# Optimal $L^p$ regularity for $\bar{\partial}$ on the Hartogs triangle

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**Abstract**

In this paper, we prove weighted $L^p$ estimates for the canonical solutions on product domains. As an application, we show that if $p \in [4, \infty)$, the $\bar{\partial}$ equation on the Hartogs triangle with $L^p$ data admits $L^p$ solutions with the desired estimates. For any $\epsilon > 0$, by constructing an example with $L^p$ data but having no $L^{p+\epsilon}$ solutions, we verify the sharpness of the $L^p$ regularity on the Hartogs triangle.

## 1 Introduction

Let $\mathbb{H}$ be the Hartogs triangle defined by

$$\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}.$$

Being a special bounded pseudoconvex domain without Lipschitz boundary, the Hartogs triangle has played an important role in complex analysis and attracted considerable attention. See [1, 7–11, 16, 17, 23] et al. Among others a well-known result by Chaumat-Chollet [2] in function theory states that the $\bar{\partial}$ problem with smooth data on $\overline{\mathbb{H}}$ has no smooth solutions in general. On the other hand, when restricted at each Hölder level, they showed that the $\bar{\partial}$ equation admits solutions in the same Hölder space as that of the data. (Note this does not contradict with the global irregularity, as the solution operators are different at different Hölder levels.)

In view of a biholomorphism between the punctured bidisc and the Hartogs triangle, a natural machinery was introduced in works of Ma-Michel [20] and Chakrabarti-Shaw [6] to treat with the $\bar{\partial}$ problem on the Hartogs triangle. That is, using the biholomorphism to pull back the data and obtain a $\bar{\partial}$ equation on the punctured bidisc. Upon solving it via available integral representations on the punctured bidisc (or, on the bidisc after extension), use the biholomorphism again to push forward the solutions onto the Hartogs triangle. See also a recent joint work [26] with Yuan for some applications to a general class of quotient domains. Since the push-forward and pull-back operators via the biholomorphism introduce a certain (nontrivial) weight, the regularity problem of $\bar{\partial}$ on the Hartogs triangle is reduced to a weighted $\bar{\partial}$ regularity problem on the underlying bidisc.

Motivated by these works and the machinery, we study the weighted optimal $L^p$ estimates for the canonical solutions on general product domains, when the weights lie in a class of Muckenhoupt spaces $A_p^*$ (see Definition 2.1). Recall that the canonical solutions to $\bar{\partial}$ are the unique square-integrable solutions satisfying the non-homogeneous Cauchy-Riemann equation $\bar{\partial} u = \mathbf{f}$ that are orthogonal to $ker(\bar{\partial})$.

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**Theorem 1.1.** *Let $\Omega := D_1 \times \cdots \times D_n$ be a bounded product domain in $\mathbb{C}^n$, where each $D_j$ has $C^2$ boundary, $j = 1, \ldots, n$. Assume $\mu \in A^*_p$, $1 < p < \infty$. Then the canonical solution operator $T_c$ to $\bar\partial$ on $\Omega$ extends as a bounded operator from $L^p(\Omega, \mu)$ into itself. Namely, there exists a constant $C$ dependent only on $\Omega$, $p$ and the $A^*_p$ constant of $\mu$ such that for any $(0,1)$ forms $\mathbf{f} \in L^p(\Omega, \mu)$,*

$$\|T_c\mathbf{f}\|_{L^p(\Omega,\mu)} \le C\|\mathbf{f}\|_{L^p(\Omega,\mu)}.$$

The main ingredient of the proof is the weighted $L^p$ estimates for some Riesz-type integrals, as well as a pointwise estimate of the canonical solution kernel established by Dong, Pan and the author [12] (see also an observation by Yuan [25]). According to an example of Kerzman (Example 1), the theorem gives the optimal weighted $L^p$ regularity on product domains in terms of the canonical solutions. In particular, since $1 \in A^*_p$ for all $p > 1$, the canonical solutions provides optimal solutions to $\bar\partial$ in the (unweighted) $L^p$ category as well (Example 2), unlike another well-investigated solution operator along the line of Henkin which by a result of Chen-McNeal [3] is unbounded in $L^p, p < 2$.

As an application of Theorem 1.1, we obtain the following (unweighted) $L^p$ regularity for $\bar\partial$ on the Hartogs triangle if $p \ge 4$.

**Theorem 1.2.** *There exists a solution operator $T$ such that for any $\bar\partial$-closed $(0,1)$ form $\mathbf{f} \in L^p(\mathbb{H}), 4 \le p < \infty$, $T\mathbf{f} \in L^p(\mathbb{H})$ and solves $\bar\partial u = \mathbf{f}$ on $\mathbb{H}$. Moreover, there exists a constant $C$ dependent only on $p$ such that for any $\bar\partial$-closed $(0,1)$ form $\mathbf{f} \in L^p(\mathbb{H})$,*

$$\|T\mathbf{f}\|_{L^p(\mathbb{H})} \le C\|\mathbf{f}\|_{L^p(\mathbb{H})}.$$

Unfortunately, our method only works in the special range $p \ge 4$. In fact, as shown in Lemma 3.2, this range allows us to deal with a technical difficulty arisen from extending the $\bar\partial$-closed forms from the punctured bidisc to the whole bidisc, where Theorem 1.1 can be applied. Remark 3.3 further demonstrates that such a $\bar\partial$-closed extension to the bidisc fails in general if $p < 4$.

At the end of the paper, we construct an example (Example 3) to demonstrate the sharpness of Theorem 1.2, in the sense that for any $\epsilon > 0$, there exists an $L^p$ datum which does not admit any $L^{p+\epsilon}$ solutions on the Hartogs triangle. This non-improving phenomenon for the $\bar\partial$ regularity on the Hartogs triangle is essentially rooted from that for product domains. Theorem 1.1 and the general framework in [26] can certainly be applied to other special domains such as proper holomorphic map images of product domains, which are left to interested readers.

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# 2 Weighted $L^p$ estimates on product domains

## 2.1 Notations and Preliminaries

We first introduce our weight space under consideration. Denote by $dV_z$ the Lebesgue integral element along the $z$ directions, and by $|S|$ the Lebesgue measure of a subset $S$ in $\mathbb{C}^n$. For $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, let $\hat z_j = (z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_n) \in \mathbb{C}^{n-1}$, where the $j$-th component of $z$ is skipped.

