VECTOR FIELDS WITH CONTINUOUS CURL BUT DISCONTINUOUS PARTIAL DERIVATIVES

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Abstract. Motivated by strong versions of Green's theorem, we give an example of a differentiable vector field for which the curl is continuous but not all the partial derivatives are continuous.

1. Introduction

A multivariable calculus exam reasonably asks: "State Green's theorem for a vector field $\vec{F}(x,y) = (P(x,y), Q(x,y))$ defined on the unit disk $D \subseteq \mathbb{R}^2$, and a rectangle $R = [a,b] \times [c,d] \subseteq D$ with positively oriented boundary ∂R." The following answer might be expected to get a lot of credit:

$$\int_{\partial R} \vec{F} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right). \tag{1}$$

The left side is the line integral around the boundary ∂R, and the right side is the double integral over R of a combination of derivatives, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$.

However, something is missing! The above answer (1) is only the conclusion of the theorem; how about a hypothesis? Without some assumption about $\vec{F}$, there's no reason for these integrals to even exist, much less be equal.

A sampling of calculus textbooks (from this century and the previous: [2, 7, 11, 12]) suggests that the following statement is the most frequently made assumption about the vector field $\vec{F} = (P, Q)$ in Green's theorem, denoted by $\mathcal{C}^1$ (for "continuous first derivatives"):

Hypothesis 1. P and Q have continuous partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ on D.

The standard calculus proof that Hypothesis 1 implies Green's formula (1) for the rectangle uses the one-variable fundamental theorem of calculus to relate the line integral to iterated integrals, and then Fubini's theorem to relate the iterated integrals to the double integral. Such a proof is valid, enlightening, and gets across the main idea that Green's theorem is a natural two-dimensional version of the fundamental theorem. Textbooks then go on to state Green's theorem more generally, for shapes and domains other than rectangles $R \subseteq D$. A different direction for generalization, which we consider here, is to

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keep working with just the rectangle, but to try to improve Green's theorem by weakening Hypothesis 1.

While Paul Cohen was a graduate student at the University of Chicago, he proved ([5, Chapter 4], later published as [6]) that the following condition on $\vec{F} = (P, Q)$ is sufficient to imply (1) for any R.

Hypothesis 2. P and Q are continuous on D, all four first partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ exist at all but countably many points in D, and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is L^1-integrable on compact subsets of D.

The technical terms in such a statement may be a bit much for an undergraduate calculus class, and Cohen's proof is also very difficult. So, we won't try to put the weakest possible hypothesis for Green's theorem on that calculus test (or this article); this remains a current topic of mathematical research ([8]).

The following assumption for Green's theorem appears in [3].

Hypothesis 3. P and Q are differentiable on D, and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous on D.

This statement has the advantage of involving only concepts already seen in multivariable calculus. In Hypothesis 3, there is no assumption about the continuity, or even integrability, of the individual terms $\frac{\partial Q}{\partial x}$ or $\frac{\partial P}{\partial y}$, so the proof given in [3] that (1) follows from Hypothesis 3 is necessarily different from the previously mentioned fundamental theorem/Fubini argument, but not as hard as Cohen's proof. The term *differentiable* here means that there exists a local approximation by a linear function (the usual textbook notion as in [2, 12]); it implies that all the first partial derivatives exist and is implied by the $\mathcal{C}^1$ property. (The conditions from all three hypotheses are stated for the open disk D rather than the closed rectangle R just to avoid technicalities about limits on boundaries.)

The three hypotheses are logically related:

$$\text{Hyp. 1 } (\mathcal{C}^1) \implies \text{Hyp. 3} \implies \text{Hyp. 2 (Cohen).}$$

These implications are standard calculus facts, not proved here; rather, a goal of this article is to show by concrete examples that their converses are false. Namely, Cohen's version of Green's theorem is strictly better than the version of [3], applying to a larger class of vector fields, including Example 1. Similarly, the version with Hypothesis 3 is strictly better than the usual $\mathcal{C}^1$ version in most calculus books, as shown by Example 2.

