Weighted Sobolev estimates of $\bar{\partial}$ on domains covered by polydiscs

Yuan Yuan and Yuan Zhang

Abstract

This paper concerns the weighted Sobolev estimate of $\bar{\partial}$ on bounded domains in $\mathbb{C}^n$ covered regularly by the polydisc. In particular, this applies to quotient domains of the polydisc, such as generalized Hartogs triangles and symmetrized polydiscs.

1 Introduction and the main theorems

The $\bar{\partial}$ problem is to study the solvability and regularity of the nonhomogeneous Cauchy-Riemann equation $\bar{\partial}u = f$ on domains in $\mathbb{C}^n$. When the domain is smoothly bounded and pseudoconvex, the $\bar{\partial}$-Neumann theory provides a powerful technique in the study of the L^2-Sobolev estimates of $\bar{\partial}$ (cf. [19, 39]). However, for general Sobolev estimates on domains with non-smooth boundary, it seems that the $\bar{\partial}$-Neumann theory is not quite applicable. The purpose of this paper is to investigate L^p-Sobolev regularity of $\bar{\partial}$ over some types of non-smooth domains.

One interesting example of non-smooth domains that attracts substantial attention is the Hartogs triangle $\mathbb{H}_{1,1}$ in $\mathbb{C}^2$, a bounded pseudoconvex domain without Lipschitz boundary. It is well-known that the $\bar{\partial}$ problem on $\mathbb{H}_{1,1}$ is not globally regular [12]. Namely, there is a $\bar{\partial}$-closed $(0,1)$-form g that is smooth on $\overline{\mathbb{H}_{1,1}}$, such that $\bar{\partial}v = g$ has no smooth solution on $\overline{\mathbb{H}_{1,1}}$. On the other hand, the works of [12, 33] show that the $\bar{\partial}$ equation on $\mathbb{H}_{1,1}$ admits Hölder solutions with desired estimates at each Hölder level via integral representations. Using the $\bar{\partial}$ theory on product domains developed in [9] and the fact that $\mathbb{H}_{1,1}$ is biholomorphic to $\Delta \times \Delta^*$, Chakrabarti-Shaw obtained the weighted L^2-Sobolev estimates of $\bar{\partial}$ on $\mathbb{H}_{1,1}$ [10]. More recently, the L^p regularity of the Bergman projection on the Hartogs triangle and its generalizations have been extensively studied by many authors (cf. [13, 14, 24, 25, 11, 6, 41]).

Another example is the so-called symmetrized polydisc $\mathbb{G}^n$, which is also a bounded pseudoconvex domain without Lipschitz boundary (cf. Proposition 5.3 in [8]). Various analytic and geometric properties on the symmetrized bidisc have been studied extensively (cf. [2, 3, 1, 26] and reference therein). Since $\mathbb{G}^n$ has a nice Stein neighborhood basis, by the well-known results in [23], Chakrabarti-Gorai showed that the $\bar{\partial}$ problem on $\mathbb{G}^n$ is globally regular [8]. However, it seems that the Sobolev estimates of the canonical solution of $\bar{\partial}$ is still missing. On the other hand, the L^p regularity of the Bergman projection on $\mathbb{G}^n$ or more general quotient domains is obtained in [16, 15].

Chakrabarti-Shaw pointed out in [10] that it will be interesting to have a general technique to deal with the Sobolev regularity of the $\bar{\partial}$-problem on singular domains such as $\mathbb{H}_{1,1}$. This is the main motivation of this paper. One obvious feature of these two aforementioned domains is that they can be viewed as quotients of the "polydisc" type domains. Nevertheless, the singularities of these two domains are quite different. On the Hartogs triangle, the singularity somehow can be considered as "product type": the Jacobian of the quotient map can separate variables; on the symmetrized polydisc, the singularity is somehow of "mixture type": the Jacobian of the quotient map does not separate variables. However, they both can be treated using the idea of the quotient maps. Since there have been intensive recent studies on the integral representation on the product of planar domains and L^p-Sobolev estimates of $\bar{\partial}$ have been obtained (cf. [17, 18, 27, 31, 22]), it is natural to wonder if the L^p-Sobolev estimates of $\bar{\partial}$ on product domains can be transformed to that of the quotient domains. More precisely, one may ask if the weighted Sobolev estimates of $\bar{\partial}$ on $\mathbb{H}_{1,1}$ obtained by Chakrabarti-Shaw can be extended to general quotient domains.

In fact, the idea to handle the quotient domains is simple. Assume $\psi : \Omega_1 \to \Omega_2$ is a quotient map. Given any $\bar{\partial}$-closed $(0,1)$-form on Ω_2, we pull it back to get a $\bar{\partial}$-closed $(0,1)$-form on Ω_1, solve the $\bar{\partial}$ equation on Ω_1, and then push forward the solution to get the solution of the $\bar{\partial}$ equation on Ω_2. However, in order to realize this idea, we need to deal with several difficulties: the weighted Sobolev estimates of $\bar{\partial}$ on Ω_1, and also the weighted Sobolev estimates of the Bergman projection on Ω_2 if we wish to estimate the canonical solutions. Fortunately, these difficulties can be overcome if the weight function is in a type of a refined Muckenhoupt's class A_p^* (see Definition 2.4). As one immediately sees, the method heavily relies on the recent important development on the integral representation on the product of planar domains for $\bar{\partial}$ (cf. [17, 18, 27]). Now we are ready to state our main theorem.

Theorem 1.1. *Let $n \geq 2$ and $\Omega \subset \mathbb{C}^n$ be a bounded domain covered regularly by the polydisc. Let $\mu := |\det J_{\mathbb{C}}(\psi)|^2$ and $\delta = \frac{1}{m}\psi_*\mu$, where m is the degree of ψ. Assume $\mu \in A_p^*, p > 1$. For any $\bar{\partial}$-closed $(0,1)$-form $g \in W^{k,p}_{(0,1)}(\Omega)$ on Ω with $k \geq n-1$, there exists a solution $v \in W^{k-n+1,p}(\Omega, \delta^l)$ of $\bar{\partial}v = g$ with $l = \max\left\{0, \frac{(2k-2n+1)p}{2}\right\}$ such that it satisfies*

$$\|v\|_{W^{k-n+1,p}(\Omega,\, \delta^l)} \lesssim \|g\|_{W^{k,p}_{(0,1)}(\Omega)}.$$

Furthermore, the canonical solution u of $\bar{\partial}u = g$ is in $W^{k-n+1,p}\left(\Omega, \delta^{\frac{3(k-n+1)p}{2}}\right)$ and satisfies

$$\|u\|_{W^{k-n+1,p}\left(\Omega,\, \delta^{\frac{3(k-n+1)p}{2}}\right)} \lesssim \|g\|_{W^{k,p}_{(0,1)}(\Omega)}.$$

We note that the method in this paper can be applied to the quotient of product of general planar domains. However, for the simplicity of the presentation, we restrict ourselves only to the case of bounded domains covered regularly by the polydisc as introduced in section 5. In particular, the main theorem applies to the Hartogs triangle and the symmetrized polydisc to give the following weighted estimates to $\bar{\partial}$ in Section 6.

Corollary 1.2. *Let $p > 2$ and $\delta = |z_2|^2$. For any $g \in W^{k,p}_{(0,1)}(\mathbb{H}_{1,1})$ be a $\bar{\partial}$-closed $(0,1)$-form on $\mathbb{H}_{1,1}$ with $k \geq 1$, the canonical solution u of $\bar{\partial}u = g$ is in $W^{k-1,p}\left(\mathbb{H}_{1,1}, \delta^{\frac{3(k-1)p}{2}}\right)$ and satisfies*

$$\|u\|_{W^{k-1,p}\left(\mathbb{H}_{1,1}, \delta^{\frac{3(k-1)p}{2}}\right)} \lesssim \|g\|_{W^{k,p}_{(0,1)}(\mathbb{H}_{1,1})}.$$

Corollary 1.3. *Let $p > n$ and $\delta = \frac{1}{n!}\psi_*\left(\prod_{j<k} |w_j - w_k|^2\right)$. For any $g \in W^{k,p}_{(0,1)}(\mathbb{G}^n)$ be a $\bar{\partial}$-closed $(0,1)$-form on $\mathbb{G}^n$ with $k \geq n-1$, the canonical solution u of $\bar{\partial}u = g$ is in $W^{k-n+1,p}\left(\mathbb{G}^n, \delta^{\frac{3(k-n+1)p}{2}}\right)$ and satisfies*

$$\|u\|_{W^{k-n+1,p}\left(\mathbb{G}^n, \delta^{\frac{3(k-n+1)p}{2}}\right)} \lesssim \|g\|_{W^{k,p}_{(0,1)}(\mathbb{G}^n)}.$$

The paper is organized as follows. In section 2, notations for function and weight spaces spaces are defined. In Section 3, we establish the weighted Sobolev estimates of $\bar{\partial}$ on planar domains. The estimates for product domains are obtained in Section 4. In Section 5, after defining domains covered regularly by polydiscs, we prove the weighted Sobolev estimates for the Bergman projection operator. This along with the estimates of the pullback and pushforward operators completes the proof of Theorem 1.1. Examples and applications of the main theorem are discussed in Section 6.

Acknowledgement: Part of the work was done when the first author was visiting BICMR in Spring 2022. He thanks the center for providing him the wonderful research environment. He would also like to thank Xinyi Yuan and Zheng Zhu for helpful discussions.

