, t+s\le k$,

$$\left\||w_2|^{-k+s}\nabla^t_{w_1}\nabla^s_{w_2}\tilde{\mathbf{f}}\right\|_{L^p(\triangle\times\triangle,|w_2|^2)}\lesssim\|\mathbf{f}\|_{W^{k,p}(\mathbb{H})}. \tag{4.4}$$

In order to prove Proposition 4.1, we need to establish a crucial weighted Hardy-type inequality on $\mathbb{C}$. We shall adopt the same notation $\mathcal{P}_k$ for the $(k-1)$-th order Taylor polynomial operator at 0 on C^{k-1} functions near $0\in\mathbb{C}$.

Lemma 4.2. *For any* $h\in W^{k,p}(\mathbb{C},|w|^2), k\in\mathbb{Z}^+, p>4$ *with* $\mathcal{P}_kh=0$, *and* $j=0,\dots,k$,

$$\int_{\mathbb{C}}|\nabla^j_wh(w)|^p|w|^{2-(k-j)p}dV_w\lesssim\int_{\mathbb{C}}|\nabla^k_wh(w)|^p|w|^2dV_w.$$

Proof. Since the $j=k$ case is trivial, we assume $j\le k-1$. We shall show that for $h\in W^{l,p}(\mathbb{C},|w|^2)$ with $\mathcal{P}_lh=0$, $l=1,\dots,k$,

$$\int_{\mathbb{C}}|h(w)|^p|w|^{2-lp}dV_w\lesssim\int_{\mathbb{C}}|\nabla_wh(w)|^p|w|^{2-(l-1)p}dV_w. \tag{4.5}$$

If so, replacing l and h by $k-j$ and ∇^j_wh in (4.5), respectively, then

$$\int_{\mathbb{C}}|\nabla^j_wh(w)|^p|w|^{2-(k-j)p}dV_w\lesssim\int_{\mathbb{C}}|\nabla^{j+1}_wh(w)|^p|w|^{2-(k-j-1)p}dV_w.$$

A standard induction on j will complete the proof of the lemma.

To show (4.5), first apply the Stokes' theorem to $|h(w)|^p|w|^{2-lp}\bar wdw$ on $\triangle_R\backslash\overline{\triangle_\epsilon}, 0<\epsilon<R<\infty$ to get

$$\begin{aligned}&\frac{1}{2i}\int_{b\triangle_R}|h(w)|^p|w|^{2-lp}\bar wdw-\frac{1}{2i}\int_{b\triangle_\epsilon}|h(w)|^p|w|^{2-lp}\bar wdw\\&=\int_{\triangle_R\backslash\overline{\triangle_\epsilon}}\bar\partial_w\left(|h(w)|^p|w|^{2-lp}\bar w\right)dV_w\\&=\left(2-\frac{lp}{2}\right)\int_{\triangle_R\backslash\overline{\triangle_\epsilon}}|h(w)|^p|w|^{2-lp}dV_w+\int_{\triangle_R\backslash\overline{\triangle_\epsilon}}\bar\partial_w\left(|h(w)|^p\right)|w|^{2-lp}\bar wdV_w.\end{aligned}$$

Since

$$\frac{1}{2i}\int_{b\triangle_R}|h(w)|^p|w|^{2-lp}\bar wdw=\frac{1}{2}\int_0^{2\pi}|h(Re^{i\theta})|^pR^{4-lp}d\theta\ge0,$$

one further has

$$\left(\frac{lp}{2}-2\right)\int_{\triangle_R\backslash\overline{\triangle_\epsilon}}|h(w)|^p|w|^{2-lp}dV_w\le\frac{1}{2i}\int_{b\triangle_\epsilon}|h(w)|^p|w|^{2-lp}\bar wdw+\int_{\triangle_R\backslash\overline{\triangle_\epsilon}}\bar\partial_w\left(|h(w)|^p\right)|w|^{2-lp}\bar wdV_w. \tag{4.6}$$

We claim that

$$\lim_{\epsilon\to0}\epsilon^{3-lp}\int_{b\triangle_\epsilon}|h(w)|^pd\sigma_w=0, \tag{4.7}$$

which is equivalent to

$$\lim_{\epsilon\to 0}\left|\int_{b\triangle_\epsilon}|h(w)|^p|w|^{2-lp}\bar{w}dw\right|=0.$$

Indeed, let q be the dual of p, i.e., $\frac{1}{p}+\frac{1}{q}=1$. For a.e. $w\in b\triangle$ and $0<\delta<\epsilon$, applying Fubini theorem in polar coordinates, one can see $h(tw)\in W^{k,p}((\delta,\epsilon))$ as a function of t. By the fundamental theorem of calculus, we have

$$h(\epsilon w)=h(\delta w)+\int_\delta^\epsilon\frac{d}{dt}h(tw)dt.$$

Letting $\delta\to 0$ in the above, we have

$$|h(\epsilon w)|\le\int_0^\epsilon|\nabla h(tw)|dt.$$

An induction process further gives

$$\begin{aligned}|h(\epsilon w)|^p\le&\left|\int_0^\epsilon\int_0^{t_1}\cdots\int_0^{t_{l-1}}|\nabla^l h(t_lw)|dt_l\cdots dt_2dt_1\right|^p\\ \le&\left|\int_0^\epsilon\int_0^{\epsilon}\cdots\int_0^{\epsilon}|\nabla^l h(t_lw)|dt_l\cdots dt_2dt_1\right|^p\\ \le&\epsilon^{(l-1)p}\left|\int_0^\epsilon|\nabla^l h(tw)|t^{\frac{3}{p}}\cdot t^{-\frac{3}{p}}dt\right|^p\\ \le&\epsilon^{(l-1)p}\int_0^\epsilon|\nabla^l h(tw)|^pt^3dt\left(\int_0^\epsilon t^{-\frac{3q}{p}}dt)\right)^{\frac{p}{q}}\\ \lesssim&\epsilon^{lp-4}\int_0^\epsilon|\nabla^l h(tw)|^pt^3dt.\end{aligned}\tag{4.8}$$

Here we used the fact that $-\frac{3q}{p}>-1$ when $p>4$ in the last inequality. Note that

$$\epsilon\int_{b\triangle}|h(\epsilon w)|^pd\sigma_w=\int_{b\triangle_\epsilon}|h(w)|^pd\sigma_w.$$

Multiplying both sides of (4.8) by ϵ^{4-lp} and integrating over $b\triangle$, one has

$$\begin{aligned}\epsilon^{3-lp}\int_{b\triangle_\epsilon}|h(w)|^pd\sigma_w\lesssim&\int_0^\epsilon\int_{b\triangle}|\nabla^l h(tw)|^pt^3d\sigma_wdt\\ =&\int_0^\epsilon\int_{b\triangle_t}|\nabla^l h(w)|^p|w|^2d\sigma_wdt\\ \le&\int_{\triangle_\epsilon}|\nabla^l h(w)|^p|w|^2dV_w\to 0\end{aligned}$$

as $\epsilon\to 0$. The claim (4.7) is thus proved.