**Definition 2.1.** *Given $1 < p < \infty$, a weight $\mu : \mathbb{C}^n \to [0, \infty)$ is said to be in $A_p^*$ if*

$$A_p^*(\mu) := \sup \left( \frac{1}{|D|} \int_D \mu(z) dV_{z_j} \right) \left( \frac{1}{|D|} \int_D \mu(z)^{\frac{1}{1-p}} dV_{z_j} \right)^{p-1} < \infty,$$

*where the supremum is taken over almost every $\hat{z}_j \in \mathbb{C}^{n-1}, j = 1, \ldots, n$, and all discs $D \subset \mathbb{C}$.*

We also recall the standard Muckenhoupt's class $A_p$, which consists of all weights $\mu : \mathbb{C}^n \to [0, \infty)$ satisfying

$$A_p(\mu) := \sup \left( \frac{1}{|B|} \int_B \mu(z) dV_z \right) \left( \frac{1}{|B|} \int_B \mu(z)^{\frac{1}{1-p}} dV_z \right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{C}^n$. See [24, Chapter V] for an introduction of the class. It is not hard to see that $A_q \subset A_p$ if $1 < q < p$. Moreover, $A_p$ spaces satisfy an open-end property: if $\mu \in A_p$ for some $1 < p < \infty$, then $\mu \in A_{\tilde{p}}$ for some $\tilde{p} < p$.

Clearly, $A_p^* = A_p$ when $n = 1$. In general, $\mu \in A_p^*$ if and only if the $\delta$-dilation $\mu_\delta(z) := \mu(\delta_1 z_1, \ldots, \delta_n z_n) \in A_p$ with a uniform $A_p$ constant for all $\delta = (\delta_1, \ldots, \delta_n) \in (\mathbb{R}^+)^n$ ( [14, pp. 454]). This in particular implies $A_p^* \subset A_p$. As will be seen in the rest of the paper, the setting of $A_p^*$ weights allows us to apply the slicing property of product domains rather effectively.

Given a non-negative weight $\mu$ and $1 < p < \infty$, the weighted function space $L^p(\Omega, \mu)$ is the set of functions $f$ on $\Omega$ such that its weighted $L^p$ norm

$$\|f\|_{L^p(\Omega,\mu)} := \left( \int_\Omega |f(z)|^p \mu(z) dV_z \right)^{\frac{1}{p}} < \infty.$$

When $\mu \equiv 1$, it is reduced to the (unweighted) $L^p(\Omega)$ space. From now on, we shall say $a \lesssim b$ if $a \le Cb$ for a constant $C > 0$ dependent only possibly on $\Omega, p$ and the $A_p^*$ (or $A_p$) constant of $\mu$. We say $a \approx b$ if and only if $a \lesssim b$ and $b \lesssim a$ at the same time.

## 2.2 Weighted $L^p$ estimates for Riesz-type integrals

We focus on a bounded product domain $\Omega = D_1 \times \cdots \times D_n \subset \mathbb{C}^n$, where $D_j$ has $C^2$ boundary. Fixing a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $0 < \alpha_j < 2, j = 1, \ldots, n$ and $1 < p < \infty$, define the following Riesz-type integral of $f \in L^p(\Omega)$

$$R_\alpha f(z) := \int_\Omega \frac{f(\zeta)}{\prod_{j=1}^n |\zeta_j - z_j|^{\alpha_j}} dV_\zeta, \quad z \in \Omega.$$

$R_\alpha$ is a bounded operator from $L^p(\Omega)$ into itself by Riesz integral theory. We shall show a weighted version of this result in $L^p(\Omega, \mu)$ under the assumption that $\mu \in A_p^*$.

Firstly we consider the $n = 1$ case below, whose proof is a slight modification of a standard trick for fractional integrals.

**Proposition 2.2.** *Let $D$ be a bounded domain in $\mathbb{C}$. Assume $0 < \alpha < 2$ and $\mu \in A_p, 1 < p < \infty$. Then $R_\alpha$ is a bounded operator from $L^p(D, \mu)$ into itself. Namely,*

$$\|R_\alpha f\|_{L^p(D,\mu)} \lesssim \|f\|_{L^p(D,\mu)}. \tag{2.1}$$

*Proof.* Without loss of generality, assume $f \geq 0$ and $f$ trivially extends to $\mathbb{C}$ by letting it be zero outside $D$. Denote by $Mf$ the Hardy-Littlewood maximal function of $f$. For each $z \in \mathbb{C}$ with $\delta > 0$ to be chosen later,

$$\begin{aligned}\int_{|\zeta - z|<\delta, \zeta\in D} \frac{f(\zeta)}{|\zeta - z|^\alpha} dV_\zeta =& \sum_{k=1}^{\infty} \int_{\frac{\delta}{2^k}<|\zeta-z|<\frac{\delta}{2^{k-1}}} \frac{f(\zeta)}{|\zeta - z|^\alpha} dV_\zeta \leq \sum_{k=1}^{\infty} \frac{2^{k\alpha}}{\delta^\alpha} \int_{|\zeta-z|<\frac{\delta}{2^{k-1}}} f(\zeta) dV_\zeta \\ \lesssim& \sum_{k=1}^{\infty} 2^{-k(2-\alpha)} \delta^{2-\alpha} Mf(z) \approx \delta^{2-\alpha} Mf(z).\end{aligned} \tag{2.2}$$

Due to the open-end property of $A_p$, we can pick some $\tilde{p} \in \left(\frac{(2-\alpha)p}{2}, p\right)$ such that $\mu \in A_{\tilde{p}}$. Then

$$\begin{aligned}\int_{|\zeta-z|>\delta,\zeta\in D} \frac{f(\zeta)}{|\zeta - z|^\alpha} dV_\zeta \leq& \left(\int_{|\zeta-z|>\delta,\zeta\in D} |f(\zeta)|^p \mu(\zeta) dV_\zeta\right)^{\frac{1}{p}} \left(\int_{|\zeta-z|>\delta,\zeta\in D} |\zeta - z|^{\frac{\alpha p}{1-p}} \mu(\zeta)^{\frac{1}{1-p}} dV_\zeta\right)^{\frac{p-1}{p}} \\ \lesssim& \|f\|_{L^p(D,\mu)} \left(\int_{|\zeta-z|>\delta,\zeta\in D} |\zeta - z|^{\frac{\alpha p}{\tilde{p}-p}} dV_\zeta\right)^{\frac{p-\tilde{p}}{p}} \left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}} dV_\zeta\right)^{\frac{\tilde{p}-1}{p}} \\ \lesssim& \delta^{2-\alpha-\frac{2\tilde{p}}{p}} \|f\|_{L^p(D,\mu)} \left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}} dV_\zeta\right)^{\frac{\tilde{p}-1}{p}}.\end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3), we have

$$R_\alpha f(z) \lesssim \delta^{2-\alpha} Mf(z) + \delta^{2-\alpha-\frac{2\tilde{p}}{p}} \|f\|_{L^p(D,\mu)} \left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}} dV_\zeta\right)^{\frac{\tilde{p}-1}{p}}.$$