2 Notations and preliminaries

1. Weighted Sobolev spaces.

   Let Ω be a bounded domain in $\mathbb{C}^n$. Given a weight $\mu : \Omega \to [0, \infty)$, a function f on Ω is said to be in $L^p(\Omega, \mu), 1 \leq p < \infty$, if its weighted L^p norm

$$\|f\|_{L^p(\Omega,\mu)} := \left(\int_\Omega |f(z)|^p \mu(z) dV(z)\right)^{\frac{1}{p}} < \infty.$$

   Here $dV(z)$ is the standard Lebesgue measure with respect to the dummy variable $z \in \Omega$. Given $k \in \mathbb{Z}^+$, the weighted Sobolev spaces $W^{k,p}(\Omega, \mu)$ is the collection of functions whose weak derivatives up to order k exist and belong to $L^p(\Omega, \mu)$. For $f \in W^{k,p}(\Omega, \mu)$, denoting $D^j f$ all j-th order (weak) derivatives of f, the norm is

$$\|f\|_{W^{k,p}(\Omega,\mu)} := \sum_{j=0}^{k} \|D^j f\|_{L^p(\Omega,\mu)}.$$

When $\mu \equiv 1$, $L^p(\Omega,\mu)$ and $W^{k,p}(\Omega,\mu)$ reduce to the standard $L^p(\Omega)$ and $W^{k,p}(\Omega)$ spaces, respectively.

Furthermore, we say a smooth $(0,1)$-form $f = \sum_{j=1}^n f_j d\bar{z}^j$ on Ω is in $W^{k,p}(\Omega,\mu)$ if all the coefficients f_j $(1 \le j \le n)$ are in $W^{k,p}(\Omega,\mu)$.

2. A_p and A_p^* weights.

We will be focusing on weights in the following Muckenhoupt's class.

Definition 2.1. A weight $\mu : \mathbb{R}^N \to [0,\infty)$ is said to be in A_p, the Muckenhoupt's class, if its A_p constant

$$A_p(\mu) := \sup \left(\frac{1}{|B|} \int_B \mu(z) dV(z) \right) \left(\frac{1}{|B|} \int_B \mu(z)^{\frac{1}{1-p}} dV(z) \right)^{p-1} < \infty, \tag{1}$$

where the supremum is taken over all balls $B \subset \mathbb{R}^N$, and $|B|$ is the Lebesgue measure of B.

Remark 2.2. One can similarly define A_p spaces restricted on a domain $\Omega \subset \mathbb{R}^N$. In fact, $\mu : \Omega \to [0,\infty)$ is said to be in $A_{p,\Omega}$, if

$$\sup \left(\frac{1}{|B|} \int_{B\cap\Omega} \mu(z) dV(z) \right) \left(\frac{1}{|B|} \int_{B\cap\Omega} \mu(z)^{\frac{1}{1-p}} dV(z) \right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^N$. According to an unpublished result of Wolff (see also [28, pp. 439], [32] etc), if there exists $\epsilon > 0$ such that $\mu^{1+\epsilon} \in A_{p,\Omega}$, then μ has an extension $\tilde{\mu} \in A_p$. Due to this extension result, for a weight μ originally defined on Ω, we say $\mu \in A_p$ if it has an extension $\tilde{\mu}$ on $\mathbb{R}^N$ such that $\tilde{\mu} \in A_p$.

It is not hard to see that $A_q \subset A_p$ if $1 \le q < p$. More properties of the Muckenhoupt's classes can be found in [38, Chapter V]. In particular, A_p spaces satisfy an open-end property: if $\mu \in A_p$ for some $p > 1$, then $\mu \in A_{\tilde{p}}$ for some $\tilde{p} < p$. We will also need the following well-known fact for our examples later on.

Example 2.3. Measures of the form $\mu = |x-c|^a \in A_p$ in $\mathbb{R}^N$ if and only if $-N < a < N(p-1)$, with the A_p constant independent of $c \in \mathbb{R}^N$.

In order to obtain the weighted Sobolev estimates for $\bar{\partial}$ on product domains, we impose additional condition on the weights such that their restriction to almost every 1-dimensional coordinate slice is A_p with a uniform A_p constant. More precisely, For any $z \in \mathbb{C}^n$, denote by $\hat{z}_j$ the point in $\mathbb{C}^{n-1}$ with the j-th component of z skipped. Namely, if $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, then $\hat{z}_j = (z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_n) \in \mathbb{C}^{n-1}$. We have

Definition 2.4. A weight $\mu : \mathbb{C}^n \to [0,\infty)$ is said to be in A_p^* if

$$A_p^*(\mu) := \sup \left(\frac{1}{|B|} \int_B \mu(z) dV(z_j) \right) \left(\frac{1}{|B|} \int_B \mu(z)^{\frac{1}{1-p}} dV(z_j) \right)^{p-1} < \infty,$$

where the supremum is taken over almost every $\hat{z}_j \in \mathbb{C}^{n-1}, j = 1, \ldots, n$, and all discs $B \subset \mathbb{C}$.

When $n = 1$, A_p^* is reduced to A_p. When $n \geq 2$, $\mu \in A_p^*$ if and only if the δ-dilation $\mu_\delta(z) := \mu(\delta_1 z_1, \ldots, \delta_n z_n) \in A_p$ with a uniform A_p constant for all $\delta = (\delta_1, \ldots, \delta_n) \in (\mathbb{R}^+)^n$. See, for instance, [28, pp. 454]. In particular, $A_p^* \subset A_p$.

3. Uniform domains.

Definition 2.5. Given $\epsilon > 0, \delta > 0$, a domain $\Omega \subset \mathbb{C}^n$ is said to be an (ϵ, δ) domain if whenever $z, z' \in \Omega$ and $|z - z'| < \delta$, there exists a rectifiable $\gamma \subset \Omega$ joining z to z' such that

$$l(\gamma) \leq \frac{1}{\epsilon}|z - z'|, \;\; dist(\xi, \Omega) \geq \frac{\epsilon|z - \xi||z' - \xi|}{|z - z'|}, \tag{2}$$

for all $\xi \in \gamma$, where $l(\gamma)$ is the arc length of γ. When $\delta = \infty$, Ω is called a uniform domain.

Roughly speaking, the first inequality in (2) says that Ω is locally connected in some quantitative manner; the second inequality says that there exists a "tube" in Ω containing γ such that the ratio of the width of the tube at $\xi \in \gamma$ with $\min\{|z-\xi|, |z'-\xi|\}$ is bounded uniformly from below. It is well-known that Lipschitz domains and products of uniform domains are uniform domains. When Ω is uniform, it was also shown in [20, Theorem 1.1] that any function in $W^{k,p}(\Omega, \mu)$ extends to a function in $W^{k,p}(\mathbb{C}^n, \mu)$ provided that $\mu \in A_p, p > 1$. In section 5, we will assume the domains under consideration are uniform domains.

Throughout the paper, we say that two quantities a and b satisfy $a \lesssim b$, if there exists a constant $C > 0$ dependent only possibly on k, p, Ω and the A_p (or A_p^*, according to the context) constant of μ such that $a \leq Cb$. We say $a \approx b$ if and only if $a \lesssim b$ and $b \lesssim a$ at the same time. $\mathbb{Z}^+$ is the positive integer set and $\mathbb{Z}^+ \cup \{0\}$ is the nonnegative integer set.

3 Solving $\bar{\partial}$ with weighted Sobolev estimates on planar domains

Throughout the section, $D \subset \mathbb{C}$ is a bounded domain with sufficiently smooth boundary. Define for $f \in L^p(D), p > 1$,

$$Tf(z) = \frac{-1}{\pi} \int_D \frac{f(\zeta)}{\zeta - z} dV(\zeta), \quad Hf(z) = p.v. \frac{-1}{\pi} \int_D \frac{f(\zeta)}{(\zeta - z)^2} dV(\zeta).$$

It is known that

$$\bar{\partial}(Tf) = f, \ \ \partial(Tf) = Hf \tag{3}$$

weakly on D. See for instance [40] etc. The goal of the section is to prove the $W^{k,p}(D,\mu)$ estimates for the solution operator T to the $\bar{\partial}$ equation on D, $k \in \mathbb{Z}^+ \cup \{0\}, \mu \in A_p, p > 1$.

3.1 Weighted L^p estimates for the Cauchy integral

In this subsection, we prove the boundedness of T and H in weighted spaces $L^p(D,\mu)$ with $\mu \in A_p, p > 1$. The boundedness of the Hilbert transform

$$\tilde{H}f(z) := p.v.\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{(\zeta - z)^2} dV(\zeta)$$

is a classical result in harmonic analysis. For the operator T, in particular for the case $p < 2$, the boundedness in $L^p(D,\mu)$ can be derived from the classical result of Muckenhoupt and Wheeden [34] on Riesz potentials. In general, we provide a self-contained proof below by modifying a standard approach for the estimates of the non-weighted fractional integrals. See also [29].

Proposition 3.1. *Assume $\mu \in A_p, p > 1$. Then T and H are bounded operators between $L^p(D,\mu)$. More precisely,*

$$\begin{aligned} \|Tf\|_{L^p(D,\mu)} &\lesssim \|f\|_{L^p(D,\mu)}; \\ \|Hf\|_{L^p(D,\mu)} &\lesssim \|f\|_{L^p(D,\mu)}. \end{aligned} \tag{4}$$

for all $f \in L^p(D,\mu)$.