Pass $\epsilon\to 0$ and $R\to\infty$ in (4.6), and then make use of (4.7). Since $\frac{lp}{2}-2>0$, we further infer

$$\begin{aligned}\int_{\mathbb{C}}|h(w)|^p|w|^{2-lp}dV_w\lesssim&\int_{\mathbb{C}}|\nabla_w h(w)||h(w)|^{p-1}|w|^{3-lp}dV_w\\ =&\int_{\mathbb{C}}|\nabla_w h(w)||w|^{\frac{2}{p}-(l-1)}\cdot|h(w)|^{p-1}|w|^{2-lp+l-\frac{2}{p}}dV_w\\ \le&\left(\int_{\mathbb{C}}|\nabla_w h(w)|^p|w|^{2-(l-1)p}dV_w\right)^{\frac{1}{p}}\left(\int_{\mathbb{C}}|h(w)|^p|w|^{2-lp}dV_w\right)^{1-\frac{1}{p}}.\end{aligned}$$

(4.5) follows by dividing both sides by $\left(\int_{\mathbb{C}} |h(w)|^p |w|^{2-lp} dV_w\right)^{1-\frac{1}{p}}$ and then taking the p-th power. $\square$

Corollary 4.3. *Let D be a uniform domain in $\mathbb{C}$ and $0 \in D$. Then for any $h \in W^{k,p}(D, |w|^2), k \in \mathbb{Z}^+, p > 4$ with $\mathcal{P}_k h = 0$, and $j = 0, \ldots, k$,*

$$\int_D |\nabla_w^j h(w)|^p |w|^{2-(k-j)p} dV_w \lesssim \int_D |\nabla_w^k h(w)|^p |w|^2 dV_w.$$

Proof. Given h satisfying the assumption of the corollary, according to [4, Theorem 1.2], one can extend h to be an element $\tilde{h} \in W^{k,p}(\mathbb{C}, |w|^2)$, such that

$$\int_{\mathbb{C}} |\nabla_w^k \tilde{h}(w)|^p |w|^2 dV_w \lesssim \int_D |\nabla_w^k h(w)|^p |w|^2 dV_w.$$

Obviously $\mathcal{P}_k \tilde{h} = 0$. Hence making use of Lemma 4.2 to $\tilde{h}$, we have

$$\begin{aligned}\int_D |\nabla_w^j h(w)|^p |w|^{2-(k-j)p} dV_w &\le \int_{\mathbb{C}} |\nabla_w^j \tilde{h}(w)|^p |w|^{2-(k-j)p} dV_w \\ &\lesssim \int_{\mathbb{C}} |\nabla_w^k \tilde{h}(w)|^p |w|^2 dV_w \lesssim \int_D |\nabla_w^k h(w)|^p |w|^2 dV_w.\end{aligned}$$

$\square$

Remark 4.4. *Recall that any domain with Lipschitz boundary is a uniform domain. As a direct consequence of Corollary 4.3, whenever $h \in W^{k,p}(\triangle, |w|^2), p > 4$ with $\mathcal{P}_k h = 0$, then $w^{-k} h \in L^p(\triangle, |w|^2)$.*

As shown in the proof of Lemma 4.2 (and thus Corollary 4.3), the assumption $p > 4$ is essential and can not be dropped. Now we are ready to prove Proposition 4.1 making use of Corollary 4.3.

Proof of Proposition 4.1: The $\bar{\partial}$-closedness of $\psi^* \mathbf{f}$ on $\triangle \times \triangle$ was checked in Corollary 1.3. Thus $\tilde{\mathbf{f}}$ is $\bar{\partial}$-closed on $\triangle \times \triangle$, and by (3.13),

$$\tilde{f}_1 = \bar{w}_2 \psi^*(f_1 - \mathcal{P}_k f_1), \quad \tilde{f}_2 = \bar{w}_1 \psi^*(f_1 - \mathcal{P}_k f_1) + \psi^*(f_2 - \mathcal{P}_k f_2). \tag{4.9}$$

(4.3) follows from the above (4.9) and (4.2).

Next we prove (4.4). For $l_1, l_2 \in \mathbb{Z}^+ \cup \{0\}$ with $l_1 + l_2 = t$,

$$\bar{\partial}_{w_1}^{l_1} \partial_{w_1}^{l_2} (\psi^* f_j) = \bar{w}_2^{l_1} w_2^{l_2} \psi^* \left(\bar{\partial}_{z_1}^{l_1} \partial_{z_1}^{l_2} f_j\right), \quad j = 1, 2.$$

Observing that

$$\nabla_{z_1}^t (\mathcal{P}_k f_j) = \mathcal{P}_{k-t}\left(\nabla_{z_1}^t f_j\right),$$

we get from (4.9) that

$$\begin{aligned}&\left\| |w_2|^{-k+s} \nabla_{w_1}^t \nabla_{w_2}^s \tilde{\mathbf{f}} \right\|_{L^p(\triangle\times\triangle, |w_2|^2)} \\ &\lesssim \sum_{j=1}^2 \left\| |w_2|^{-k+s} \nabla_{w_2}^s \nabla_{w_1}^t \left(\psi^*(f_j - \mathcal{P}_k f_j)\right) \right\|_{L^p(\triangle\times\triangle, |w_2|^2)} \\ &\lesssim \sum_{j=1}^2 \sum_{l_1+l_2=t} \left\| |w_2|^{-k+s} \nabla_{w_2}^s \left(\bar{w}_2^{l_1} w_2^{l_2} \psi^*\left(\nabla_{z_1}^t f_j - \mathcal{P}_{k-t}\left(\nabla_{z_1}^t f_j\right)\right)\right) \right\|_{L^p(\triangle\times\triangle, |w_2|^2)} \\ &\lesssim \sum_{1\le j\le 2} \sum_{0 \le l \le s} \left\| |w_2|^{-k+t+l} \nabla_{w_2}^l \left(\psi^*\left(\nabla_{z_1}^t f_j - \mathcal{P}_{k-t}\left(\nabla_{z_1}^t f_j\right)\right)\right) \right\|_{L^p(\triangle\times\triangle, |w_2|^2)}.\end{aligned}$$

Thus we only need to estimate $\left\| |w_2|^{-k+t+l} \nabla_{w_2}^l \left(\psi^* \left(\nabla_{z_1}^t f_j - \mathcal{P}_{k-t} \left(\nabla_{z_1}^t f_j \right) \right) \right) \right\|_{L^p(\triangle\times\triangle, |w_2|^2)}, 0 \le l \le s$.