Choosing $\delta = \left(\frac{\|f\|_{L^p(D,\mu)} \left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}} dV_\zeta\right)^{\frac{\tilde{p}-1}{p}}}{Mf}\right)^{\frac{p}{2\tilde{p}}}$ in the above, we further get

$$R_\alpha f(z) \lesssim \|f\|_{L^p(D,\mu)}^{\frac{(2-\alpha)p}{2\tilde{p}}} \left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}} dV_\zeta\right)^{\frac{(2-\alpha)(\tilde{p}-1)}{2\tilde{p}}} Mf(z)^{\frac{2\tilde{p}-(2-\alpha)p}{2\tilde{p}}}.$$

Note that $2\tilde{p} - (2-\alpha)p > 0$ by the choice of $\tilde{p}$. Making use of the boundedness of the maximal function operator in $L^p(\mathbb{C}, \mu)$,

$$\begin{aligned}\|R_\alpha f\|_{L^{\frac{2p\tilde{p}}{2\tilde{p}-(2-\alpha)p}}(D,\mu)} \lesssim& \|f\|_{L^p(D,\mu)}^{\frac{(2-\alpha)p}{2\tilde{p}}} \left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}} dV_\zeta\right)^{\frac{(2-\alpha)(\tilde{p}-1)}{2\tilde{p}}} \|Mf\|_{L^p(\mathbb{C},\mu)}^{\frac{2\tilde{p}-(2-\alpha)p}{2\tilde{p}}} \\ \lesssim& \|f\|_{L^p(D,\mu)} \left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}} dV_\zeta\right)^{\frac{(2-\alpha)(\tilde{p}-1)}{2\tilde{p}}}.\end{aligned}$$

Lastly, since $\mu \in A_{\tilde{p}}$, we have

$$\begin{aligned}\|R_\alpha f\|_{L^p(D,\mu)}^p \leq& \|R_\alpha f\|_{L^{\frac{2p\tilde{p}}{2\tilde{p}-(2-\alpha)p}}(D,\mu)}^p \left(\int_D \mu(\zeta) dV_\zeta\right)^{\frac{(2-\alpha)p}{2\tilde{p}}} \\ \lesssim& \|f\|_{L^p(D,\mu)}^p \left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}} dV_\zeta\right)^{\frac{(2-\alpha)(\tilde{p}-1)p}{2\tilde{p}}} \left(\int_D \mu(\zeta) dV_\zeta\right)^{\frac{(2-\alpha)p}{2\tilde{p}}} \lesssim \|f\|_{L^p(D,\mu)}^p.\end{aligned}$$

∎

It is worth pointing out that $\mu \in A_p$ can not be dropped in Proposition 2.2. Indeed, letting $\triangle$ be the unit disc on $\mathbb{C}$, a function $f \in L^2(\triangle, |z|^2)$ was constructed in [26] such that $R_1 f \notin L^2(\triangle, |z|^2)$. Note that $|z|^2 \notin A_2$.

**Theorem 2.3.** *Let $\Omega$ be a bounded product domain in $\mathbb{C}^n, n \geq 1$. Assume $\alpha = (\alpha_1, \dots, \alpha_n)$ with $0 < \alpha_j < 2, j = 1, \dots, n$, and $\mu \in A_p^*, 1 < p < \infty$. Then $R_\alpha$ is a bounded operator from $L^p(\Omega, \mu)$ into itself. Namely,*

$$\|R_\alpha f\|_{L^p(D,\mu)} \lesssim \|f\|_{L^p(D,\mu)}.$$

*Proof.* $n = 1$ case is due to Proposition 2.2. We shall prove $n \geq 2$ cases by induction. Denote by $\alpha'$ the first $n-1$ components of $\alpha$. Similarly define $z', \zeta'$ and $\Omega'$. Write $\|R_\alpha f\|^p_{L^p(\Omega,\mu)}$ as

$$\int_{D_1 \times \cdots D_{n-1}} \int_{D_n} \left| \int_{D_n} \frac{1}{|\zeta_n - z_n|^{\alpha_n}} \left( \int_{D_1 \times \cdots D_{n-1}} \frac{f(\zeta)}{|\zeta' - z'|^{\alpha'}} dV_{\zeta'} \right) dV_{\zeta_n} \right|^p \mu(z', z_n) dV_{z_n} dV_{z'}.$$

For almost everywhere fixed $z' \in \Omega'$, note that $\mu(z', \cdot) \in A_p$ by definition. Applying (2.1) to $\int_{D_1 \times \cdots D_{n-1}} \frac{f(\zeta)}{|\zeta' - z'|^{\alpha'}} dV_{\zeta'}$ on $D_n$, we have

$$\begin{aligned}
&\int_{D_n} \left| \int_{D_n} \frac{1}{|\zeta_n - z_n|^{\alpha_n}} \left( \int_{D_1 \times \cdots D_{n-1}} \frac{f(\zeta)}{|\zeta' - z'|^{\alpha'}} dV_{\zeta'} \right) dV_{\zeta_n} \right|^p \mu(z', z_n) dV_{z_n} \\
\lesssim &\int_{D_n} \left| \int_{D_1 \times \cdots D_{n-1}} \frac{f(\zeta', z_n)}{|\zeta' - z'|^{\alpha'}} dV_{\zeta'} \right|^p \mu(z) dV_z.
\end{aligned}$$

Thus

$$\begin{aligned}
\|R_\alpha f\|^p_{L^p(\Omega,\mu)} &\lesssim \int_{D_1 \times \cdots D_{n-1} \times D_n} \left| \int_{D_1 \times \cdots D_{n-1}} \frac{f(\zeta', z_n)}{|\zeta' - z'|^{\alpha'}} dV_{\zeta'} \right|^p \mu(z', z_n) dV_z \\
&= \int_{D_n} \left( \int_{D_1 \times \cdots \times D_{n-1}} \left| \int_{D_1 \times \cdots D_{n-1}} \frac{f(\zeta', z_n)}{|\zeta' - z'|^{\alpha'}} dV_{\zeta'} \right|^p \mu(z', z_n) dV_{z'} \right) dV_{z_n} \\
&\lesssim \cdots \\
&\lesssim \int_{D_1 \times \cdots D_{n-1} \times D_n} |f(z)|^p \mu(z) dV_z = \|f\|^p_{L^p(\Omega,\mu)},
\end{aligned}$$

where in the omitted part, we have employed a standard induction to the term inside the parenthesis for almost everywhere fixed $z_n \in D_n$.