Proof. By the singluar operator theory (see [38] pp. 205 for instance), $\tilde{H}$ is bounded from $L^p(\mathbb{C},\mu)$ into itself. Given $f \in L^p(D,\mu)$, extend f to be in $L^p(\mathbb{C},\mu)$ trivially by letting $f = 0$ on $\mathbb{C} \setminus D$, denoted by $\tilde{f}$. Then $Hf = \tilde{H}\tilde{f}$ on D and

$$\|Hf\|_{L^p(D,\mu)} \le \|\tilde{H}\tilde{f}\|_{L^p(\mathbb{C},\mu)} \lesssim \|\tilde{f}\|_{L^p(\mathbb{C},\mu)} = \|f\|_{L^p(D,\mu)}.$$

For the weighted boundedness of T, first by extending f to be zero outside D if necessary, we assume that D is a disc. Consider

$$T^+f(z) := \int_D \frac{f(\zeta)}{|\zeta - z|} dV(\zeta), \quad z \in D.$$

We shall show that for any $f \in L^p(D,\mu)$,

$$\|T^+f\|_{L^p(D,\mu)} \lesssim \|f\|_{L^p(D,\mu)}. \tag{5}$$

From this (4) follows immediately.

For each $z \in \mathbb{C}$ with $\delta > 0$ to be chosen later, write

$$T^+ f(z) = \left(\int_{|\zeta - z| < \delta, \zeta \in D} + \int_{|\zeta - z| > \delta, \zeta \in D} \right) \frac{f(\zeta)}{|\zeta - z|} dV(\zeta) =: I + II.$$

Denote by Mf the Hardy-Littlewood maximal function of f. For I,

$$\begin{aligned} I =& \sum_{k=1}^{\infty} \int_{\frac{\delta}{2^k} < |\zeta - z| < \frac{\delta}{2^{k-1}}} \frac{f(\zeta)}{|\zeta - z|} dV(\zeta) \leq \sum_{k=1}^{\infty} \frac{2^k}{\delta} \int_{|\zeta - z| < \frac{\delta}{2^{k-1}}} f(\zeta) dV(\zeta) \\ =& \sum_{k=1}^{\infty} \frac{2^k}{\delta} \left| D_{\frac{\delta}{2^{k-1}}} \right| \left(\frac{1}{\left| D_{\frac{\delta}{2^{k-1}}} \right|} \int_{|\zeta - z| < \frac{\delta}{2^{k-1}}} f(\zeta) dV(\zeta) \right) \\ \leq& \sum_{k=1}^{\infty} 2^{-k+2} \pi \delta Mf(z) \approx \delta Mf(z). \end{aligned}$$

To estimate II, first note that since $\mu \in A_p$. By open-end property there exists $\frac{p}{2} < \tilde{p} < p$ such that $\mu \in A_{\tilde{p}}$. By Hölder inequality,

$$\begin{aligned} II \leq& \left(\int_{|\zeta - z| > \delta, \zeta \in D} |f(\zeta)|^p \mu(\zeta) dV(\zeta) \right)^{\frac{1}{p}} \left(\int_{|\zeta - z| > \delta, \zeta \in D} |\zeta - z|^{\frac{p}{1-p}} \mu(\zeta)^{\frac{1}{1-p}} dV(\zeta) \right)^{\frac{p-1}{p}} \\ \lesssim& \|f\|_{L^p(D,\mu)} \left(\int_{|\zeta - z| > \delta, \zeta \in D} |\zeta - z|^{\frac{p}{\tilde{p}-p}} dV(\zeta) \right)^{\frac{p-\tilde{p}}{p}} \left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}} dV(\zeta) \right)^{\frac{\tilde{p}-1}{p}} \\ \lesssim& \|f\|_{L^p(D,\mu)} \left(\int_{\delta}^{\infty} s^{\frac{\tilde{p}}{\tilde{p}-p}} ds \right)^{\frac{p-\tilde{p}}{p}} \left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}} dV(\zeta) \right)^{\frac{\tilde{p}-1}{p}} \\ =& \frac{p - \tilde{p}}{2\tilde{p} - p} \delta^{1 - \frac{2\tilde{p}}{p}} \|f\|_{L^p(D,\mu)} \left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}} dV(\zeta) \right)^{\frac{\tilde{p}-1}{p}}. \end{aligned}$$

Thus we have

$$T^+ f(z) \lesssim \delta Mf(z) + \delta^{1 - \frac{2\tilde{p}}{p}} \|f\|_{L^p(D,\mu)} \left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}} dV(\zeta) \right)^{\frac{\tilde{p}-1}{p}}.$$

After choosing $\delta = \left(\frac{\|f\|_{L^p(D,\mu)} \left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}} dV(\zeta) \right)^{\frac{\tilde{p}-1}{p}}}{Mf} \right)^{\frac{p}{2\tilde{p}}}$ in the above, we further get

$$T^+ f(z) \lesssim \|f\|_{L^p(D,\mu)}^{\frac{p}{2\tilde{p}}} \left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}} dV(\zeta) \right)^{\frac{\tilde{p}-1}{2\tilde{p}}} Mf(z)^{\frac{2\tilde{p}-p}{2\tilde{p}}}.$$

Then

$$\begin{aligned}\|T^+f\|_{L^{\frac{2p\tilde{p}}{2\tilde{p}-p}}(D,\mu)} \lesssim & \|f\|_{L^p(D,\mu)}^{\frac{p}{2\tilde{p}}}\left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}}dV(\zeta)\right)^{\frac{\tilde{p}-1}{2\tilde{p}}}\|Mf\|_{L^p(\mathbb{C},\mu)}^{\frac{2\tilde{p}-p}{2\tilde{p}}}\\ \lesssim & \|f\|_{L^p(D,\mu)}^{\frac{p}{2\tilde{p}}}\left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}}dV(\zeta)\right)^{\frac{\tilde{p}-1}{2\tilde{p}}}\|f\|_{L^p(\mathbb{C},\mu)}^{\frac{2\tilde{p}-p}{2\tilde{p}}}\\ = & \|f\|_{L^p(D,\mu)}\left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}}dV(\zeta)\right)^{\frac{\tilde{p}-1}{2\tilde{p}}}.\end{aligned}$$

Here we used the boundedness of the maximal function operator in $L^p(\mathbb{C},\mu)$ with $\mu \in A_p$ in the second inequality. Finally, by Hölder inequality,

$$\begin{aligned}\|T^+f\|^p_{L^p(D,\mu)} \le & \|T^+f\|^p_{L^{\frac{2p\tilde{p}}{2\tilde{p}-p}}(D,\mu)}\left(\int_D \mu(\zeta)dV(\zeta)\right)^{\frac{p}{2\tilde{p}}}\\ \lesssim & \|f\|^p_{L^p(D,\mu)}\left(\int_D \mu(\zeta)dV(\zeta)\right)^{\frac{p}{2\tilde{p}}}\left(\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}}dV(\zeta)\right)^{\frac{(\tilde{p}-1)p}{2\tilde{p}}}\\ = & |D|^{\frac{p}{2}}\left(\left(\frac{1}{|D|}\int_D \mu(\zeta)dV(\zeta)\right)\left(\frac{1}{|D|}\int_D \mu(\zeta)^{\frac{1}{1-\tilde{p}}}dV(\zeta)\right)^{\tilde{p}-1}\right)^{\frac{p}{2\tilde{p}}}\|f\|^p_{L^p(D,\mu)}.\end{aligned}$$

(5) is thus proved due to the fact that $\mu \in A_{\tilde{p}}$.

∎

The $\mu \in A_p$ assumption in Proposition 3.1 is known to be necessary for the boundedness of H in $L^p(\Delta,\mu)$. We remark that this assumption can not be dropped for the boundedness of T in $L^p(\Delta,\mu)$, either. In fact, the following example shows that T in general does not even send $L^p(\Delta,\mu)$ into itself if $\mu \notin A_p$. This is in strong contrast to the smoothing property of T in the trivial $\mu \equiv 1$ case.

Example 3.2. Let $\mu = |z|^2$ and $f(z) = -\frac{1}{|z|^2 \ln|z|}$ on Δ. Then $\mu \notin A_2$ and $f \in L^2(\Delta,|z|^2)$. However, $Tf \notin L^2(\Delta,|z|^2)$.

Proof. A direct computation shows that

$$\int_\Delta |f(z)|^2|z|^2dV(z) \approx \int_0^1 \frac{1}{r|\ln r|^2}dr < \infty.$$

Namely, $f \in L^2(\Delta,|z|^2)$. Consequently, $zf \in L^2(\Delta)$, and $T(zf) \in L^2(\Delta)$ by standard complex analysis theory. Assume by contradiction that $Tf \in L^2(\Delta,|z|^2)$. Then $zTf \in L^2(\Delta)$. In particular,

$zTf(z)-T(zf)(z) \in L^2(\Delta)$. However, this would contradict with the following fact that for almost all $z \in \Delta$,

$$zTf(z) - T(zf)(z) \equiv -\frac{1}{\pi}\int_\Delta f(z)dV(z) \approx \int_0^1 \frac{1}{r|\ln r|}dr = \infty.$$

∎

3.2 Weighted Sobolev estimates for the Cauchy integral

In this subsection, we investigate the weighted Sobolev estimate for T and H in $W^{k,p}(D,\mu), k \in \mathbb{Z}^+, p > 1$. When $\mu \equiv 1$, the boundedness of H in $W^{k,p}(D)$ is due to a technical result of Prats [36]. For general weights, we will make use of the following relation between T and H:

Theorem 3.3. *[40] For $f \in W^{1,p}(D), p > 1$, the following formula holds weakly in D:*

$$\partial Tf (= Hf) = T\left(\frac{\partial f}{\partial \zeta}\right) - \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)d\bar\zeta}{\zeta - \cdot}. \tag{6}$$

Proof. When $f \in C^{1,\alpha}(D), 0 < \alpha < 1$, the formula was proved in [40, pp. 60-61] pointwisely. In particular, if $f \in C_0^\infty(D)$,

$$\partial Tf = T\partial f. \tag{7}$$

We will verify (6) if $f \in W^{1,p}(D)$. Note that by trace theorem $W^{1,p}(D) \hookrightarrow L^p(\partial D)$, the last integral in (6) is well defined.