For each fixed $w_1 \in \triangle$, let

$$h_{w_1} := \psi^* \left(\nabla_{z_1}^t f_j - \mathcal{P}_{k-t} \left(\nabla_{z_1}^t f_j \right) \right) (w_1, \cdot).$$

Then $\mathcal{P}_{k-t} h_{w_1} = 0$ by (4.2). Applying Corollary 4.3 to h_{w_1} on $\triangle$, we have for $0 \le l (\le s) \le k-t$,

$$\begin{aligned} & \left\| |w_2|^{-k+t+l} \nabla_{w_2}^l \left(\psi^* \left(\nabla_{z_1}^t f_j - \mathcal{P}_{k-t} \left(\nabla_{z_1}^t f_j \right) \right) \right) \right\|^p_{L^p(\triangle\times\triangle, |w_2|^2)} \\ \le & \int_\triangle \int_\triangle |w_2|^{2-(k-t-l)p} \left| \nabla_{w_2}^l \left(\psi^* \left(\nabla_{z_1}^t f_j - \mathcal{P}_{k-t} \left(\nabla_{z_1}^t f_j \right) \right) \right) (w_1, w_2) \right|^p dV_{w_2} dV_{w_1} \\ \lesssim & \int_\triangle \int_\triangle |w_2|^2 \left| \nabla_{w_2}^{k-t} \left(\psi^* \left(\nabla_{z_1}^t f_j - \mathcal{P}_{k-t} \left(\nabla_{z_1}^t f_j \right) \right) \right) (w_1, w_2) \right|^p dV_{w_2} dV_{w_1}. \end{aligned} \tag{4.10}$$

On the other hand, note that for any function $h \in W^{k-t,p}(\mathbb{H})$, $l_1 + l_2 = k - t$,

$$\bar\partial_{w_2}^{l_1} \partial_{w_2}^{l_2} \psi^* h = \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} C_{m_1,m_2,l_1,l_2} \bar w_1^{m_1} w_1^{m_2} \psi^* \left(\bar\partial_{z_1}^{m_1} \bar\partial_{z_2}^{l_1-m_1} \partial_{z_1}^{m_2} \partial_{z_2}^{l_2-m_2} h \right)$$

for some constants C_{m_1,m_2,l_1,l_2} dependent only on m_1, m_2, l_1 and l_2. Thus

$$\begin{aligned} & \left| \nabla_{w_2}^{k-t} \left(\psi^* \left(\nabla_{z_1}^t f_j - \mathcal{P}_{k-t} \left(\nabla_{z_1}^t f_j \right) \right) \right) \right| \\ \lesssim & \sum_{m=0}^{k-t} |w_1|^m \left| \psi^* \left(\nabla_{z_1}^{t+m} \nabla_{z_2}^{k-t-m} f_j \right) - \psi^* \left(\nabla_{z_1}^m \nabla_{z_2}^{k-t-m} \left(\mathcal{P}_{k-t} \left(\nabla_{z_1}^t f_j \right) \right) \right) \right| \\ \le & \sum_{m=0}^{k-t} \left| \psi^* \left(\nabla_{z_1}^{t+m} \nabla_{z_2}^{k-t-m} f_j \right) \right|. \end{aligned}$$

Here we used in the last equality the fact that $\nabla_z^{k-t} \left(\mathcal{P}_{k-t} \left(\nabla_{z_1}^t f_j \right) \right) = 0$. Hence by a change of variables (4.10) is further estimated as follows.

$$\begin{aligned} & \left\| |w_2|^{-k+t+l} \nabla_{w_2}^l \left(\psi^* \left(\nabla_{z_1}^t f_j - \mathcal{P}_{k-t} \left(\nabla_{z_1}^t f_j \right) \right) \right) \right\|^p_{L^p(\triangle\times\triangle, |w_2|^2)} \\ \lesssim & \sum_{m=0}^{k-t} \int_\triangle \int_\triangle |w_2|^2 \left| \psi^* \left(\nabla_{z_1}^{t+m} \nabla_{z_2}^{k-t-m} f_j \right) (w_1, w_2) \right|^p dV_{w_2} dV_{w_1} \\ \lesssim & \| \psi^* (\nabla_z^k f_j) \|^p_{L^p(\triangle\times\triangle, |w_2|^2)} \lesssim \| \nabla_z^k f_j \|^p_{L^p(\mathbb{H})} \le \| \mathbf{f} \|^p_{W^{k,p}(\mathbb{H})}. \end{aligned}$$

The proof of (4.4) is complete. That $\tilde{\mathbf{f}} \in W^{k,p}(\triangle \times \triangle, |w_2|^2)$ is a direct consequence of (4.4). $\square$

4.2 Truncating solutions on the bidisc

Given $\tilde{\mathbf{f}}$ in Proposition 4.1, let u^* be the solution to $\bar\partial u^* = \tilde{\mathbf{f}}$ on $\triangle \times \triangle$ obtained in Theorem 1.2 with

$$\|u^*\|_{W^{k,p}(\triangle\times\triangle, |w_2|^2)} \lesssim \|\tilde{\mathbf{f}}\|_{W^{k,p}(\triangle\times\triangle, |w_2|^2)}. \tag{4.11}$$

Consider

$$\begin{aligned}\tilde{u}(w_1, w_2) :=&u^*(w_1, w_2) - \tilde{\mathcal{P}}_{2,k}u^*(w_1, w_2)\\ =&u^*(w_1, w_2) - \sum_{l=0}^{k-1}\frac{1}{l!}w_2^l\partial_{w_2}^l u^*(w_1, 0), \quad (w_1, w_2) \in \triangle\times\triangle,\end{aligned} \tag{4.12}$$

where $\tilde{\mathcal{P}}_{2,k}$ is the $(k-1)$-th order holomorphic Taylor polynomial operator in the w_2 variable at $w_2 = 0$. $\tilde{u}$ is well defined, due to the facts that for each fixed $w_1 \in \triangle$, $l \le k-1$, $\partial_{w_2}^l u^*(w_1, \cdot) \in W^{1,p}(\triangle, |w_2|^2)$, and when $p > 4$,

$$W^{1,p}(\triangle, |w_2|^2) \subset W^{1,q}(\triangle) \subset C^\alpha(\triangle) \tag{4.13}$$

for some $q > 2$, and $\alpha = 1-\frac{2}{q}$. Here the last inclusion $W^{1,q}(\triangle) \subset C^\alpha(\triangle)$ is the Sobolev embedding theorem; the inclusion $W^{1,p}(\triangle, |w_2|^2) \subset W^{1,q}(\triangle)$ can be seen as follows. Choose some $r \in (\frac{2}{p}, \frac{1}{2})$ and let $q = pr$. Then $q > 2$ and $\frac{r}{1-r} < 1$. For any $h \in W^{1,p}(\triangle, |w_2|^2)$,

$$\int_\triangle |h(w)|^q dV_w = \int_\triangle |h(w)|^q|w_2|^{2r}|w_2|^{-2r}dV \le \left(\int_\triangle |h(w)|^q|w_2|^2dV\right)^r \left(\int_\triangle |w_2|^{-\frac{2r}{1-r}}dV_w\right)^{1-r} < \infty,$$

and similarly $|\nabla h| \in L^q(\triangle)$. The goal of this subsection is to show that $\tilde{u}$ satisfies the following refined weighted estimate.