The scalar quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ can be called the two-dimensional curl of the planar vector field $\vec{F} = (P, Q)$ (following [2, §17.4]). The term *curl* is often used only for a vector quantity derived from a three-dimensional vector field. Of course, the plane vector field extends to $\vec{F}(x, y, z) = (P(x, y), Q(x, y), 0)$ in $\mathbb{R}^3$, and the curl vector is $(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})$. So, the planar vector field constructed in Example 2 can be extended to a differentiable vector field in space such that the curl vector has continuous components but not all of the partial derivatives of the components of $\vec{F}$ are continuous.

2. The Examples.

The first example is adapted from [1, §15.1].

Example 1. Consider a vector field $\vec{F}(x, y) = (P(x, y), Q(x, y))$, defined at the origin by $\vec{F}(0, 0) = (0, 0)$, and everywhere else in the unit disk D by

$$\begin{aligned} P &= y\sqrt{-\ln(x^2 + y^2)}; \\ Q &= x\sqrt{-\ln(x^2 + y^2)}. \end{aligned}$$

The functions P and Q are continuous on D, but some partial derivatives do not exist at the origin. From the definition of partial derivative, we find

$$\left.\frac{\partial P}{\partial y}\right|_{(0,0)} = \lim_{y\to 0} \frac{P(0, y) - P(0, 0)}{y - 0} = \lim_{y\to 0} \frac{y\sqrt{-\ln(y^2)}}{y} = +\infty$$

and similarly for $\frac{\partial Q}{\partial x}$. For $(x, y) \neq (0, 0)$, all four partial derivatives exist. In particular, $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are calculated using the product rule and chain rule as follows.

$$\begin{aligned} \frac{\partial P}{\partial y} &= \sqrt{-\ln(x^2 + y^2)} + \frac{y^2}{x^2 + y^2} \cdot \frac{-1}{\sqrt{-\ln(x^2 + y^2)}}; \\ \frac{\partial Q}{\partial x} &= \sqrt{-\ln(x^2 + y^2)} + \frac{x^2}{x^2 + y^2} \cdot \frac{-1}{\sqrt{-\ln(x^2 + y^2)}}. \end{aligned}$$

The $\sqrt{-\ln(x^2 + y^2)}$ terms are unbounded as $(x, y) \to (0, 0)$ while the second terms approach limit 0, so these partial derivatives cannot be extended to be continuous on D. However, there is a cancellation when evaluating the curl:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{y^2 - x^2}{x^2 + y^2} \cdot \frac{1}{\sqrt{-\ln(x^2 + y^2)}}. \tag{2}$$

The first factor $\frac{y^2-x^2}{x^2+y^2}$ is bounded and the other factor $\frac{1}{\sqrt{-\ln(x^2+y^2)}}$ approaches 0, so

$$\lim_{(x,y)\to(0,0)} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$

The curl thus becomes continuous if we extend its value to 0 at the origin. This $\vec{F}$ satisfies Cohen's Hypothesis 2, so Green's theorem can be applied to evaluate (1) for any rectangle $R \subseteq D$. Hypothesis 3 is not satisfied: P and Q are not differentiable. Moreover, the curl has a removable discontinuity, so it is locally integrable, but not actually continuous, because $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$ fail to exist at the origin.

Recall from one-variable calculus that a function $f : \mathbb{R} \to \mathbb{R}$ can have a derivative f' that exists everywhere but is discontinuous. For example, $f(x) = x^2 \sin(1/x)$, extended to $f(0) = 0$, is differentiable, with $f'(0) = 0$, but $f'(x)$ is discontinuous because it oscillates as $x \to 0$. Example 2 gives a two-dimensional version of this phenomenon: a vector field

$\vec{F}(x,y) = (P(x,y), Q(x,y))$ such that the partial derivatives of P and Q exist everywhere but at least one is discontinuous at $(0,0)$, so the $\mathcal{C}^1$ property does not hold. The challenge is to find such an example where $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous.