For any testing function $\phi \in C_0^\infty(D)$,

$$\langle \partial Tf, \phi\rangle = -\langle Tf, \bar\partial\phi\rangle = \int_D f(\zeta)T(\partial\bar\phi)(\zeta)dV(\zeta). \tag{8}$$

On the other hand, denote the right hand side of (6) by Rf. We then have

$$\begin{aligned}\langle Rf, \phi\rangle =& \frac{-1}{\pi}\int_D \overline{\phi(z)}\int_D \frac{\partial_\zeta f(\zeta)}{\zeta - z}dV(\zeta)dV(z) - \frac{1}{2\pi i}\int_D \overline{\phi(z)}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\bar\zeta dV(z)\\ =& \frac{-1}{\pi}\int_D \partial_\zeta f(\zeta)\int_D \frac{\overline{\phi(z)}}{\zeta - z}dV(z)dV(\zeta) - \frac{1}{2\pi i}\int_{\partial D} f(\zeta)\int_D \frac{\overline{\phi(z)}}{\zeta - z}dV(z)d\bar\zeta\\ =& -\int_D \partial_\zeta f(\zeta)T\bar\phi(\zeta)dV(\zeta) - \frac{1}{2i}\int_{\partial D} f(\zeta)T\bar\phi(\zeta)d\bar\zeta.\end{aligned}$$

Thus

$$\langle Rf, \phi\rangle = \int_D f(\zeta)\partial_\zeta T\bar\phi(\zeta)dV(\zeta) = \int_D f(\zeta)T(\partial\bar\phi)(\zeta)dV(\zeta), \tag{9}$$

where we used Stokes' theorem in the first equality since $fT\bar\phi \in W^{1,1}(D)$, and used (7) in the second equality. The proof of the theorem is complete in view of (8) and (9).

∎

For each $k \in \mathbb{Z}^+$, the operator

$$J_k f := \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta) d\bar{\zeta}}{(\zeta - \cdot)^k}$$

is well defined for $f \in W^{1,p}(D)$, with $J_k f \in C^\infty(D)$. As an immediate consequence of the above theorem and (3), one further has following recursive formula for higher derivatives of H.

Corollary 3.4. *For $f \in W^{1,p}(D)$, $p > 1$, we have for all $k \in \mathbb{Z}^+$, the following holds weakly on D:*

$$\partial^k T f = \partial^{k-1} T \left(\frac{\partial f}{\partial \zeta} \right) - J_k f.$$

Before estimating the higher derivatives of T, we first observe the following inductive formula for $J_k f$ if in addition $f \in W^{k,p}(D)$.

Lemma 3.5. *Let $f \in W^{k,p}(D), k \in \mathbb{Z}^+, p > 1$. Then*

$$J_k f = J_1 \tilde{f},$$

where $\tilde{f} = \sum_{j=0}^{k-1} c_j f^{(j)}$ for some functions $c_j \in C^\infty(\bar{D})$ dependent only on D.

Proof. Let ζ be a parameterization of ∂D in terms of the arc length s with the total length s_0. So $\zeta'(s) = \left(\bar{\zeta}'(s)\right)^{-1}$. A direct integration-by-parts computation gives

$$\begin{aligned}
(k-1) \int_{\partial D} \frac{f(\zeta) d\bar{\zeta}}{(\zeta - z)^k} &= \int_0^{s_0} \partial_z \left(\frac{1}{(\zeta(s) - z)^{k-1}} \right) f(\zeta(s)) \bar{\zeta}'(s) ds \\
&= - \int_0^{s_0} \partial_s \left(\frac{1}{(\zeta(s) - z)^{k-1}} \right) \frac{f(\zeta(s)) \bar{\zeta}'(s)}{\zeta'(s)} ds \\
&= \int_0^{s_0} \frac{1}{(\zeta(s) - z)^{k-1}} \partial_s (f(\zeta(s)) \bar{\zeta}'^2(s)) ds \\
&= \int_0^{s_0} \frac{1}{(\zeta(s) - z)^{k-1}} (f'(\zeta(s)) \bar{\zeta}'(s) + 2 f(\zeta(s)) \bar{\zeta}'(s) \bar{\zeta}''(s)) ds \\
&= \int_{\partial D} \frac{f'(\zeta) + 2\bar{\zeta}''(s) f(\zeta)}{(\zeta - z)^{k-1}} d\bar{\zeta} =: \int_{\partial D} \frac{\tilde{f}(\zeta)}{(\zeta - z)^{k-1}} d\bar{\zeta}
\end{aligned}$$

where $\tilde{f} = f' + 2\bar{\zeta}''(s) f$. The remaining part of the lemma follows by induction. ∎

We next extend J_k to be an operator defined on the weighted Sobolev spaces $W^{1,p}(D, \mu)$ with $\mu \in A_p$. In fact, let $\tilde{p} < p$ be such that $\mu \in A_{\tilde{p}}$. Then $q := p/\tilde{p} > 1$. For any function $f \in L^p(D, \mu)$, by Hölder inequality

$$\int_D |f|^q dA = \int_D |f|^q \mu^{\frac{1}{\tilde{p}}} \mu^{-\frac{1}{\tilde{p}}} dA \le \|f\|_{L^p(D,\mu)} \left(\int_D \mu^{\frac{1}{1-\tilde{p}}} dA \right)^{\frac{\tilde{p}-1}{\tilde{p}}} \lesssim \|f\|_{L^p(D,\mu)}.$$

Thus $L^p(D,\mu) \hookrightarrow L^q(D)$, and further $W^{1,p}(D,\mu) \hookrightarrow W^{1,q}(D)$ if $\mu \in A_p$. Consequently, Theorem 3.3, Corollary 3.4 as well as Lemma 3.5 passes onto $W^{1,p}(D,\mu)$ seamlessly. In particular, J_k satisfies the following estimate on $W^{k,p}(D,\mu)$.

Proposition 3.6. *Given $\mu \in A_p, p > 1$. For each $k \in \mathbb{Z}^+$, J_k is well defined on $W^{1,p}(D,\mu)$. Moreover, if $f \in W^{k,p}(D,\mu)$, then*

$$\|J_k f\|_{L^p(D,\mu)} \lesssim \|f\|_{W^{k,p}(D,\mu)}.$$

Proof. If $f \in W^{1,p}(D,\mu)$, $J_1 f = T(\frac{\partial f}{\partial z}) - Hf$ by Theorem 3.3. Hence

$$\|J_1 f\|_{L^p(D,\mu)} \leq \left\| T\left(\frac{\partial f}{\partial z}\right)\right\|_{L^p(D,\mu)} + \|Hf\|_{L^p(D,\mu)} \lesssim \|f\|_{W^{1,p}(D,\mu)}.$$

When $k \geq 2$, one makes use of the induction formula in Lemma 3.5 to reduce $J_k f$ to $J_1 \tilde{f}$, for some $\tilde{f} \in W^{1,p}(D,\mu)$ as defined there. The proof is complete. ∎

We are ready to prove the following wighted Sobolev estimates of T and H.

Theorem 3.7. *Let $\mu \in A_p, p > 1$. For each $k \in \mathbb{Z}^+ \cup \{0\}$,*

$$\|Tf\|_{W^{k,p}(D,\mu)} \lesssim \|f\|_{W^{k,p}(D,\mu)}$$

for all $f \in W^{k,p}(D,\mu)$; and for each $k \in \mathbb{Z}^+$,

$$\|Hf\|_{W^{k,p}(D,\mu)} \lesssim \|f\|_{W^{k+1,p}(D,\mu)}$$

for all $f \in W^{k+1,p}(D,\mu)$.

Proof. It suffices to show the inequality for T by (3). $k = 0$ case is a consequence of Proposition 3.1. When $k \geq 1$, let $D^\gamma = \partial^l \bar{\partial}^j$ with $l + j = k$. If $j \geq 1$, then $D^\gamma Tf = \partial^l \bar{\partial}^{j-1} f$ and thus $\|D^\gamma Tf\|_{L^p(D,\mu)} \lesssim \|f\|_{W^{k-1,p}(D,\mu)}$. Otherwise, we have $D^\gamma Tf = \partial^k Tf$. Its estimate then follows from Corollary 3.4 and Proposition 3.6 by induction on k. ∎

4 Weighted Sobolev estimates for $\bar{\partial}$ on product domains

Let $\Omega = D_1 \times \cdots \times D_n \subset \mathbb{C}^n$ be a Cartesian product of planar domains D_j with smooth boundary. The $\bar{\partial}$ problem and the corresponding regularity have been investigated since the seminal work of Nijenhuis and Woolf [35]. See also [30, 9, 17, 18, 27, 31, 22] and the references therein.