□

## 2.3 Proof of Theorem 1.1

The (unweighted) $\bar\partial$ theory on product domains has been thoroughly understood through the works, for instance, [3–5, 13, 15, 18, 21, 22, 27] and the references therein. In particular, it was proved in [12] that there exists a family of functions $e_w$ on $\Omega$ such that

$$T_c \mathbf{f}(z) = \sum_{s=1}^n \sum_{1 \leq i_1 < \cdots < i_s \leq n} \sum_{k=1}^s \int_{D_{i_1} \times \cdots \times D_{i_s}} f_{i_k}(\zeta', z'') \frac{\partial^{s-1} e_w^{k, i_1, \dots, i_s}(\zeta)}{\partial \bar\zeta_{i_1} \cdots \partial \bar\zeta_{i_{k-1}} \partial \bar\zeta_{i_{k+1}} \cdots \partial \bar\zeta_{i_s}} \tag{2.4}$$

is the canonical solution to $\bar\partial u = \mathbf{f} (= \sum_{j=1}^n f_j d\bar z_j)$ on $\Omega$. In fact, after the preprint [12] was submitted to a journal, we observed no boundary integrals should be involved in the solution representation originally constructed in [12, Section 5], due to the vanishing property of the kernels. This leads to the above simplified expression of the canonical solution. Moreover, formula (5.2) in [12] with $s = m + 1$ states that there exists some constant $0 < \alpha_j < 2$ such that $e_w$ satisfies

$$\left| \frac{\partial^{s-1} e_w^{k,i_1,\ldots,i_s}}{\partial \bar{\zeta}_{i_1} \cdots \partial \bar{\zeta}_{i_{k-1}} \partial \bar{\zeta}_{i_{k+1}} \cdots \partial \bar{\zeta}_{i_s}} \right| \lesssim \frac{1}{\prod_{r=1}^{s} |\zeta_{i_r} - w_{i_r}|^{\alpha_j}} \tag{2.5}$$

on $\Omega$. The unweighted $L^p$ theory for the canonical solution operator on product domains follows immediately from (2.5) by Young's inequality. Recently the same observation on the vanishing of the boundary integrals was made in [25] as well (we contacted the author right away and provided with our email record in 2021 on the observation and the completed proof to the $L^p$ estimates).

It turns out our weighted $L^p$ estimate Theorem 1.1 is essentially a consequence of (2.5) and Theorem 2.3.

*Proof of Theorem 1.1:* We estimate the term in (2.4) with $(i_1, \ldots, i_s) = (1, \ldots, s)$ and $i_k = s$. Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ satisfying (2.5). Denote $z' = (z_1, \ldots, z_s)$ and $z'' = (z_{s+1}, \ldots, z_n)$, and similarly define $D'$ and $D''$. By Fubini theorem and (2.5),

$$\begin{aligned} &\left\| \int_{D_{i_1} \times \cdots \times D_{i_s}} f_{i_k}(\zeta', z'') \frac{\partial^{s-1} e_w^{k,i_1,\ldots,i_s}(\zeta)}{\partial \bar{\zeta}_{i_1} \cdots \partial \bar{\zeta}_{i_{k-1}} \partial \bar{\zeta}_{i_{k+1}} \cdots \partial \bar{\zeta}_{i_s}} \right\|^p_{L^p(\Omega,\mu)} \\ &\lesssim \int_{D''} \int_{D'} |R_\alpha f_s(z', z'')|^p \mu(z', z'') dV_{z'} dV_{z''} \\ &\lesssim \int_{D''} \int_{D'} |f_s(z', z'')|^p \mu(z', z'') dV_{z'} dV_{z''} \lesssim \|\mathbf{f}\|^p_{L^p(\Omega,\mu)}. \end{aligned}$$

Here we used Theorem 2.3 on $D'$ in the second inequality for almost every fixed $z'' \in D''$. The rest of the terms in the sum in $T_c\mathbf{f}$ are proved similarly. The proof is complete.

□

The following example along the line of Kerzman [19] demonstrates that given a weighted $L^p$ data, the $\bar{\partial}$ problem on product domains in general does not admit weighted $L^{p+\epsilon}, \epsilon > 0$ solutions. Thus Theorem 1.1 gives the optimal weighted $L^p$ regularity on product domains in terms of the canonical solutions.

**Example 1.** *For each $1 < p < \infty, \epsilon > 0$ and any $r \in \left(\frac{2}{1+\epsilon}, 2\right)$, consider $\mathbf{f} = (z_2 - 1)^{-r} d\bar{z}_1$ on $\triangle^2$, $\frac{1}{2}\pi < \arg(z_2 - 1) < \frac{3}{2}\pi$ and $\mu = |z_2 - 1|^{r(p-1)}$. Then $\mu \in A_p^*$, $\mathbf{f} \in L^p(\triangle^2, \mu)$ and is $\bar{\partial}$-closed on $\triangle^2$. However, there does not exist a solution $u \in L^{p+\epsilon}(\triangle^2, \mu)$ to $\bar{\partial} u = \mathbf{f}$ on $\triangle^2$.*

*Proof.* Clearly $\mathbf{f}$ is $\bar{\partial}$-closed on $\triangle^2$. Since $r < 2$, we can also verify directly that $\mu \in A_p^*$ and $\mathbf{f} \in L^p(\triangle^2, \mu)$. Suppose there exists some $u \in L^{p+\epsilon}(\triangle^2, \mu)$ satisfying $\bar{\partial} u = \mathbf{f}$ on $\triangle^2$. Noting that $L^p(\triangle^2, \mu) \subset L^1(\triangle^2)$, an application of Weyl's lemma gives the existence of some holomorphic function $h$ on $\triangle^2$, such that $u = (z_2 - 1)^{-r} \bar{z}_1 + h \in L^{p+\epsilon}(\triangle^2, \mu)$.