Proposition 4.5. *Let $\tilde{u}$ be defined in (4.12). Then $\tilde{u} \in W^{k,p}(\triangle\times\triangle, |w_2|^2)$. Moreover, for each $s, t \in \mathbb{Z}^+ \cup \{0\}$ with $s + t \le k$, we have*

$$\left\||w_2|^{-k+s}\partial_{w_1}^t\partial_{w_2}^s\tilde{u}\right\|_{L^p(\triangle\times\triangle,|w_2|^2)} \lesssim \|\mathbf{f}\|_{W^{k,p}(\mathbb{H})}.$$

We begin by first proving $\tilde{u} \in W^{k,p}(\triangle\times\triangle, |w_2|^2)$ below. It is worth pointing out that, arguing similarly as in (4.13), one has when $k \in \mathbb{Z}^+, p > 4$,

$$W^{k,p}(\triangle, |w_2|^2) \subset W^{k,q}(\triangle) \subset C^{k-1,\alpha}(\triangle).$$

for some $q > 2$ and $\alpha > 0$. In particular, for any $h \in W^{k,p}(\triangle\times\triangle, |w_2|^2), k \in \mathbb{Z}^+, p > 4$, we have $h(w_1, \cdot) \in C^{k-1,\alpha}(\triangle)$ for a.e. fixed $w_1 \in \triangle$.

Lemma 4.6. *Let $\tilde{u}$ be defined in (4.12). For each $l = 0, \ldots, k-1$, $\partial_{w_2}^l u^*(w_1, 0) \in W^{k,p}(\triangle\times\triangle, |w_2|^2)$ with*

$$\left\|\partial_{w_2}^l u^*(w_1, 0)\right\|_{W^{k,p}(\triangle\times\triangle,|w_2|^2)} \lesssim \|\mathbf{f}\|_{W^{k,p}(\mathbb{H})}. \tag{4.14}$$

Consequently, $\tilde{u} \in W^{k,p}(\triangle\times\triangle, |w_2|^2)$ satisfying

$$\mathcal{P}_{2,k}\tilde{u} = 0, \tag{4.15}$$

$$\bar\partial\tilde{u} = \tilde{\mathbf{f}} \quad on \quad \triangle\times\triangle \tag{4.16}$$

and

$$\|\tilde{u}\|_{W^{k,p}(\triangle\times\triangle,|w_2|^2)} \lesssim \|\mathbf{f}\|_{W^{k,p}(\mathbb{H})}. \tag{4.17}$$

Proof. We first show that $\sum_{l=0}^{k-1} w_2^l \partial_{w_2}^l u^*(w_1,0)$ is holomorphic on $\triangle \times \triangle$, from which (4.16) follows. Clearly, it is holomorphic in the w_2 variable. For the holomorphy in the w_1 variable, note that

$$\bar\partial_{w_1}\partial_{w_2}^l u^* = \partial_{w_2}^l \tilde f_1$$

in the weak sense. On the other hand, for fixed $w_1 \in \triangle$, $\partial_{w_2}^l \tilde f_1(w_1,\cdot) \in C^\alpha(\triangle)$ for some $\alpha > 0$ by (4.13), and $\partial_{w_2}^l \tilde f_1(w_1, 0) = 0$ by (4.3). Thus $\bar\partial_{w_1}\partial_{w_2}^l u^* \in C^\alpha(\triangle)$ with $\bar\partial_{w_1}\partial_{w_2}^l u^*(w_1,0) = 0$.

Next we prove (4.14). By the holomorphy of $\partial_{w_2}^l u^*(w_1, 0)$ above, it suffices to estimate $\left\|\partial_{w_1}^t \partial_{w_2}^l u^*(w_1,0)\right\|_{L^p(\triangle\times\triangle,|w_2|^2)}$ for $t = 0,\ldots,k$ and $l = 0,\ldots,k-1$. Let χ be a smooth function on $\triangle$ such that $\chi = 1$ in $\triangle_{\frac12}$ and $\chi = 0$ outside $\triangle$. By (3.8) (or directly verifying $u^* = T_1\tilde f_1 + T_2 S_1 \tilde f_2 = T_2 \tilde f_2 + T_1 S_2 \tilde f_1$), we have

$$\begin{aligned}\partial_{w_1}^t\partial_{w_2}^l u^* =& \partial_{w_2}^l T_2\left((1-\chi(w_2))\partial_{w_1}^t \tilde f_2\right) + \partial_{w_1}^t\partial_{w_2}^l T_2\left(\chi(w_2)\tilde f_2\right) + \partial_{w_2}^l S_2\left(\partial_{w_1}^t T_1 \tilde f_1\right)\\ =:& A_1 + A_2 + A_3.\end{aligned}$$

For A_3, let $h := \partial_{w_1}^t T_1 \tilde f_1$. Since $t \le k$, by (3.5) $h \in W^{1,p}(\triangle\times\triangle,|w_2|^2)$, with $\|h\|_{W^{1,p}(\triangle\times\triangle,|w_2|^2)} \lesssim \|\tilde f_1\|_{W^{k,p}(\triangle\times\triangle,|w_2|^2)}$. Note that for $w_1 \in \triangle$,

$$A_3(w_1,0) = \frac{l!}{2\pi i}\int_{b\triangle}\frac{h(w_1,\zeta)}{\zeta^{l+1}}d\zeta.$$

Hence

$$\begin{aligned}\|A_3(w_1,0)\|^p_{L^p(\triangle\times\triangle,|w_2|^2)} &\lesssim \int_\triangle\left|\int_{b\triangle}|h(w_1,\zeta)|d\sigma_\zeta\right|^p dV_{w_1}\int_\triangle |w_2|^2 dV_{w_2}\\ &\lesssim \int_\triangle\left|\int_\triangle |h(w_1,w_2)| + |\nabla_{w_2}h(w_1,w_2)|dV_{w_2}\right|^p dV_{w_1}\\ &\lesssim \|h\|^p_{W^{1,p}(\triangle\times\triangle,|w_2|^2)} \lesssim \|\tilde f_1\|^p_{W^{k,p}(\triangle\times\triangle,|w_2|^2)}\\ &\lesssim \|\mathbf{f}\|^p_{W^{k,p}(\mathbb{H})}.\end{aligned}\tag{4.18}$$

Here in the second line we used the trace theorem for $W^{1,1}(\triangle) \subset L^1(\partial\triangle)$; in the third line we used Hölder inequality and the fact that $|w_2|^2 \in A_p$ (or directly that $|w_2|^{-\frac{2}{p-1}} \in L^1(\triangle)$); in the fourth line we used Proposition 4.1.