For each $j = 1, \dots, n$ and $f \in L^1(\Omega)$, let

$$T_j f(z) := -\frac{1}{2\pi i} \int_{D_j} \frac{f(z_1, \dots, z_{j-1}, \zeta_j, z_{j+1}, \dots, z_n)}{\zeta_j - z_j} d\bar{\zeta}_j \wedge d\zeta_j.$$

Following Chen-McNeal [18] and Fassina-Pan [27], given a $\bar{\partial}$-closed $(0,1)$ form f with $W^{n-1,1}$ coefficients on Ω, we define

$$Tf := \sum_{s=1}^{n} (-1)^{s-1} \sum_{1 \le i_1 < \cdots < i_s \le n} T_{i_1} \cdots T_{i_s} \left(\frac{\partial^{s-1} f_{i_s}}{\partial \bar{z}_{i_1} \cdots \partial \bar{z}_{i_{s-1}}} \right). \tag{10}$$

Then T solves $\bar{\partial} u = f$ weakly on Ω. The following theorem extends the estimate of T to weighted Sobolev spaces, provided that $\mu \in A_p^*$. Recall the definition for A_p^* in Definition 2.4.

Theorem 4.1. *Assume $\mu \in A_p^*, p > 1$ and an integer $k \ge n-1$. Then T defined in (10) is a bounded operator from $W^{k,p}(\Omega, \mu)$ into $W^{k-n+1,p}(\Omega, \mu)$. Namely,*

$$\|Tf\|_{W^{k-n+1,p}(\Omega,\mu)} \lesssim \|f\|_{W^{k,p}(\Omega,\mu)}$$

for all $f \in W^{k,p}(\Omega, \mu)$.

Proof. We first show T_j is bounded from $W^{k,p}(\Omega, \mu)$ into $W^{k,p}(\Omega, \mu)$, $j = 1, \dots, n$.

$$\|T_j f\|_{W^{k,p}(\Omega,\mu)} \lesssim \|f\|_{W^{k,p}(\Omega,\mu)} \tag{11}$$

for $f \in W^{k,p}(\Omega, \mu)$. Indeed, for $j = 1$, write $\Omega' = D_2 \times \cdots \times D_n$. Given any n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n) := (\alpha_1, \alpha')$ with $|\alpha| = k$, and $f \in W^{k,p}(\Omega, \mu)$, we have $D^\alpha T_1 f = D_{z_1}^{\alpha_1} T_1 \tilde{f}$ with $\tilde{f} := D_{z'}^{\alpha'} f \in W^{\alpha_1,p}(\Omega, \mu)$. For almost all fixed $z' \in \Omega'$, note that $\tilde{f}(\cdot, z') \in W^{\alpha_1,p}(D_1, \mu(\cdot, z'))$, and $\mu(\cdot, z') \in A_p$. By Theorem 3.7, $D^\alpha T_1 f(\cdot, z') \in L^p(D_1, \mu(\cdot, z'))$ and

$$\int_{D_1} |D^\alpha T_1 f(z)|^p \mu(z) dV(z_1) \lesssim \sum_{0 \le k \le \alpha_1} \int_{D_1} |D_{z_1}^k \tilde{f}(z)|^p \mu(z) dV(z_1).$$

Thus

$$\int_\Omega |D^\alpha T_1 f(z)|^p \mu(z) dV(z) = \int_{\Omega'} \int_{D_1} |D^{\alpha_1} T_1 \tilde{f}(z)|^p \mu(z) dV(z_1) dV(z') \lesssim \|f\|_{W^{k,p}(\Omega,\mu)}^p.$$

So (11) is proved for $j = 1$. The rest of the cases for (11) are proved similarly.

For any $f \in W^{k,p}(\Omega, \mu)$, and for $1 \le i_1 < \cdots < i_s \le n, s \le n$, we apply (11) inductively to obtain

$$\left\| T_{i_1} \cdots T_{i_s} \left(\frac{\partial^{s-1} f_{i_s}}{\partial \bar{z}_{i_1} \cdots \partial \bar{z}_{i_{s-1}}} \right) \right\|_{W^{k-n+1,p}(\Omega,\mu)} \lesssim \left\| \frac{\partial^{s-1} f_{i_s}}{\partial \bar{z}_{i_1} \cdots \partial \bar{z}_{i_{s-1}}} \right\|_{W^{k-n+1,p}(\Omega,\mu)} \lesssim \|f\|_{W^{k,p}(\Omega,\mu)}.$$

Finally, the theorem follows from (10) and the above inequality. ∎

5 $\bar{\partial}$ equation on bounded domains covered by the polydisc

5.1 Bounded domains covered regularly by the polydisc

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain such that there exists a surjective proper holomorphic map $\psi : (\Delta^n)^* \to \Omega$, where $(\Delta^n)^*$ is a domain obtained from Δ^n minus an analytic subvariety. It follows from the Remmert proper mapping theorem that ψ is a ramified covering of order m. More precisely, letting S be the analytic subset in $(\Delta^n)^*$ where ψ is ramified, then $\psi : (\Delta^n)^* \setminus S \to \Omega \setminus \tilde{S}$ is a regular covering of order m, where $\tilde{S} = \psi(S)$. We are interested in solving $\bar{\partial}$ in the following class of domains.

Definition 5.1. A bounded domain $\Omega \subset \mathbb{C}^n$ is called *a domain covered regularly by the polydisc* if there exists a surjective proper holomorphic map $\psi : (\Delta^n)^* \to \Omega$ such that the pair $\{(\Delta^n)^*, \psi\}$ satisfies the following assumptions:

- $(\Delta^n)^*$ is a uniform domain.
- ψ extends smoothly to $\overline{\Delta^n}$.
- ψ is a Galois covering. Namely there exists a group G of order m with $((\Delta^n)^* \setminus S)/G = \psi((\Delta^n)^* \setminus S)$ such that $\psi|_{(\Delta^n)^* \setminus S}$ is G-invariant.
- the action of G on $(\Delta^n)^* \setminus S$ extends smoothly to $\overline{\Delta^n}$. Namely, for any $\sigma \in G$, there exists a smooth map $\hat{\sigma} : \overline{\Delta^n} \to \overline{\Delta^n}$ such that $\hat{\sigma}|_{(\Delta^n)^* \setminus S} = \sigma$.
- for any $\sigma \in G$, $\det \mathrm{J}_{\mathbb{C}}(\sigma)$ extends to a non-vanishing smooth function on $\overline{\Delta^n}$. Consequently,
$$|\det \mathrm{J}_{\mathbb{C}}(\sigma)|^2 \approx 1.$$

In the following context, we will not distinguish $\hat{\sigma}$ from σ. We also denote the local inverse maps of ψ by $\phi_1, \cdots, \phi_m$ in the sense that

- $\psi \circ \phi_j(z) = z$ for all $z \in \Omega$;
- for every $w \in (\Delta^n)^* \setminus S$ and $1 \leq j \leq m$, there exists $\sigma \in G$, such that $\phi_j \circ \psi(w) = \sigma(w)$.

As a consequence, for any $w \in (\Delta^n)^* \setminus S$ and each j, there exists some $\sigma \in G$, such that

$$\det \mathrm{J}_{\mathbb{C}}(\phi_j)(\psi(w)) \cdot \det \mathrm{J}_{\mathbb{C}}(\psi)(\sigma(w)) = 1. \tag{12}$$

Moreover, for the fixed w, when j runs from 1 to m, σ exactly realizes all elements in G. Let $\mu = |\det \mathrm{J}_{\mathbb{C}}(\psi)|^2$. Define $\delta = \frac{1}{m}\psi_*\mu$, the average of the push-forward of μ in the sense of the distribution.

Lemma 5.2. *For any $u \in L^1_{loc}((\Delta^n)^*)$, $\psi_* u = \sum_{i=1}^m \phi_i^* u$ almost everywhere on Ω. In particular, $\psi_*\psi^* v = mv$ almost everywhere on Ω, for any $v \in L^1_{loc}(\Omega)$.*

Proof. For any $z \in \Omega \setminus \tilde{S}$, let U be a open neighborhood of z such that $\phi_k : U \to (\Delta^n)^*$ is a biholomorphism to the image for every $1 \leq k \leq m$. Assume that χ is a smooth (n,n)-form with compact support and the support is contained in U. Therefore,

$$\int_U (\psi_* u)\,\chi = \int_\Omega (\psi_* u)\,\chi = \int_{(\Delta^n)^*} u\,(\psi^*\chi) = \sum_{k=1}^m \int_{\phi_k(U)} u\,(\psi^*\chi) = \sum_{k=1}^m \int_U u(\phi_k(z))\chi.$$

The lemma thus follows as χ is arbitrary. ∎

By Lemma 5.2, $\delta = \frac{1}{m}\sum_{i=1}^m |\det \mathrm{J}_{\mathbb{C}}(\psi)|^2 \circ \phi_i$ almost everywhere on Ω. Note that $\psi(\eta) = \psi(\sigma(\eta))$ yields

$$\det \mathrm{J}_{\mathbb{C}}(\psi)(\eta) = \det \mathrm{J}_{\mathbb{C}}(\psi)(\sigma(\eta)) \cdot \det \mathrm{J}_{\mathbb{C}}(\sigma)(\eta). \tag{13}$$

It follows from the assumption that for $i, j = 1, \ldots, m$,

$$|\det \mathrm{J}_{\mathbb{C}}(\psi)|^2 \circ \phi_j \approx |\det \mathrm{J}_{\mathbb{C}}(\psi)|^2 \circ \phi_i$$

holds almost everywhere on Ω. Therefore, for all $j = 1, \ldots, m$,

$$|\det \mathrm{J}_{\mathbb{C}}(\psi)|^2 \circ \phi_j \approx \delta. \tag{14}$$

5.2 Bergman projection with Sobolev estimates

We will use w, η for the Euclidean coordinates on $(\Delta^n)^*$, and z, ξ for those on Ω. We further use $D^\alpha_{w,\bar{w}}$ and $D^\beta_{z,\bar{z}}$ to represent the derivatives with multi-index α, β on $(\Delta^n)^*$ and Ω, respectively. For simplicity of exposition, we assume that ψ extends smoothly to a open neighborhood of $\overline{\Delta^n}$.