For almost everywhere $(r, z_2) \in U := (0, 1) \times \triangle \subset \mathbb{R}^3$, consider

$$v(r, z_2) := \int_{|z_1|=r} u(z_1, z_2) dz_1.$$

By Hölder inequality, Fubini theorem and the fact that $p > 1$,

$$\begin{aligned}
\|v\|^{p+\epsilon}_{L^{p+\epsilon}(U,\mu)} &= \int_U \left| \int_{|z_1|=r} u(z_1, z_2) dz_1 \right|^{p+\epsilon} \mu(z_2) dV_{z_2,r} \\
&\le \int_{|z_2|<1} \int_{r<1} \left| r \int_0^{2\pi} |u(re^{i\theta}, z_2)| d\theta \right|^{p+\epsilon} dr \mu(z_2) dV_{z_2} \\
&\lesssim \int_{|z_2|<1} \int_{r<1} \int_0^{2\pi} |u(re^{i\theta}, z_2)|^{p+\epsilon} d\theta r dr \mu(z_2) dV_{z_2} \\
&= \int_{|z_2|<1, |z_1|<1} |u(z)|^{p+\epsilon} \mu(z_2) dV_z = \|u\|^{p+\epsilon}_{L^{p+\epsilon}(\triangle^2,\mu)} < \infty.
\end{aligned}$$

Thus $v \in L^{p+\epsilon}(U, \mu)$. On the other hand, by Cauchy's theorem, for almost everywhere $(r, z_2) \in U$,

$$v(r, z_2) = \int_{|z_1|=r} (z_2 - 1)^{-r} \bar{z}_1 dz_1 = (z_2 - 1)^{-r} \int_{|z_1|=r} \frac{r^2}{z_1} dz_1 = 2\pi r^2 i (z_2 - 1)^{-r},$$

which is not in $L^{p+\epsilon}(U, \mu)$ by the choice of $r > \frac{2}{1+\epsilon}$. This is a contradiction! The example is thus verified.

$\square$

In the case when $\mu \equiv 1$, one can verify similarly the following example to conclude that the $\bar{\partial}$ problem on product domains does not improve regularity in $L^p$ space, either. Thus the canonical solutions provide optimal $L^p$ solutions on product domains. Example 2 can easily be tailored to show that the $\bar{\partial}$ operator does not improve regularity in unweighted $W^{k,p}$ spaces as well. This phenomenon is consistent with that in Hölder spaces ( [22, 27]).

**Example 2.** *For each $1 < p < \infty$, let $\mathbf{f} = (z_2 - 1)^{-\frac{2}{p}} d\bar{z}_1$ on $\triangle^2$, $\frac{1}{2}\pi < \arg(z_2 - 1) < \frac{3}{2}\pi$. Then $\mathbf{f} \in L^{\tilde{p}}(\triangle^2)$ for all $1 < \tilde{p} < p$ and is $\bar{\partial}$-closed on $\triangle^2$. However, there does not exist a solution $u \in L^p(\triangle^2)$ to $\bar{\partial} u = \mathbf{f}$ on $\triangle^2$.*

*Proof.* The proof is similar to that of Example 1 with $\mu \equiv 1$ instead, so we only sketch it here. Clearly $\mathbf{f} \in L^{\tilde{p}}(\triangle^2)$ for all $1 < \tilde{p} < p$. Suppose there exists some $u \in L^p(\triangle^2)$ satisfying $\bar{\partial} u = \mathbf{f}$ on $\triangle^2$. Then for some holomorphic function $h$ on $\triangle^2$ we have $u = (z_2 - 1)^{-\frac{2}{p}} \bar{z}_1 + h \in L^p(\triangle^2)$. Consider $v(r, z_2) := \int_{|z_1|=r} u(z_1, z_2) dz_1$ for almost everywhere $(r, z_2) \in U := (0, 1) \times \triangle \subset \mathbb{R}^3$. Then $v \in L^p(U)$. However, by Cauchy's theorem, $v(r, z_2) = 2\pi r^2 i (z_2 - 1)^{-\frac{2}{p}}$ almost everywhere in $U$, which contradicts with the fact that $v \in L^p(U)$. $\square$

# 3 $L^p$ estimates on the Hartogs triangle

Denote by $\triangle^* := \triangle \setminus \{0\}$, the punctured disc on $\mathbb{C}$. Then $\psi : \triangle \times \triangle^* \to \mathbb{H}$ given by

$$(w_1, w_2) \mapsto (z_1, z_2) = \psi(w) = (w_1 w_2, w_2)$$

is a biholomorphism, with its inverse $\phi : \mathbb{H} \to \triangle \times \triangle^*$ given by

$$(z_1, z_2) \mapsto (w_1, w_2) = \phi(z) = \left( \frac{z_1}{z_2}, z_2 \right).$$

This biholomorphism allows us to pull back and push forward between $\mathbb{H}$ and $\triangle \times \triangle^*$. Due to the explicit and simple form of $\psi$, we shall be self-contained and chase concretely how the singularity affects the $\bar{\partial}$-closedness, the pull-back of the data and push-forward of (solution) functions. The general framework can be founded in [26]. In fact, for any $\mathbf{f} = \sum_{j=1}^{2} f_j(z)d\bar{z}_j \in L^p(\mathbb{H})$, making use of change of variables formula we have the pull-back

$$\psi^*\mathbf{f} = f_1 \circ \psi \cdot \bar{w}_2 d\bar{w}_1 + (f_1 \circ \psi \cdot \bar{w}_1 + f_2 \circ \psi)\, d\bar{w}_2 \in L^p(\triangle^2, |w_2|^2) \tag{3.1}$$

with

$$\|\psi^*\mathbf{f}\|^p_{L^p(\triangle^2,|w_2|^2)} \lesssim \sum_{j=1}^{2} \int_{\triangle^2} |f_j \circ \psi(w)|^p |w_2|^2 dV_w = \sum_{j=1}^{2} \int_{\mathbb{H}} |f_j(z)|^p dV_z = \|\mathbf{f}\|^p_{L^p(\mathbb{H})}. \tag{3.2}$$

The inverse $\phi$ is used to push forward any function $\tilde{u} \in L^p(\triangle^2, |w_2|^2)$ to be in $L^p(\mathbb{H})$ with

$$\|\tilde{u} \circ \phi\|^p_{L^p(\mathbb{H})} = \int_{\mathbb{H}} |\tilde{u} \circ \phi(z)|^p dV_z = \int_{\triangle^2} |\tilde{u}(w)|^p |w_2|^2 dV_w = \|\tilde{u}\|^p_{L^p(\triangle^2,|w_2|^2)}. \tag{3.3}$$

Note that $|w_2|^2 \in A_p^*$, $p > 2$. In order to apply the weighted $L^p$ estimates in Theorem 1.1 (where each portion $D_j$ are assumed to have $C^2$ boundary), we need to justify that the pull-back data is $\bar{\partial}$-closed on $\triangle^2$.