For A_1, by the choice of χ, we have

$$A_1(w_1,0) = -\frac{l!}{2\pi i}\int_\triangle \frac{(1-\chi(\zeta))\partial_{w_1}^t\tilde f_2(w_1,\zeta)}{\zeta^{l+1}}d\bar\zeta\wedge d\zeta,$$

with $\left|\frac{1-\chi(\zeta)}{\zeta^{l+1}}\right| \lesssim 1$ on $\triangle$. Thus by Proposition 4.1 and the fact that $|w_2|^2 \in A_p$ similarly,

$$\begin{aligned}\|A_1(w_1,0)\|^p_{L^p(\triangle\times\triangle,|w_2|^2)} &\lesssim \int_\triangle\left|\int_\triangle\left|\partial_{w_1}^t \tilde f_2(w_1,\zeta)\right|dV_\zeta\right|^p dV_{w_1}\int_\triangle |w_2|^2 dV_{w_2}\\ &\lesssim \int_\triangle\left|\int_\triangle\left|\partial_{w_1}^t\tilde f_2(w_1,w_2)\right|dV_{w_2}\right|^p dV_{w_1}\\ &\lesssim \int_\triangle\int_\triangle\left|\partial_{w_1}^t\tilde f_2(w_1,w_2)\right|^p|w_2|^2 dV_{w_2}dV_{w_1}\\ &\le \|\tilde{\mathbf{f}}\|^p_{W^{k,p}(\triangle\times\triangle,|w_2|^2)} \lesssim \|\mathbf{f}\|^p_{W^{k,p}(\mathbb{H})}.\end{aligned}\tag{4.19}$$

Now we treat A_2. With a change of variables, rewrite it as

$$
\begin{aligned}
A_2(w_1,0) &= -\frac{1}{2\pi i}\partial_{w_1}^t\partial_{w_2}^l \int_{\mathbb{C}} \frac{\chi(\zeta+w_2)\tilde f_2(w_1,\zeta+w_2)}{\zeta}d\bar\zeta\wedge d\zeta\Bigg|_{w_2=0}\\
&= -\frac{1}{2\pi i}\partial_{w_1}^t \int_{\mathbb{C}} \frac{\partial_\zeta^l\left(\chi(\zeta+w_2)\tilde f_2(w_1,\zeta+w_2)\right)}{\zeta}d\bar\zeta\wedge d\zeta\Bigg|_{w_2=0}\\
&= -\frac{1}{2\pi i}\partial_{w_1}^t \int_{\mathbb{C}} \frac{\partial_\zeta^l\left(\chi(\zeta)\tilde f_2(w_1,\zeta)\right)}{\zeta}d\bar\zeta\wedge d\zeta.
\end{aligned}
$$

Note that $\chi(\cdot)\tilde f_2(w_1,\cdot)\in C_c^{k-1,\alpha}(\triangle)$ for some $\alpha>0$ with $\mathcal{P}_k\left(\chi(\cdot)\tilde f_2(w_1,\cdot)\right)=0$. In particular, for $j=0,\dots,l$, $\left|\partial_\zeta^j\left(\chi(\zeta)\tilde f_2(w_1,\zeta)\right)\right|\lesssim|\zeta|^{k-1-j+\alpha}$ near 0. With a repeated application of Stokes' theorem, we have

$$
\begin{aligned}
A_2(w_1,0) &= -\frac{l!}{2\pi i}\partial_{w_1}^t\int_{\mathbb{C}}\frac{\chi(\zeta)\tilde f_2(w_1,\zeta)}{\zeta^{l+1}}d\bar\zeta\wedge d\zeta\\
&= -\frac{l!}{2\pi i}\int_{\triangle}\frac{\chi(\zeta)\partial_{w_1}^t\tilde f_2(w_1,\zeta)}{\zeta^{l+1}}d\bar\zeta\wedge d\zeta.
\end{aligned}
$$

Since $l\le k-1$, making use of Proposition 4.1 with $s=0$ and the fact that $|w_2|^2\in A_p$ again, we get

$$
\begin{aligned}
\|A_2(w_1,0)\|^p_{L^p(\triangle\times\triangle,|w_2|^2)} &\lesssim \int_\triangle\left|\int_\triangle|\zeta|^{-(l+1)}\left|\partial_{w_1}^t\tilde f_2(w_1,\zeta)\right|dV_\zeta\right|^p dV_{w_1}\int_\triangle|w_2|^2dV_{w_2}\\
&\lesssim \int_\triangle\left|\int_\triangle|w_2|^{-(l+1)}\left|\partial_{w_1}^t\tilde f_2(w_1,w_2)\right|dV_{w_2}\right|^p dV_{w_1}\\
&\lesssim \int_\triangle\int_\triangle|w_2|^{-(l+1)p}\left|\partial_{w_1}^t\tilde f_2(w_1,w_2)\right|^p|w_2|^2dV_{w_2}dV_{w_1}\\
&\lesssim \left\||w_2|^{-k}\partial_{w_1}^t\tilde f_2\right\|^p_{L^p(\triangle\times\triangle,|w_2|^2)}\lesssim\|\mathbf{f}\|^p_{W^{k,p}(\mathbb{H})}.
\end{aligned}
\tag{4.20}
$$

Combining (4.18)-(4.20), we have the desired inequality (4.14).

(4.17) follows from (4.14) and (4.11). To see (4.15), we shall verify that $\bar\partial_{w_2}^m\partial_{w_2}^l\tilde u(w_1,0)=0$ for all $l,m\in\mathbb{Z}^+\cup\{0\}, l+m\le k-1$. Note that $\bar\partial_{w_2}^m\partial_{w_2}^l\tilde u(w_1,\cdot)\in C^\alpha(\triangle)$ for some $\alpha>0$ by (4.17). If $m=0$, then $\partial_{w_2}^l\tilde u(w_1,0)=0$ by its definition. If $m\ge1$, since $\bar\partial_{w_2}\tilde u=\tilde f_2$ by (4.16),

$$
\bar\partial_{w_2}^m\partial_{w_2}^l\tilde u(w_1,0)=\bar\partial_{w_2}^{m-1}\partial_{w_2}^l\tilde f_2(w_1,0)=0,
$$

where we used (4.3) in the last equality. Thus (4.15) is proved, and the proof of the lemma is complete.

□

In order to derive the refined weighted estimate of $\tilde u$ in Proposition 4.5, we also need the following modified identities/formulas for $W^{k,p}$ functions on $\triangle$ with vanishing $(k-1)$-th Taylor polynomials.

Lemma 4.7. *Let $h \in W^{k,p}(\triangle, |w|^2), k \in \mathbb{Z}^+, p > 4$ with $\mathcal{P}_k h = 0$. Then for a.e. $w \in \triangle$,*
i).

$$2\pi i w^{-k} h(w) = \int_{b\triangle} \frac{h(\zeta)}{\zeta^k(\zeta - w)} d\zeta - \int_{\triangle} \frac{\bar\partial h(\zeta)}{\zeta^k(\zeta - w)} d\bar\zeta \wedge d\zeta;$$

ii).

$$Th(w) - \tilde{\mathcal{P}}_k(Th)(w) = w^k T\left(w^{-k}h\right)(w),$$

where $\tilde{\mathcal{P}}_k$ is the $(k-1)$-th order holomorphic Taylor polynomial operator at 0.