Example 2. Consider a vector field $\vec{F}(x,y) = (P(x,y), Q(x,y))$, defined at the origin by $\vec{F}(0,0) = (0,0)$, and everywhere else in the unit disk D by

$$\begin{aligned} P &= xy\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right); \\ Q &= y^2\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right). \end{aligned}$$

Clearly, P and Q are continuous at the origin. By the definition of partial derivative, $\frac{\partial P}{\partial x}|_{(0,0)} = \frac{\partial P}{\partial y}|_{(0,0)} = 0$. Thus

$$\lim_{(x,y)\to(0,0)} \frac{\left|P(x,y) - P(0,0) - x\frac{\partial P}{\partial x}\big|_{(0,0)} - y\frac{\partial P}{\partial y}\big|_{(0,0)}\right|}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{|P(x,y)|}{\sqrt{x^2+y^2}} = 0.$$

Hence P (and similarly Q) is differentiable at the origin. For $(x,y) \neq (0,0)$, the partial derivatives are calculated using the product rule and chain rule:

$$\begin{aligned} \frac{\partial P}{\partial x} &= y\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x^2y}{(x^2+y^2)^{3/2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right); \\ \frac{\partial P}{\partial y} &= x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{xy^2}{(x^2+y^2)^{3/2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right); \\ \frac{\partial Q}{\partial x} &= -\frac{xy^2}{(x^2+y^2)^{3/2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right); \\ \frac{\partial Q}{\partial y} &= 2y\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{y^3}{(x^2+y^2)^{3/2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right). \end{aligned}$$

All of these partial derivatives are continuous away from $(0,0)$ (establishing the differentiability property everywhere) but are discontinuous at $(0,0)$. In particular, they oscillate while approaching $(0,0)$ along the line $y = x$. However, similar to Example 1, there is a cancellation when evaluating the curl:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right),$$

so the curl exists and is continuous on D. To conclude, this $\vec{F}$ is differentiable with continuous curl everywhere on D, but does not have continuous partial derivatives on D.

Example 2 exhibits the phenomenon claimed in the title, and shows that Green's theorem with Hypothesis 3 is better than the usual $\mathcal{C}^1$ version.

3. A complex view.

The vector fields from Examples 1 and 2 and their derivatives can be reformulated in terms of complex numbers, which simplifies the expressions and calculations. The conversion from real coordinates (x, y) to complex coordinates uses the notation $z = x+iy$, $\bar{z} = x - iy$, and the derivative operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The two-dimensional curl then appears in this identity for differentiable functions P and Q:

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}}(Q + iP) &= \frac{1}{2}\left[\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) + i\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\right] && (3)\\
&= \frac{1}{2}\left[\mathrm{curl}(P, Q) + i\mathrm{curl}(-Q, P)\right]\\
\implies \mathrm{curl}(P, Q) &= 2\mathrm{Re}\left(\frac{\partial}{\partial \bar{z}}(Q + iP)\right). && (4)
\end{aligned}$$

The ordering of the terms $Q+iP$ in (3) is chosen just to make some $\pm$ signs more convenient.

Recalling Example 1, the conversion to complex coordinates gives, for $z \neq 0$, the following expression (as it originally appears in [1]):

$$\begin{aligned}
(P, Q) &= \left(y\sqrt{-\ln(x^2 + y^2)}, x\sqrt{-\ln(x^2 + y^2)}\right)\\
\implies \frac{\partial}{\partial \bar{z}}(Q + iP) &= \frac{\partial}{\partial \bar{z}}\left(z\sqrt{-\ln(z\bar{z})}\right)\\
&= \frac{-z}{2\bar{z}\sqrt{-\ln(z\bar{z})}}. && (5)
\end{aligned}$$

In the same way as the curl computed in (2), this derivative fails to exist at the origin, but (5) has a removable discontinuity.