**Proposition 3.1.** *Let $\mathbf{f} \in L^p(\mathbb{H})$ be a $\bar{\partial}$-closed $(0,1)$ form on $\mathbb{H}$. If $4 \le p < \infty$, then $\psi^*\mathbf{f}$ lies in $L^p(\triangle^2, |w_2|^2)$ and is a $\bar{\partial}$-closed $(0,1)$ form on $\triangle^2$.*

The proof of the proposition boils down to showing the following Harvey-Polking type extension (or resolution) of $\bar{\partial}$-closed $L^p(\triangle^2, |w_2|^2)$ forms from $\triangle \times \triangle^*$ to $\triangle^2$, $p \ge 4$. We remark that if the forms lie in $W^{1,p}(\triangle, |w_2|^2)$ in addition, then this range of $p$ can be relaxed to $p > 2$. See, for instance, [26, Proposition 5.10].

**Lemma 3.2.** *Suppose a $(0,1)$ form $\mathbf{h} \in L^p(\triangle^2, |w_2|^2)$ is $\bar{\partial}$-closed on $\triangle \times \triangle^*$. If $4 \le p < \infty$, then $\mathbf{h}$ is $\bar{\partial}$-closed on $\triangle^2$.*

*Proof.* Write $\mathbf{h}(w) = h_1(w)d\bar{w}_1 + h_2(w)d\bar{w}_2$. Let $\eta = \eta(w)dw_1 \wedge dw_2$ be a smooth $(2,0)$-form in $\triangle^2$ with compact support. We shall show

$$-\left\langle \bar{\partial} h, \eta \right\rangle_{\triangle^2} := \int_{\triangle^2} h_1(w)\frac{\partial \eta(w)}{\partial \bar{w}_2} - h_2(w)\frac{\partial \eta(w)}{\partial \bar{w}_1} dV_w = 0.$$

Denote by $\triangle_r$ the disc centered at 0 with radius $r > 0$. Choose a cut-off function $\chi \in C_c^\infty(\triangle)$ such that $\chi = 1$ on $\triangle_{\frac{1}{2}}$ and $|\nabla \chi| < \frac{1}{3}$ on $\triangle$. Letting $\chi_k(w_2) = \chi(kw_2)$ on $\triangle$, then $\chi_k$ is supported on $\triangle_{\frac{1}{k}}$ and $|\nabla \chi_k| \lesssim k$. Consequently,

$$\begin{aligned}
&\left| \int_{\triangle^2} h_2(w)\frac{\partial \eta(w)}{\partial \bar{w}_1} dV_w - \int_{\triangle^2} h_1(w)\frac{\partial\left((1-\chi_k(w_2))\eta(w)\right)}{\partial \bar{w}_2} dV_w \right| \\
&\le \int_{\triangle^2} \left| h_2(w)\frac{\partial\left(\chi_k(w_2)\eta(w)\right)}{\partial \bar{w}_1} \right| dV_w \\
&\quad + \left| \int_{\triangle^2} h_2(w)\frac{\partial\left((1-\chi_k(w_2))\eta(w)\right)}{\partial \bar{w}_1} - h_1(w)\frac{\partial\left((1-\chi_k(w_2))\eta(w)\right)}{\partial \bar{w}_2} dV_w \right|.
\end{aligned}$$

Since $(1-\chi_k(w_2))\eta(w)$ has compact support on $\triangle\times\triangle^*$, the last line in the above is zero by the $\bar\partial$-closedness of $\mathbf{h}$ on $\triangle\times\triangle^*$. For the first term, since $p>2$,

$$\begin{aligned}\int_{\triangle^2}\left|h_2(w)\frac{\partial\left(\chi_k(w_2)\eta(w)\right)}{\partial\bar w_1}\right|dV_w\lesssim&\int_{\triangle\times\triangle_{\frac1k}}|h_2(w)||w_2|^{\frac2p}|w_2|^{-\frac2p}|dV_w\\ \lesssim&\|h_2\|_{L^p(\triangle^2,|w_2|^2)}\left(\int_{\triangle_{\frac1k}}|w_2|^{-\frac{2}{p-1}}dV_{w_2}\right)^{\frac{p-1}{p}}\to0\end{aligned}\tag{3.4}$$

as $k\to\infty$. Hence as $k\to\infty$,

$$\left|\int_{\triangle^2}h_2(w)\frac{\partial\eta(w)}{\partial\bar w_1}dV_w-\int_{\triangle^2}h_1(w)\frac{\partial\left((1-\chi_k(w_2))\eta(w)\right)}{\partial\bar w_2}dV_w\right|\to0.\tag{3.5}$$

On the other hand,

$$\left|\int_{\triangle^2}h_1(w)\frac{\partial\left(\chi_k(w_2)\eta(w)\right)}{\partial\bar w_2}dV_w\right|\lesssim\int_{\triangle\times\triangle_{\frac1k}}\left|h_1(w)\frac{\partial\left(\chi_k(w_2)\right)}{\partial\bar w_2}\right|dV_w+\int_{\triangle\times\triangle_{\frac1k}}|h_1(w)|dV_w.$$

By the same reasoning as in (3.4), the last term goes to 0 as $k\to\infty$. For the first term in the right hand side of the last line, making use of the fact that $\left|\frac{\partial(\chi_k(w_2))}{\partial\bar w_2}\right|\lesssim k$ on $\triangle_{\frac1k}$, we get

$$\begin{aligned}\left|\int_{\triangle\times\triangle_{\frac1k}}h_1(w)\frac{\partial\left(\chi_k(w_2)\right)}{\partial\bar w_2}dV_w\right|\lesssim&k\int_{\triangle\times\triangle_{\frac1k}}|h_1(w)||w_2|^{\frac2p}|w_2|^{-\frac2p}|dV_w\\ \lesssim&k\|h_1\|_{L^p(\triangle\times\triangle_{\frac1k},|w_2|^2)}\left(\int_{\triangle_{\frac1k}}|w_2|^{-\frac{2}{p-1}}dV_{w_2}\right)^{\frac{p-1}{p}}\\ \lesssim&k^{\frac4p-1}\|h_1\|_{L^p(\triangle\times\triangle_{\frac1k},|w_2|^2)}.\end{aligned}$$

Since $p\ge4$, as $k\to\infty$ the last term goes to zero, and thus

$$\left|\int_{\triangle^2}h_1(w)\frac{\partial\left(\chi_k(w_2)\eta(w)\right)}{\partial\bar w_2}dV_w\right|\to0.\tag{3.6}$$

The proof of the proposition is complete by combining (3.5) and (3.6).