Proof. For part i), applying the Cauchy-Green formula to $w^{-k}h$ on $\triangle \setminus \overline{\triangle_\epsilon}$, we have for each fixed $w \neq 0$,

$$2\pi i w^{-k} h(w) = \int_{b\triangle} \frac{h(\zeta)}{\zeta^k(\zeta - w)} d\zeta - \int_{b\triangle_\epsilon} \frac{h(\zeta)}{\zeta^k(\zeta - w)} d\zeta - \int_{\triangle\setminus\overline{\triangle_\epsilon(0)}} \frac{\bar\partial h(\zeta)}{\zeta^k(\zeta - w)} d\bar\zeta \wedge d\zeta. \tag{4.21}$$

We claim that

$$\lim_{\epsilon\to 0} \int_{b\triangle_\epsilon} \frac{h(\zeta)}{\zeta^k(\zeta - w)} d\zeta = 0.$$

Indeed, let $g_w(\zeta) := (\zeta - w)^{-1}h(\zeta)$. Since $w \neq 0$, $g_w \in W^{k,p}(\triangle_\epsilon, |\zeta|^2), p > 4$ with ϵ sufficiently small and $\mathcal{P}_k g_w = 0$. In particular, $g_w \in C^{k-1,\alpha}(\triangle_\epsilon)$ for some $\alpha > 0$, with $|g_w(\zeta)| \lesssim |\zeta|^{k-1+\alpha}$ near 0. Thus

$$\lim_{\epsilon\to 0} \left| \int_{b\triangle_\epsilon} \frac{h(\zeta)}{\zeta^k(\zeta - w)} d\zeta \right| \le \lim_{\epsilon\to 0} \epsilon^{-k} \int_{b\triangle_\epsilon} |g_w(\zeta)| d\sigma_\zeta \lesssim \lim_{\epsilon\to 0} \epsilon^\alpha = 0. \tag{4.22}$$

The claim is proved. Part i) follows from the claim by letting $\epsilon \to 0$ in (4.21).

For ii), let χ be a smooth function which is 1 near 0, and vanishes outside $\triangle_{\frac{1}{2}}$. A direct computation gives that

$$\begin{aligned}
-2\pi i \partial Th(0) &= \partial \int_{\triangle} \frac{\chi(\zeta)h(\zeta)}{\zeta - w} d\bar\zeta \wedge d\zeta \Big|_{w=0} + \partial \int_{\triangle} \frac{(1-\chi(\zeta))h(\zeta)}{\zeta - w} d\bar\zeta \wedge d\zeta \Big|_{w=0} \\
&= \int_{\mathbb{C}} \frac{\partial_w\left(\chi(\zeta+w)h(\zeta+w)\right)}{\zeta} d\bar\zeta \wedge d\zeta \Big|_{w=0} + \int_{\triangle} \frac{(1-\chi(\zeta))h(\zeta)}{\zeta^2} d\bar\zeta \wedge d\zeta \\
&= \int_{\mathbb{C}} \frac{\partial_\zeta\left(\chi(\zeta)h(\zeta)\right)}{\zeta} d\bar\zeta \wedge d\zeta + \int_{\triangle} \frac{(1-\chi(\zeta))h(\zeta)}{\zeta^2} d\bar\zeta \wedge d\zeta \\
&= \int_{\mathbb{C}} \frac{\chi(\zeta)h(\zeta)}{\zeta^2} d\bar\zeta \wedge d\zeta + \int_{\triangle} \frac{(1-\chi(\zeta))h(\zeta)}{\zeta^2} d\bar\zeta \wedge d\zeta = \int_{\triangle} \frac{h(\zeta)}{\zeta^2} d\bar\zeta \wedge d\zeta.
\end{aligned}$$

Here in the fourth line above we used Stokes' theorem and a similar argument as in (4.22) (with $k = 1$ there). Consequently with an induction,

$$\tilde{\mathcal{P}}_k Th = -\sum_{l=0}^{k-1} \frac{w^l}{2\pi i} \int_{\triangle} \frac{h(\zeta)}{\zeta^{l+1}} d\bar\zeta \wedge d\zeta.$$

Note that each term in the right hand side of the above is well defined due to Remark 4.4.

Making use of the following elementary identity for the Cauchy kernel:

$$\frac{1}{\zeta - w} - \sum_{l=0}^{k-1} \frac{w^l}{\zeta^{l+1}} = \frac{w^k}{\zeta^k(\zeta - w)}, \quad \text{for all } \zeta \neq w \text{ nor } 0,$$

we immediately get

$$Th(w) - \tilde{\mathcal{P}}_k Th(w) = -\frac{w^k}{2\pi i}\int_{\triangle}\frac{h(\zeta)}{\zeta^k(\zeta - w)}d\bar{\zeta}\wedge d\zeta = w^k T\left(w^{-k}h\right), \qquad w \in \triangle.$$

$\square$

Lemma 4.8. *If $h \in W^{2,p}(\triangle, |w|^2), p > 4,$ then*

$$S\partial h = \partial S h + S(\bar{w}^2\bar{\partial}h) \quad on \quad \triangle.$$

Proof. Note that $h \in W^{2,p}(\triangle, |w|^2) \subset C^{1,\alpha}(\triangle)$ for some $\alpha > 0$. So both sides of the above equality are actually in the strong sense. The lemma follows from a direct computation below. For $w \in \triangle$,

$$\begin{aligned}
S\partial h(w) &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{\partial_\zeta h(e^{i\theta})ie^{i\theta}}{e^{i\theta} - w}d\theta = \frac{1}{2\pi i}\int_0^{2\pi}\frac{\partial_\theta\left(h(e^{i\theta})\right) + i\bar{\partial}_\zeta h(e^{i\theta})e^{-i\theta}}{e^{i\theta} - w}d\theta\\
&= -\frac{1}{2\pi i}\int_0^{2\pi}\partial_\theta\left(\frac{1}{e^{i\theta} - w}\right)h(e^{i\theta})d\theta + \frac{1}{2\pi i}\int_0^{2\pi}\frac{\bar{\partial}_\zeta h(e^{i\theta})e^{-2i\theta}}{e^{i\theta} - w}ie^{i\theta}d\theta\\
&= \frac{1}{2\pi i}\int_0^{2\pi}\partial_w\left(\frac{1}{e^{i\theta} - w}\right)h(e^{i\theta})ie^{i\theta}d\theta + \frac{1}{2\pi i}\int_{b\triangle}\frac{\bar{\partial}_\zeta h(\zeta)\bar{\zeta}^2}{\zeta - w}d\zeta\\
&= \frac{1}{2\pi i}\int_{b\triangle}\partial_w\left(\frac{1}{\zeta - w}\right)h(\zeta)d\zeta + S\left(\bar{w}^2\bar{\partial}h\right) = \partial S h(w) + S\left(\bar{w}^2\bar{\partial}h\right)(w).
\end{aligned}$$

$\square$

Proof of Proposition 4.5: In view of Lemma 4.6, we only need to prove the estimate in the proposition when $s \le k-1$. First consider the case when $0 \le t \le k-1$. For fixed $w_1 \in \triangle$, $h_{w_1} := \partial^s_{w_2}\tilde{u}(w_1, \cdot) \in W^{k-s,p}(\triangle, |w_2|^2)$, $\mathcal{P}_{k-s}h_{w_1} = 0$ by (4.15), and $\bar{\partial}_{w_2}h_{w_1} = \partial^s_{w_2}\tilde{f}_2$. We apply Lemma 4.7, part i) to h_{w_1} and obtain

$$2\pi i w_2^{-k+s}\partial^s_{w_2}\tilde{u}(w_1, w_2) = \int_{b\triangle}\frac{\partial^s_\zeta\tilde{u}(w_1,\zeta)}{\zeta^{k-s}(\zeta - w_2)}d\zeta - \int_{\triangle}\frac{\partial^s_\zeta\tilde{f}_2(w_1,\zeta)}{\zeta^{k-s}(\zeta - w_2)}d\bar{\zeta}\wedge d\zeta.$$