Example 2 has this complex expression for $z \neq 0$:

$$\begin{aligned}
(P, Q) &= \left(xy\sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right), y^2\sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right)\right)\\
\implies \frac{\partial}{\partial \bar{z}}(Q + iP) &= \frac{\partial}{\partial \bar{z}}\left(\frac{1}{2}(z - \bar{z})\bar{z}\sin((z\bar{z})^{-1/2})\right)\\
&= \frac{z - 2\bar{z}}{2}\sin((z\bar{z})^{-1/2}) - \frac{z - \bar{z}}{4(z\bar{z})^{1/2}}\cos((z\bar{z})^{-1/2}). && (6)
\end{aligned}$$

The first term of (6) is continuous, and the second term is discontinuous but has purely imaginary values. So the curl of (P,Q) is continuous because, by (4), it is twice the real part of (6).

The identity (3) suggests another regularity property for vector fields:

Hypothesis 4. The functions P and Q are differentiable on the unit disk D, and satisfy any of the following equivalent properties:

- $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ and $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ are continuous;
- The vector fields (P,Q) and $(-Q,P)$ both have continuous curl;
- $\frac{\partial}{\partial \bar{z}}(Q+iP)$ is continuous.

Clearly, this new hypothesis is intermediate between Hypothesis 1 and Hypothesis 3:

$$\text{Hyp. 1 } (\mathcal{C}^1) \implies \text{Hyp. 4} \implies \text{Hyp. 3},$$

and the calculation (6) shows that Example 2 has the property that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous but $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 2\mathrm{Im}\left(\frac{\partial}{\partial \bar{z}}(Q+iP)\right)$ is not, so Hypothesis 3 does not imply Hypothesis 4.

Hypothesis 4 is used in [3] to state the following complex version of Green's theorem. The approach used in [3] to the proof (and to the real version with Hypothesis 3) is not that much different from nor more difficult than the well-known proof of the Cauchy–Goursat theorem from complex analysis ([10]).

Proposition 1. *For a complex-valued function $f(x+iy) = Q(x,y) + iP(x,y)$ with (P,Q) satisfying Hypothesis 4, the complex line integral equals the double integral:*

$$\int_{\partial R} f(z)dz = 2i \iint_R \frac{\partial f}{\partial \bar{z}}.$$

The only remaining implication among the four hypotheses, that the $\mathcal{C}^1$ property (Hypothesis 1) implies the continuity of the $\bar{z}$-derivative (Hypothesis 4), is trivial, but the converse is false and this is both difficult and interesting.

It is interesting because some conditions on the $\bar{z}$-derivative do have consequences for the other partial derivatives. For example, a fundamental fact from complex analysis is that if a complex function h is differentiable with $\frac{\partial h}{\partial \bar{z}} \equiv 0$, then the real and imaginary parts $\mathrm{Re}(h(z))$ and $\mathrm{Im}(h(z))$ are $\mathcal{C}^1$, and in fact infinitely differentiable. An even stronger result is the Looman–Menchoff theorem, which we recall here from [9, 10] as Proposition 2, and which was the motivation for Cohen's proof of his version of Green's theorem (before he moved on to Fields Medal–winning work on a mostly unrelated topic in set theory).

Proposition 2. *Suppose $h = Q + iP : D \to \mathbb{C}$ is continuous and all four first partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ exist at all but countably many points in D. If $\frac{\partial h}{\partial \bar{z}} = 0$ almost everywhere in D, then h is complex analytic on D.*

However, replacing "$\frac{\partial h}{\partial \bar{z}}$ is identically zero" with "$\frac{\partial h}{\partial \bar{z}}$ is continuous" does not lead to any conclusion about the continuity of the partial derivatives with respect to x or y or z (with $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$). An example of a complex-valued differentiable function h such that $\partial h/\partial \bar{z}$ is continuous and $\partial h/\partial z$ exists but is discontinuous recently appeared in [4, §3], although it involves a lengthy construction. We have not been able to find any such example with an *elementary* expression as in Examples 1 and 2, so we leave that as an open question.

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