$\square$

**Remark 3.3.** *The $p\ge4$ assumption in Lemma 3.2 can not be relaxed. For instance, $\mathbf{h}(w)=\frac{1}{w_2}d\bar w_1$ is $\bar\partial$-closed on $\triangle\times\triangle^*$ and lies in $L^p(\triangle^2,|w_2|^2)$ for all $1<p<4$. However, $\mathbf{h}(w)$ is not $\bar\partial$-closed on $\triangle^2$. In fact, given a smooth $(2,0)$-form $\eta=\eta(w)dw_1\wedge dw_2$ in $\triangle^2$ with compact support, by Stokes' theorem and Residue theorem,*

$$\begin{aligned}-\langle\bar\partial\mathbf{h},\eta\rangle_{\triangle^2}=&\int_{\triangle^2}\frac{1}{w_2}\frac{\partial\eta(w)}{\partial\bar w_2}dV_w=\int_{\triangle}\lim_{\epsilon\to0}\int_{\triangle\setminus\triangle_\epsilon}\frac{1}{w_2}\frac{\partial\eta(w)}{\partial\bar w_2}dV_{w_2}dV_{w_1}\\ =&\frac12\int_{\triangle}\lim_{\epsilon\to0}\int_{\triangle\setminus\triangle_\epsilon}\bar\partial\left(\frac{\eta(w)}{w_2}dw_2\right)dV_{w_1}=-\frac12\int_{\triangle}\lim_{\epsilon\to0}\int_{\partial\triangle_\epsilon}\frac{\eta(w)}{w_2}dw_2dV_{w_1}\\ =&-\pi i\int_{\triangle}\eta(w_1,0)dV_{w_1},\end{aligned}$$

*which is not zero in general.*

*Proof of Proposition 3.1:* In view of Lemma 3.2 and (3.2), we only need to verify that $\psi^*\mathbf{f}$ is $\bar\partial$-closed on $\triangle\times\triangle^*$. Indeed, for any smooth function $\chi$ with compact support on $\mathbb{H}$, by $\bar\partial$-closedness of $\mathbf{f}$ on $\mathbb{H}$, we have

$$\int_{\mathbb{H}} f_1(z)\frac{\partial\chi}{\partial\bar z_2}(z) - f_2(z)\frac{\partial\chi}{\partial\bar z_1}(z)dV_z = 0. \tag{3.7}$$

For any $(2,0)$ form $\eta$ with compact support on $\triangle\times\triangle^*$, by (3.1), chain rule and change of variables formula,

$$\begin{aligned}
-\langle\bar\partial\psi^*\mathbf{f},\eta\rangle_{\triangle^2} &= \int_{\triangle^2} f_1\circ\psi(w)\bar w_2\frac{\partial\eta(w)}{\partial\bar w_2} - \left(f_1\circ\psi(w)\bar w_1 + f_2\circ\psi(w)\right)\frac{\partial\eta(w)}{\partial\bar w_1}dV_w\\
&= \int_{\mathbb{H}}\left(f_1(z)\bar z_2\left(\frac{\partial\eta\circ\phi(z)}{\partial\bar z_1}\frac{\bar z_1}{\bar z_2} + \frac{\partial\eta\circ\phi(z)}{\partial\bar z_2}\right) - \left(f_1(z)\frac{\bar z_1}{\bar z_2} + f_2(z)\right)\frac{\partial\eta\circ\phi(z)}{\partial\bar z_1}\bar z_2\right)\frac{1}{|z_2|^2}dV_z\\
&= \int_{\mathbb{H}} f_1(z)\frac{\partial}{\partial\bar z_2}\left(\frac{\eta\circ\phi(z)}{z_2}\right) - f_2(z)\frac{\partial}{\partial\bar z_1}\left(\frac{\eta\circ\phi(z)}{z_2}\right)dV_z = 0,
\end{aligned}$$

where the last line is due to (3.7) and the fact that $\frac{\eta\circ\phi(z)}{z_2}$ is smooth with compact support on $\mathbb{H}$. $\square$

*Proof of Theorem 1.2:* Given a $\bar\partial$-closed $(0,1)$ form $\mathbf{f}\in L^p(\mathbb{H})$, $\psi^*\mathbf{f}\in L^p(\triangle^2,|w_2|^2)$ and is $\bar\partial$-closed on $\triangle^2$ by Proposition 3.1. As $|w_2|^2\in A_p^*$, an application of Theorem 1.1 gives a solution $\tilde u\in L^p(\triangle^2,|w_2|^2)$ to $\bar\partial\tilde u = \psi^*\mathbf{f}$ on $\triangle^2$. Namely, for any smooth and compactly supported $(2,1)$ form $\eta = (\eta_1 d\bar w_1 + \eta_2 d\bar w_2)\wedge dw_1\wedge dw_2$ on $\triangle^2$,

$$\begin{aligned}
&\int_{\triangle^2}\tilde u(w)\left(\frac{\partial\eta_1(w)}{\partial\bar w_2} - \frac{\partial\eta_2(w)}{\partial\bar w_1}\right)dV_w = -\langle\bar\partial\tilde u,\eta\rangle_{\triangle^2} = -\langle\psi^*\mathbf{f},\eta\rangle_{\triangle^2}\\
&= \int_{\triangle^2} f_1\circ\psi(w)\bar w_2\eta_2(w) - \left(f_1\circ\psi(w)\bar w_1 + f_2\circ\psi(w)\right)\eta_1(w)dV_w.
\end{aligned}\tag{3.8}$$