Consequently,

$$\begin{aligned}
w_2^{-k+s}\partial^t_{w_1}\partial^s_{w_2}\tilde{u}(w_1, w_2) =& \frac{1}{2\pi i}\left(\partial^t_{w_1}\int_{b\triangle}\frac{\partial^s_\zeta\left(\bar{\zeta}^{k-s}\tilde{u}(w_1,\zeta)\right)}{\zeta - w_2}d\zeta - \int_{\triangle}\frac{\zeta^{-k+s}\partial^t_{w_1}\partial^s_\zeta\tilde{f}_2(w_1,\zeta)}{\zeta - w_2}d\bar{\zeta}\wedge d\zeta\right)\\
=& \partial^t_{w_1}S_2\left(\partial^s_{w_2}\left(\bar{w}_2^{k-s}\tilde{u}\right)\right) + T_2\left(w_2^{-k+s}\partial^t_{w_1}\partial^s_{w_2}\tilde{f}_2\right)\\
=:& B_1 + B_2.
\end{aligned}$$

By (3.2) and Proposition 4.1,

$$\|B_2\|_{L^p(\triangle\times\triangle,|w_2|^2)} \lesssim \left\|T_2\left(w_2^{-k+s}\partial^t_{w_1}\partial^s_{w_2}\tilde{f}_2\right)\right\|_{L^p(\triangle\times\triangle,|w_2|^2)} \lesssim \left\|w_2^{-k+s}\partial^t_{w_1}\partial^s_{w_2}\tilde{f}_2\right\|_{L^p(\triangle\times\triangle,|w_2|^2)} \lesssim \|\mathbf{f}\|_{W^{k,p}(\mathbb{H})}.$$

For B_1, if $s = 0$, then $B_1 = S_2\left(\bar{w}_2^k\partial^t_{w_1}\tilde{u}\right)$, where $\bar{w}_2^k\partial^t_{w_1}\tilde{u} \in W^{1,p}(\triangle\times\triangle, |w_2|^2)$ as $t \le k-1$. Then (3.3) and Lemma 4.6 give

$$\begin{aligned}
\|B_1\|_{L^p(\triangle\times\triangle,|w_2|^2)} \lesssim& \left\|S_2\left(\bar{w}_2^k\partial^t_{w_1}\tilde{u}\right)\right\|_{L^p(\triangle\times\triangle,|w_2|^2)} \lesssim \left\|\bar{w}_2^k\partial^t_{w_1}\tilde{u}\right\|_{W^{1,p}(\triangle\times\triangle,|w_2|^2)}\\
\lesssim& \|\tilde{u}\|_{W^{k,p}(\triangle\times\triangle,|w_2|^2)} \lesssim \|\mathbf{f}\|_{W^{k,p}(\mathbb{H})}.
\end{aligned}$$

For the case $s \geq 1$, since $s \leq k-1$, $\partial_{w_2}^{s-1}\left(\bar{w}_2^{k-s}\tilde{u}\right)(w_1,\cdot) \in W^{2,p}(\triangle, |w_2|^2)$ for fixed $w_1 \in \triangle$. Applying Lemma 4.8 to $\partial_{w_2}^{s-1}\left(\bar{w}_2^{k-s}\tilde{u}\right)(w_1,\cdot)$ and using the fact that $\bar{\partial}_{w_2}\tilde{u} = \tilde{f}_2$, we further write

$$\begin{aligned}
B_1 =&\partial_{w_1}^t \partial_{w_2} S_2\left(\partial_{w_2}^{s-1}\left(\bar{w}_2^{k-s}\tilde{u}\right)\right) + \partial_{w_1}^t S_2\left(\bar{w}_2^2 \partial_{w_2}^{s-1}\left((k-s)\bar{w}_2^{k-s-1}\tilde{u} + \bar{w}_2^{k-s}\tilde{f}_2\right)\right)\\
=&\partial_{w_2} S_2\left(\partial_{w_1}^t \partial_{w_2}^{s-1}\left(\bar{w}_2^{k-s}\tilde{u}\right)\right) + (k-s) S_2\left(\partial_{w_1}^t \partial_{w_2}^{s-1}\left(\bar{w}_2^{k-s+1}\tilde{u}\right)\right) + S_2\left(\partial_{w_1}^t \partial_{w_2}^{s-1}\left(\bar{w}_2^{k-s+2}\tilde{f}_2\right)\right).
\end{aligned}$$

Note that $\partial_{w_1}^t \partial_{w_2}^{s-1}\left(\bar{w}_2^{l}\tilde{u}\right) \in W^{1,p}(\triangle\times\triangle, |w_2|^2)$ for $l = k-s, k-s+1, k-s+2$. By (3.6), Proposition 4.1 and (4.17),

$$\begin{aligned}
\|B_1\|_{L^p(\triangle\times\triangle,|w_2|^2)} \lesssim& \left\|\partial_{w_1}^t \partial_{w_2}^{s-1}\left(\bar{w}_2^{k-s}\tilde{u}\right)\right\|_{W^{1,p}(\triangle\times\triangle,|w_2|^2)} + \left\|\partial_{w_1}^t \partial_{w_2}^{s-1}\left(\bar{w}_2^{k-s+1}\tilde{u}\right)\right\|_{W^{1,p}(\triangle\times\triangle,|w_2|^2)}\\
&+ \left\|\partial_{w_1}^t \partial_{w_2}^{s-1}\left(\bar{w}_2^{k-s+2}\tilde{f}_2\right)\right\|_{W^{1,p}(\triangle\times\triangle,|w_2|^2)}\\
\lesssim& \|\tilde{u}\|_{W^{k,p}(\triangle\times\triangle,|w_2|^2)} + \left\|\tilde{f}_2\right\|_{W^{k,p}(\triangle\times\triangle,|w_2|^2)} \lesssim \|\mathbf{f}\|_{W^{k,p}(\mathbb{H})}.
\end{aligned}$$

Finally, we treat the case when $t = k$ (and so $s = 0$). According to the definition of $\tilde{u}$,

$$\begin{aligned}
\tilde{u} =& T_1\tilde{f}_1 + S_1 T_2 \tilde{f}_2 - T_1 \tilde{\mathcal{P}}_{2,k}\tilde{f}_1 - S_1\tilde{\mathcal{P}}_{2,k} T_2 \tilde{f}_2\\
=& T_1\tilde{f}_1 + S_1\left(T_2 - \tilde{\mathcal{P}}_{2,k} T_2\right)\tilde{f}_2\\
=& T_1\tilde{f}_1 + S_1\left(w_2^k T_2\left(w_2^{-k}\tilde{f}_2\right)\right).
\end{aligned}$$