To study the Bergman projection, we make use of an expression of the Bergman kernel of Ω obtained from the Bergman transformation formula of Bell [5].

Lemma 5.3. *Let B_{Δ^n} and B_Ω be the Bergman kernel functions on Δ^n and Ω, respectively. Then for $w, \eta \in (\Delta^n)^*$,*

$$\sum_{\sigma \in G} B_{\Delta^n}(w, \sigma(\eta)) \cdot \left(\overline{\det J_{\mathbb{C}}(\psi)(\sigma(\eta))}\right)^{-1} = \det J_{\mathbb{C}}(\psi)(w) \cdot B_\Omega(\psi(w), \psi(\eta))$$

Proof. Recalling the Bergman kernel transformation formula by Bell [5], we have for $\xi \in \Omega, w \in (\Delta^n)^*$,

$$\sum_{k=1}^m B_{(\Delta^n)^*}(w, \phi_k(\xi)) \cdot \overline{\det \mathrm{J}_{\mathbb{C}}(\phi_k)(\xi)} = \det \mathrm{J}_{\mathbb{C}}(\psi)(w) \cdot B_\Omega(\psi(w), \xi).$$

Note that $B_{(\Delta^n)^*} = B_{\Delta^n}$. Replacing ξ by $\psi(\eta)$, and by (12) and the succeeding explanation, we have

$$\begin{aligned}\det \mathrm{J}_{\mathbb{C}}(\psi)(w) \cdot B_{\Omega}(\psi(w), \psi(\eta)) &= \sum_{k=1}^{m} B_{\Delta^n}(w, \phi_k(\psi(\eta))) \cdot \overline{\det \mathrm{J}_{\mathbb{C}}(\phi_k)(\psi(\eta))} \\ &= \sum_{\sigma \in G} B_{\Delta^n}(w, \sigma(\eta)) \cdot \left(\overline{\det \mathrm{J}_{\mathbb{C}}(\psi)(\sigma(\eta))}\right)^{-1}.\end{aligned}$$

∎

Denote by $\mathcal{B}_{\Omega}$ and $\mathcal{B}_{\Delta^n}$ the Bergman projection operator on Ω and Δ^n, respectively. Recall that $\mathcal{B}_{\Delta^n} = \mathcal{B}_{\Delta} \circ \cdots \circ \mathcal{B}_{\Delta}$, where the j-th element in the composition is the Bergman projection operator on the j-th portion of Δ^n. When $n = 1$,

$$\mathcal{B}_{\Delta} f(z) = \frac{1}{\pi^2} \int_{\Delta} \frac{f(w)}{(1 - z\bar{w})^2} dV(w), \quad f \in L^2(\Delta).$$

We also define

$$\mathcal{B}_{\Delta}^{+} f(z) = \frac{1}{\pi^2} \int_{\Delta} \frac{f(w)}{|1 - z\bar{w}|^2} dV(w).$$

When $n \geq 1$, similarly define $\mathcal{B}_{\Delta^n}^{+} := \mathcal{B}_{\Delta}^{+} \circ \cdots \circ \mathcal{B}_{\Delta}^{+}$.

The following theorem generalizes Theorem 1.1 in [15] to bounded domains in $\mathbb{C}^n$ covered regularly by the polydisc. The key idea of the proof is the holomorphic integration by parts in [7].

Theorem 5.4. *Assume $\mathcal{B}_{\Delta^n}^{+} : L^p(\Delta^n, \mu) \to L^p(\Delta^n, \mu)$ is bounded, $p > 1$. Then for each $k \in \mathbb{Z}^+ \cup \{0\}$,*

$$\|\psi^* \mathcal{B}_{\Omega}(\psi_* f)\|_{W^{k,p}\left(\Delta^n, \mu^{\frac{(k+1)p+2}{2}}\right)} \lesssim \|f\|_{W^{k,p}(\Delta^n, \mu)}$$

for all $f \in W^{k,p}(\Delta^n, \mu)$. If in addition $p \geq 2$, then

$$\|\psi^* \mathcal{B}_{\Omega}(\psi_* f)\|_{W^{k,p}\left(\Delta^n, \mu^{\frac{(k+1)p}{2}}\right)} \lesssim \|f\|_{W^{k,p}(\Delta^n, \mu)}.$$

Proof. Noting $\psi^* \psi_* f(w) = \sum_{\tau \in G} f(\tau(w))$, we have by Lemma 5.3,

$$\begin{aligned}\psi^* \mathcal{B}_{\Omega}\left(\psi_* f\right)(w) &= \int_{\Omega} B_{\Omega}(\psi(w), \xi)\left(\psi_* f\right)(\xi) \mathrm{dV}(\xi) \\ &= \sum_{\tau \in G} \int_{\Delta^n} B_{\Omega}(\psi(w), \psi(\eta)) f(\tau(\eta)) \mu(\eta) \mathrm{dV}(\eta) \\ &= (\det \mathrm{J}_{\mathbb{C}}(\psi)(w))^{-1} \sum_{\sigma, \tau \in G} \int_{\Delta^n} B_{\Delta^n}(w, \sigma(\eta)) \left(\overline{\det \mathrm{J}_{\mathbb{C}}(\psi)(\sigma(\eta))}\right)^{-1} f(\tau(\eta)) \mu(\eta) \mathrm{dV}(\eta).\end{aligned}$$

We first treat the derivative terms. For $0 < |\alpha| \leq k$, since $\left|D_w^\alpha\left((\det \mathrm{J}_{\mathbb{C}}(\psi)(w))^{-1}\right)\right| \lesssim \mu^{-\frac{(k+1)}{2}}(w)$ and $\mu \lesssim 1$, we have

$$
\begin{aligned}
&\int_{\Delta^n} |D_w^\alpha \psi^* \mathcal{B}_\Omega(\psi_* f)|^p(w) \mu^{\frac{(k+1)p+2}{2}}(w) \mathrm{dV}(w) \\
\lesssim& \sum_{|\beta| \leq k} \sum_{\sigma,\tau \in G} \int_{\Delta^n} \left| D_w^\beta \left((\det \mathrm{J}_{\mathbb{C}}(\psi)(w))^{-1} \int_{\Delta^n} B_{\Delta^n}(w, \sigma(\eta)) \left(\overline{\det \mathrm{J}_{\mathbb{C}}(\psi)(\sigma(\eta))}\right)^{-1} f(\tau(\eta)) \mu(\eta) \mathrm{dV}(\eta) \right) \right|^p \\
&\qquad \cdot \mu^{\frac{(k+1)p+2}{2}}(w) \mathrm{dV}(w) \\
\lesssim& \sum_{|\beta| \leq k} \sum_{\sigma,\tau \in G} \int_{\Delta^n} \left| D_w^\beta \left(\int_{\Delta^n} B_{\Delta^n}(w, \sigma(\eta)) \left(\overline{\det \mathrm{J}_{\mathbb{C}}(\psi)(\sigma(\eta))}\right)^{-1} f(\tau(\eta)) \mu(\eta) \mathrm{dV}(\eta) \right) \right|^p (w) \mu(w) \mathrm{dV}(w).
\end{aligned}
$$

On the other hand, it follows from (13) that

$$
\left(\overline{\det \mathrm{J}_{\mathbb{C}}(\psi)(\sigma(\eta))}\right)^{-1} \mu(\eta) = \det \mathrm{J}_{\mathbb{C}}(\psi)(\sigma(\eta)) |\det \mathrm{J}_{\mathbb{C}}(\sigma)(\eta)|^2.
$$

Noting the right hand side is a non-vanishing smooth function on $\overline{\Delta^n}$ by the fact that Ω is covered regularly by the polydisc, we may write

$$
\left(\overline{\det \mathrm{J}_{\mathbb{C}}(\psi)(\sigma(\eta))}\right)^{-1} \mu(\eta) \mathrm{dV}(\eta) =: g(\sigma(\eta)) \mathrm{dV}(\sigma(\eta)),
$$

for some smooth function $g(\sigma(\eta))$ on $\overline{\Delta^n}$. By the change of variables, it follows that

$$
\begin{aligned}
&\int_{\Delta^n} B_{\Delta^n}(w, \sigma(\eta)) \left(\overline{\det \mathrm{J}_{\mathbb{C}}(\psi)(\sigma(\eta))}\right)^{-1} f(\tau(\eta)) \mu(\eta) \mathrm{dV}(\eta) \\
=& \int_{\Delta^n} B_{\Delta^n}(w, \sigma(\eta)) f(\tau(\sigma^{-1}(\sigma(\eta)))) g(\sigma(\eta)) \mathrm{dV}(\sigma(\eta)) \\
=& \int_{\Delta^n} B_{\Delta^n}(w, \eta) f(\tau(\sigma^{-1}(\eta))) g(\eta) \mathrm{dV}(\eta).
\end{aligned}
$$