We now verify that $u := \tilde u\circ\phi$ solves $\bar\partial u = \mathbf{f}$ on $\mathbb{H}$. Indeed, for any smooth $(2,1)$ form $\chi = (\chi_1 d\bar z_1 + \chi_2 d\bar z_2)\wedge dz_1\wedge dz_2$ with compact support on $\mathbb{H}$, by chain rule and change of variables,

$$\begin{aligned}
-\langle\bar\partial u,\chi\rangle_{\mathbb{H}} &= \int_{\mathbb{H}}\tilde u\circ\phi(z)\left(\frac{\partial\chi_1(z)}{\partial\bar z_2} - \frac{\partial\chi_2(z)}{\partial\bar z_1}\right)dV_z\\
&= \int_{\triangle^2}\tilde u(w)\left(-\frac{\partial\chi_1\circ\psi(w)}{\partial\bar w_1}\frac{\bar w_1}{\bar w_2} + \frac{\partial\chi_1\circ\psi(w)}{\partial\bar w_2} - \frac{\partial\chi_2\circ\psi(w)}{\partial\bar w_1}\frac{1}{\bar w_2}\right)|w_2|^2dV_w\\
&= \int_{\triangle^2}\tilde u(w)\left(-\frac{\partial\chi_1\circ\psi(w)}{\partial\bar w_1}\bar w_1 w_2 + \frac{\partial\chi_1\circ\psi(w)}{\partial\bar w_2}|w_2|^2 - \frac{\partial\chi_2\circ\psi(w)}{\partial\bar w_1}w_2\right)dV_w\\
&= \int_{\triangle^2}\tilde u(w)\left(\frac{\partial}{\partial\bar w_2}\left(\chi_1\circ\psi(w)|w_2|^2\right) - \frac{\partial}{\partial\bar w_1}\left(\chi_2\circ\psi(w)w_2 + \chi_1\circ\psi(w)\bar w_1 w_2\right)\right)dV_w.
\end{aligned}$$

Note that $\eta_1(w) := \chi_1\circ\psi(w)|w_2|^2, \eta_2(w) := \chi_2\circ\psi(w)w_2 + \chi_1\circ\psi(w)\bar w_1 w_2$ are both smooth with compact supports on $\triangle\times\triangle^*$ and thus on $\triangle^2$. Making use of (3.8), we further simplify the above to get

$$\begin{aligned}
-\langle\bar\partial u,\chi\rangle_{\mathbb{H}} &= \int_{\triangle^2}\left(f_1\circ\psi(w)\chi_2\circ\psi(w) - f_2\circ\psi(w)\chi_1\circ\psi(w)\right)|w_2|^2dV_w\\
&= \int_{\mathbb{H}} f_1(z)\chi_2(z) - f_2(z)\chi_1(z)dV_z = -\langle\mathbf{f},\chi\rangle_{\mathbb{H}}.
\end{aligned}$$

Altogether, by (3.2)-(3.3) and Theorem 1.1, the solution $u = \tilde{u} \circ \phi$ satisfies

$$\|u\|_{L^p(\mathbb{H})} = \|\tilde{u}\|_{L^p(\triangle^2, |w_2|^2)} \lesssim \|\psi^* \mathbf{f}\|_{L^p(\triangle^2, |w_2|^2)} \lesssim \|\mathbf{f}\|_{L^p(\mathbb{H})}.$$

The proof is complete.

□

The following example shows Theorem 1.2 is optimal, in the sense that solutions can not lie in a better Lebesgue space than that of the data in general.

**Example 3.** *For each $1 < p < \infty$, let $\mathbf{f} = (z_2 - 1)^{-\frac{2}{p}} d\bar{z}_1$ on $\mathbb{H}$, $\frac{1}{2}\pi < \arg(z_2 - 1) < \frac{3}{2}\pi$. Then $\mathbf{f} \in L^{\tilde{p}}(\mathbb{H})$ for all $1 < \tilde{p} < p$ and is $\bar{\partial}$-closed on $\mathbb{H}$. However, there does not exist a solution $u \in L^p(\mathbb{H})$ to $\bar{\partial} u = \mathbf{f}$ on $\mathbb{H}$.*

*Proof.* Clearly $\mathbf{f} \in L^{\tilde{p}}(\mathbb{H})$ for all $1 < \tilde{p} < p$ and $\mathbf{f}$ is $\bar{\partial}$-closed on $\mathbb{H}$. Arguing by contradiction, suppose there exists some $u \in L^p(\mathbb{H})$ satisfying $\bar{\partial} u = \mathbf{f}$ on $\mathbb{H}$. In particular, since $\triangle_{\frac{1}{2}} \times (\triangle \setminus \triangle_{\frac{1}{2}}) \subset \mathbb{H}$, there exists some holomorphic function $h$ on $\triangle_{\frac{1}{2}} \times (\triangle \setminus \triangle_{\frac{1}{2}})$ such that $u|_{\triangle_{\frac{1}{2}} \times (\triangle \setminus \triangle_{\frac{1}{2}})} = (z_2 - 1)^{-\frac{2}{p}} \bar{z}_1 + h \in L^p(\triangle_{\frac{1}{2}} \times (\triangle \setminus \triangle_{\frac{1}{2}}))$.

For almost everywhere fixed $(r, z_2) \in U := \left(0, \frac{1}{2}\right) \times \left(\triangle \setminus \triangle_{\frac{1}{2}}\right) \subset \mathbb{R}^3$, consider

$$v(r, z_2) := \int_{|z_1|=r} \tilde{u}(z_1, z_2) dz_1.$$

Similarly as in the proof of Example 1 (with $\mu = 1$ and $\triangle^2$ replaced by $\triangle_{\frac{1}{2}} \times (\triangle \setminus \triangle_{\frac{1}{2}})$), we see that $v \in L^p(U)$. Note that $h(\cdot, z_2)$ is holomorphic on $\triangle_{\frac{1}{2}}$ for each fixed $z_2 \in \triangle \setminus \triangle_{\frac{1}{2}}$. Thus for almost everywhere fixed $(r, z_2) \in U$, Cauchy's theorem gives

$$v(r, z_2) = \int_{|z_1|=r} z_2 (z_2 - 1)^{-\frac{2}{p}} \bar{z}_1 dz_1 = 2\pi r^2 i z_2 (z_2 - 1)^{-\frac{2}{p}},$$

which does not belong to $L^p(U)$. Contradiction!

□

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