Here we used the fact that $\mathcal{P}_{2,k}\tilde{f}_1 = 0$ by (4.3) in the second equality, and Lemma 4.7 part ii) in the third equality for each fixed $w_1 \in \triangle$. Consequently,

$$w_2^{-k}\partial_{w_1}^k \tilde{u} = \partial_{w_1}^k T_1\left(w_2^{-k}\tilde{f}_1\right) + T_2\left(\partial_{w_1}^k S_1\left(w_2^{-k}\tilde{f}_2\right)\right) =: C_1 + C_2.$$

For C_1, by (3.4) and Proposition 4.1 (with $s = 0$ there),

$$\|C_1\|_{L^p(\triangle\times\triangle,|w_2|^2)} \lesssim \sum_{j=0}^{k-1}\left\|w_2^{-k}\nabla_{w_1}^j \tilde{f}_2\right\|_{L^p(\triangle\times\triangle,|w_2|^2)} \lesssim \|\mathbf{f}\|_{W^{k,p}(\mathbb{H})}.$$

For C_2, by (3.4) (with $k = 1$ there), (3.6) and Proposition 4.1 (with $s = 0$ there).

$$\|C_2\|_{L^p(\triangle\times\triangle,|w_2|^2)} \lesssim \left\|\partial_{w_1}^k S_1\left(w_2^{-k}\tilde{f}_2\right)\right\|_{L^p(\triangle\times\triangle,|w_2|^2)} \lesssim \sum_{j=0}^{k}\left\|w_2^{-k}\nabla_{w_1}^j \tilde{f}_2\right\|_{L^p(\triangle\times\triangle,|w_2|^2)} \lesssim \|\mathbf{f}\|_{W^{k,p}(\mathbb{H})}.$$

The proof of the proposition is thus complete.

□

4.3 Proof of the main theorem

Proof of Theorem 1.1: Let $\mathcal{T}_k\mathbf{f} := \phi^*\tilde{u} + u_k$ on $\mathbb{H}$, where $\tilde{u}$ is defined in (4.12), and u_k satisfies (4.1). Then $\bar{\partial}\mathcal{T}_k\mathbf{f} = \mathbf{f}$ on $\mathbb{H}$. To show the desired estimate for $\|\mathcal{T}_k\mathbf{f}\|_{W^{k,p}(\mathbb{H})}$, since the anti-holomorphic derivatives of $\mathcal{T}_k\mathbf{f}$ are shifted to that of $\mathbf{f}$, we only need to estimate $\left\|\partial_{z_1}^{l_1}\partial_{z_2}^{l_2}(\phi^*\tilde{u})\right\|_{L^p(\mathbb{H})}$, $l_1, l_2 \in \mathbb{Z}^+\cup\{0\}, l_1 + l_2 \leq k$. Note that

$$\partial_{z_1}^{l_1}\partial_{z_2}^{l_2}(\phi^*\tilde{u}) = \sum_{s+t\leq l_1+l_2, t\geq l_1} C_{l_1,l_2,t,s} z_1^{t-l_1} z_2^{-t-l_2+s}\left(\partial_{w_1}^t \partial_{w_2}^s \tilde{u}\right)\left(\frac{z_1}{z_2}, z_2\right)$$

for some constants $C_{l_1,l_2,t,s}$ dependent on l_1, l_2, t, s, and $|z_1| \le |z_2|$ on $\mathbb{H}$. Then by a change of variables,

$$\begin{aligned}\left\|\partial_{z_1}^{l_1}\partial_{z_2}^{l_2}\left(\phi^*\tilde{u}\right)\right\|_{L^p(\mathbb{H})} &\lesssim \sum_{s+t\le l_1+l_2, t\ge l_1} \left\||w_2|^{-l_1-l_2+s}\partial_{w_1}^t\partial_{w_2}^s\tilde{u}\left(w_1,w_2\right)\right\|_{L^p(\triangle\times\triangle,|w_2|^2)}\\ &\le \sum_{s+t\le k}\left\||w_2|^{-k+s}\partial_{w_1}^t\partial_{w_2}^s\tilde{u}\left(w_1,w_2\right)\right\|_{L^p(\triangle\times\triangle,|w_2|^2)}.\end{aligned}$$

The rest of the proof follows from Proposition 4.5.

□

The following Kerzman-type example demonstrates that the $\bar\partial$ problem on $\mathbb{H}$ with $W^{k,p}$ data in general does not expect solutions in $W^{k,p+\epsilon}$, $\epsilon > 0$, which verifies the optimality of Theorem 1.1.

Example 2. *For each $k \in \mathbb{Z}^+$ and $2 < p < \infty$, let $\mathbf{f} = (z_2-1)^{k-\frac{2}{p}}d\bar z_1$ on $\mathbb{H}$, $\frac{1}{2}\pi < \arg(z_2-1) < \frac{3}{2}\pi$. Then $\mathbf{f} \in W^{k,\tilde p}(\mathbb{H})$ for all $2 < \tilde p < p$ and is $\bar\partial$-closed on $\mathbb{H}$. However, there does not exist a solution $u \in W^{k,p}(\mathbb{H})$ to $\bar\partial u = \mathbf{f}$ on $\mathbb{H}$.*

Proof. Clearly $\mathbf{f} \in W^{k,\tilde p}(\mathbb{H})$ for all $2 < \tilde p < p$ and is $\bar\partial$-closed on $\mathbb{H}$. Arguing by contradiction, suppose there exists some $u \in W^{k,p}(\mathbb{H})$ satisfying $\bar\partial u = \mathbf{f}$ on $\mathbb{H}$. In particular, since $\triangle_{\frac{1}{2}} \times (\triangle \setminus \overline{\triangle_{\frac{1}{2}}}) \subset \mathbb{H}$, there exists some holomorphic function h on $\triangle_{\frac{1}{2}} \times (\triangle \setminus \overline{\triangle_{\frac{1}{2}}})$ such that $u|_{\triangle_{\frac{1}{2}}\times(\triangle\setminus\overline{\triangle_{\frac{1}{2}}})} = (z_2-1)^{k-\frac{2}{p}}\bar z_1 + h \in W^{k,p}(\triangle_{\frac{1}{2}} \times (\triangle \setminus \overline{\triangle_{\frac{1}{2}}}))$.

For each fixed $(r, z_2) \in U := \left(0, \frac{1}{2}\right) \times \left(\triangle \setminus \overline{\triangle_{\frac{1}{2}}}\right) \subset \mathbb{R} \times \mathbb{C}$, consider

$$v(r, z_2) := \int_{|z_1|=r} \tilde u(z_1, z_2)dz_1.$$

Then with a similar argument as in the proof of Example 1, one can see that $v \in W^{k,p}(U)$. Note that $h(\cdot, z_2)$ is holomorphic on $\triangle_{\frac{1}{2}}$ for each fixed $z_2 \in \triangle\setminus\overline{\triangle_{\frac{1}{2}}}$. Thus for fixed $(r, z_2) \in U$, Cauchy's theorem gives

$$v(r, z_2) = \int_{|z_1|=r} z_2(z_2-1)^{k-\frac{2}{p}}\bar z_1 dz_1 = 2\pi r^2 i z_2(z_2-1)^{k-\frac{2}{p}},$$

which does not belong to $W^{k,p}(U)$. A contradiction!

□

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