Given $\eta \in \mathbb{C} \setminus \{0\}$, define a differential operator $T_\eta := \eta \frac{\partial}{\partial \eta} - \bar{\eta} \frac{\partial}{\partial \bar{\eta}}$. Therefore,

$$
\begin{aligned}
&\left| D_w^\beta \left(\int_{\Delta^n} B_{\Delta^n}(w, \sigma(\eta)) \left(\overline{\det \mathrm{J}_{\mathbb{C}}(\psi)(\sigma(\eta))}\right)^{-1} f(\tau(\eta)) \mu(\eta) \mathrm{dV}(\eta) \right) \right| \\
=& \left| \int_{\Delta^n} \left(D_w^\beta B_{\Delta^n}(w, \eta) \right) \left(f(\tau(\sigma^{-1}(\eta))) g(\eta) \right) \mathrm{dV}(\eta) \right| \\
=& \left| \frac{1}{w^\beta} \int_{\Delta^n} B_{\beta_1}(w_1, \eta_1) \cdots B_{\beta_n}(w_n, \eta_n) T_{\eta_1}^{\beta_1} \cdots T_{\eta_n}^{\beta_n} \left(f(\tau(\sigma^{-1}(\eta))) g(\eta) \right) \mathrm{dV}(\eta) \right| \\
\leq& \int_{\Delta^n} B_{\Delta^n}^+(w, \eta) \left| T_{\eta_1}^{\beta_1} \cdots T_{\eta_n}^{\beta_n} \left(f(\tau(\sigma^{-1}(\eta))) g(\eta) \right) \right| \mathrm{dV}(\eta),
\end{aligned}
$$

where the second equality and the inequality follow from (2.15) in [25] (the second equality from Corollary 3.5 in [15] and the inequality from (3.8) in [15] as well). It thus follows from the assumption that

$$\begin{aligned}
&\int_{\Delta^n} \left| D_w^\beta \left(\int_{\Delta^n} B_{\Delta^n}(w,\sigma(\eta)) \left(\overline{\det \mathrm{J}_{\mathbb{C}}(\psi)(\sigma(\eta))} \right)^{-1} f(\tau(\eta))\mu(\eta)\mathrm{dV}(\eta) \right) \right|^p (w)\mu(w)\mathrm{dV}(w) \\
&\le \int_{\Delta^n} \mu(w) \left(\int_{\Delta^n} B^+_{\Delta^n}(w,\eta) \left| T_{\eta_1}^{\beta_1} \cdots T_{\eta_n}^{\beta_n} \left(f(\tau(\sigma^{-1}(\eta)))g(\eta) \right) \right| \mathrm{dV}(\eta) \right)^p \mathrm{dV}(w) \\
&\lesssim \int_{\Delta^n} \left| T_{\eta_1}^{\beta_1} \cdots T_{\eta_n}^{\beta_n} \left(f(\tau(\sigma^{-1}(\eta)))g(\eta) \right) \right|^p \mu(\eta)\mathrm{dV}(\eta) \\
&\lesssim \sum_{|\beta|\le k} \int_{\Delta^n} \left| D_\eta^\beta f(\tau(\sigma^{-1}(\eta))) \right|^p \mu(\eta)\mathrm{dV}(\eta) \lesssim \|f\|^p_{W^{k,p}(\Delta^n,\,\mu)}.
\end{aligned}$$

Finally, for the weighted L^p-norm of $\psi^*\mathcal{B}_\Omega(\psi_* f)$, noting that $|\det \mathrm{J}_{\mathbb{C}}(\psi)(w)|^{-p} = \mu^{-\frac{p}{2}}(w)$,

$$\begin{aligned}
&\int_{\Delta^n} |\psi^*\mathcal{B}_\Omega(\psi_* f)|^p(w)\mu^{\frac{p+2}{2}}(w)\mathrm{dV}(w) \\
&= \sum_{\sigma,\tau\in G} \int_{\Delta^n} \left| \int_{\Delta^n} B_{\Delta^n}(w,\sigma(\eta)) \left(\overline{\det \mathrm{J}_{\mathbb{C}}(\psi)(\sigma(\eta))} \right)^{-1} f(\tau(\eta))\mu(\eta)\mathrm{dV}(\eta) \right|^p \mu(w)\mathrm{dV}(w) \\
&\lesssim \sum_{\sigma,\tau\in G} \int_{\Delta^n} \left| \int_{\Delta^n} B_{\Delta^n}(w,\sigma(\eta)) f(\tau(\eta)) g(\sigma(\eta))\mathrm{dV}(\sigma(\eta)) \right|^p \mu(w)\mathrm{dV}(w) \\
&\lesssim \sum_{\sigma,\tau\in G} \int_{\Delta^n} \left| \int_{\Delta^n} B_{\Delta^n}(w,\eta) f(\tau(\sigma^{-1}(\eta)))\mathrm{dV}(\eta) \right|^p \mu(w)\mathrm{dV}(w) \\
&\lesssim \sum_{\sigma,\tau\in G} \int_{\Delta^n} \left| \int_{\Delta^n} B^+_{\Delta^n}(w,\eta) |f(\tau(\sigma^{-1}(\eta)))|\mathrm{dV}(\eta) \right|^p \mu(w)\mathrm{dV}(w).
\end{aligned}$$

Using the assumption on $\mathcal{B}^+_{\Delta^n}$ and the fact that Ω is covered regularly by Δ^n, we further have

$$\int_{\Delta^n} |\psi^*\mathcal{B}_\Omega(\psi_* f)|^p(w)\mu^{\frac{p+2}{2}}(w)\mathrm{dV}(w) \lesssim \sum_{\sigma,\tau\in G} \|f\circ\tau\circ\sigma^{-1}\|^p_{L^p(\Delta^n,\,\mu)} \lesssim \|f\|^p_{L^p(\Delta^n,\,\mu)},$$

Combining the two parts, the first part of the theorem is proved.

If in addition $p \geq 2$, we only need to estimate the following term when $|\alpha| = k$. In fact,

$$
\begin{aligned}
&\int_{\Delta^n} |D_w^\alpha \psi^* \mathcal{B}_\Omega(\psi_* f)|^p(w) \mu^{\frac{(k+1)p}{2}}(w) \mathrm{dV}(w) \\
\lesssim &\sum_{\sigma,\tau\in G} \int_{\Delta^n} \left(\left| D_w^\alpha \left(\det \mathrm{J}_{\mathbb{C}}(\psi)(w)\right)^{-1} \int_{\Delta^n} B_{\Delta^n}(w,\eta) f(\tau(\sigma^{-1}(\eta))) g(\eta) \mathrm{dV}(\eta) \right|^p \mu^{\frac{(k+1)p}{2}}(w) \mathrm{dV}(w) \right. \\
&+ \left. \sum_{|\alpha'|\leq k-1, \alpha'+\alpha''=\alpha} \left| D_w^{\alpha'} \left(\det \mathrm{J}_{\mathbb{C}}(\psi)(w)\right)^{-1} D_w^{\alpha''} \int_{\Delta^n} B_{\Delta^n}(w,\eta) f(\tau(\sigma^{-1}(\eta))) g(\eta) \mathrm{dV}(\eta) \right|^p \mu^{\frac{(k+1)p}{2}}(w) \mathrm{dV}(w) \right) \\
\lesssim &\sum_{\sigma,\tau\in G} \int_{\Delta^n} \left| \int_{\Delta^n} B_{\Delta^n}(w,\eta) f(\tau(\sigma^{-1}(\eta))) g(\eta) \mathrm{dV}(\eta) \right|^p \mathrm{dV}(w) \\
&+ \sum_{\sigma,\tau\in G} \int_{\Delta^n} \mu^{\frac{p}{2}}(w) \sum_{|\alpha''|\leq k} \left| D_w^{\alpha''} \int_{\Delta^n} B_{\Delta^n}(w,\eta) f(\tau(\sigma^{-1}(\eta))) g(\eta) \mathrm{dV}(\eta) \right|^p \mathrm{dV}(w) := I + II.
\end{aligned}
$$

The term II is handled similarly as before, in view of the fact that $\mu^{\frac{p}{2}} \lesssim \mu$ when $p \geq 2$. For the term I, we observe that

$$
\begin{aligned}
I &\lesssim \sum_{\sigma,\tau\in G} \int_{\Delta^n} \left| f(\tau(\sigma^{-1}(\eta))) g(\eta) \right|^p \mathrm{dV}(\eta) \\
&\lesssim \sum_{\sigma,\tau\in G} \int_{\Delta^n} \left| f(\tau(\sigma^{-1}(\eta))) \right|^p \mu(\eta) \mathrm{dV}(\eta) \lesssim \|f\|^p_{W^{k,p}(\Delta^n,\mu)}.
\end{aligned}
$$

Here we used the known unweighted boundedness of $\mathcal{B}_{\triangle^n}$ in $L^p(\triangle^n)$ in the first inequality, and the fact that $|g|^p \approx \mu^{\frac{p}{2}} \lesssim \mu$ when $p \geq 2$ in the second inequality. The proof is complete.

∎

Next, we study the boundedness assumption on $\mathcal{B}^+_{\Delta^n}$ in Theorem 5.4. Given $z \in \Delta$, the Carleson tent over z is defined to be

$$
T_z := \left\{ w \in \Delta : \left| 1 - \bar{w} \frac{z}{|z|} \right| < 1 - |z| \right\},
$$

and the Carleson tent over 0 is Δ. It was shown in [4, 37] that

$$
\|\mathcal{B}^+_\Delta : L^p(\Delta,\mu) \to L^p(\Delta,\mu)\| \lesssim B_p(\mu), \tag{15}
$$

where

$$
B_p(\mu) := \sup_{z\in\Delta} \frac{1}{|T_z|} \int_{T_z} \mu(w) dV(w) \left(\frac{1}{|T_z|} \int_{T_z} \mu^{\frac{1}{1-p}}(w) dV(w) \right)^{p-1}.
$$

The following proposition shows that the assumption on the boundedness of $\mathcal{B}^+_{\Delta^n}$ in $L^p(\Delta^n,\mu)$ can be warranted whenever $\mu \in A^*_p$.

Proposition 5.5. *Suppose $\mu \in A_p^*$, $p > 1$. Then $\mathcal{B}_{\Delta^n}^+ : L^p(\Delta^n, \mu) \to L^p(\Delta^n, \mu)$ is bounded. In particular, the Bergman projection operator $\mathcal{B}_{\Delta^n} : L^p(\Delta^n, \mu) \to L^p(\Delta^n, \mu)$ is bounded.*

Proof. When $n = 1$, a direct computation shows that $T_z = \left\{ w \in \Delta : \left| w - \frac{z}{|z|} \right| < 1 - |z| \right\}$, for any $z \in \Delta$. Let B_z be the disc in $\mathbb{C}$ centered at $z/|z|$ with radius $1 - |z|^2$. Then $T_z \subset B_z$ and

$$|T_z| \approx (1 - |z|)^2 \approx |B_z|.$$

Consequently, by definition of B_p and A_p, we have

$$\begin{aligned} B_p(\mu) &\leq \sup_{z \in \Delta} \frac{1}{|T_z|} \int_{B_z} \mu(w) dV(w) \left(\frac{1}{|T_z|} \int_{B_z} \mu^{\frac{1}{1-p}}(w) dV(w) \right)^{p-1} \\ &\approx \sup_{z \in \Delta} \frac{1}{|B_z|} \int_{B_z} \mu(w) dV(w) \left(\frac{1}{|B_z|} \int_{B_z} \mu^{\frac{1}{1-p}}(w) dV(w) \right)^{p-1} \leq A_p(\mu). \end{aligned}$$

Hence for all $f \in L^p(\Delta, \mu)$, by (15) we have

$$\left\| \mathcal{B}_\Delta^+ f \right\|_{L^p(\Delta, \mu)} \lesssim \|f\|_{L^p(\Delta, \mu)}. \tag{16}$$

When $n \geq 2$, write $(z', z_n) \in \Delta^{n-1} \times \Delta$. For any $f \in L^p(\Delta^n, \mu)$,

$$\left\| \mathcal{B}_{\Delta^n}^+ f \right\|_{L^p(\Delta^n,\mu)}^p = \int_{\Delta^{n-1}} \int_\Delta \left| \mathcal{B}_\Delta^+ \circ \left(\mathcal{B}_{\Delta^{n-1}}^+ f(z', z_n) \right) \right|^p \mu(z', z_n) dV(z_n) dV(z').$$

Since $\mu \in A_p^*$, $\mu(z', \cdot) \in A_p$ for almost all fixed $z' \in \Delta^{n-1}$. Noting that $f(z', \cdot) \in L^p(\Delta, \mu(z', \cdot))$ for almost all fixed $z' \in \Delta^{n-1}$, we obtain from (16) that

$$\int_\Delta \left| \mathcal{B}_\Delta^+ \circ \left(\mathcal{B}_{\Delta^{n-1}}^+ f(z', z_n) \right) \right|^p \mu(z', z_n) dV(z_n) \lesssim \int_\Delta \left| \left(\mathcal{B}_{\Delta^{n-1}}^+ f(z', z_n) \right) \right|^p \mu(z', z_n) dV(z_n).$$

We thus have

$$\left\| \mathcal{B}_{\Delta^n}^+ f \right\|_{L^p(\Delta^n, \mu)}^p \lesssim \int_{\Delta^n} | \left(\mathcal{B}_{\Delta^{n-1}}^+ f(z', z_n) \right) |^p \mu(z', z_n) dV(z).$$

A standard induction gives the desired boundedness of $\mathcal{B}_{\Delta^n}^+$ in $L^p(\Delta^n, \mu)$. ∎

Combining Theorem 5.4 with Proposition 5.5, we immediately obtain

Corollary 5.6. *Assume $\mu \in A_p^*, p > 1$. Then for each $k \in \mathbb{Z}^+ \cup \{0\}$,*

$$\|\psi^* \mathcal{B}_\Omega(\psi_* f)\|_{W^{k,p}\left(\Delta^n, \mu^{\frac{(k+1)p+2}{2}}\right)} \lesssim \|f\|_{W^{k,p}(\Delta^n, \mu)}$$

for all $f \in W^{k,p}(\Delta^n, \mu)$. If in addition $p \geq 2$, then

$$\|\psi^* \mathcal{B}_\Omega(\psi_* f)\|_{W^{k,p}\left(\Delta^n, \mu^{\frac{(k+1)p}{2}}\right)} \lesssim \|f\|_{W^{k,p}(\Delta^n, \mu)}.$$

Remark 5.7. When $\det \mathrm{J}_{\mathbb{C}}(\psi)(w) = w^J$ for multi-index $J \in \mathbb{Z}_+^n$, one may obtain a better range for p. The key difference is that, in this case, we get an additional $\mu^{\frac{p}{2}}$ on the fourth line of (5.2). For example, when $\Omega = \mathbb{H}_{1,1}$ is the Hartogs triangle, since $|w_2|^{2-p} \in A_p^*$ for any $p > \frac{4}{3}$ by Example 2.3, by Proposition 5.5 and the argument in Theorem 5.4, one has

$$\|\psi^* \mathcal{B}_{\mathbb{H}_{1,1}}(\psi_* f)\|_{W^{k,p}\left(\Delta^n,\, \mu^{\frac{(k-1)p+4}{2}}\right)} \lesssim \|f\|_{W^{k,p}(\Delta^n,\, \mu)}$$

for any $p > \frac{4}{3}$. This is consistent with the estimates in [13].

5.3 Solving $\bar{\partial}$ with weighted Sobolev estimates

In order to use the integral representation on Δ^n to solve $\bar{\partial}$ on Ω, we first state two lemmas to transfer data between Ω and $(\Delta^n)^*$ (cf. [10, 14, 15]).

Lemma 5.8. *For any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_j \geq 0$ for all j and $|\alpha| := \sum_{j=1}^n \alpha_j \geq 1$,*

$$D_z^\alpha = (\det J_{\mathbb{C}}(\psi))^{-2|\alpha|+1} \sum_{1 \leq |\beta| \leq |\alpha|} P_{\alpha,\beta}(w) D_w^\beta \tag{17}$$

where $P_{\alpha,\beta}(w)$ are bounded holomorphic functions on Δ^n;

$$D_{z,\bar{z}}^\alpha = \sum_{1 \leq |\beta| \leq |\alpha|} (\det J_{\mathbb{C}}(\psi))^{-s_{\alpha,\beta,1}} \left(\overline{\det J_{\mathbb{C}}(\psi)}\right)^{-s_{\alpha,\beta,2}} \hat{P}_{\alpha,\beta}(w, \bar{w}) D_{w,\bar{w}}^\beta \tag{18}$$

with $s_{\alpha,\beta,1} + s_{\alpha,\beta,2} \leq 2|\alpha| - 1$, where $\hat{P}_{\alpha,\beta}(w,\bar{w})$ are bounded smooth functions on Δ^n; On the other hand, for any multi-index $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$,

$$D_{w,\bar{w}}^\beta = \sum_{0 \leq |\alpha| \leq |\beta|} \tilde{P}_{\alpha,\beta}(w, \bar{w}) \frac{\partial^\alpha}{\partial z_1^{\alpha_1} \partial \bar{z}_1^{\alpha_2} \partial z_2^{\alpha_3} \partial \bar{z}_2^{\alpha_4}}, \tag{19}$$

where $\tilde{P}_{\alpha,\beta}(w,\bar{w})$ are bounded smooth functions on Δ^n.

Proof. The proof is similar to the proof of Lemma 3.2 in [15] and we only point out the difference here. By the holomorphic change of variables under ψ, we have

$$\frac{\partial}{\partial w_j} = \frac{\partial \psi_k}{\partial w_j} \frac{\partial}{\partial z_k},$$

for all $1 \leq j, k \leq n$. It then follows that

$$\frac{\partial}{\partial z_k} = \left(\frac{\partial \psi_k}{\partial w_j}\right)^{-1} \frac{\partial}{\partial w_j} = (\mathrm{J}_{\mathbb{C}}(\psi))^{-1} \mathrm{J}_{\mathbb{C}}(\psi) \left(\frac{\partial \psi_k}{\partial w_j}\right)^{-1} \frac{\partial}{\partial w_j},$$

for all $1 \leq j, k \leq n$, where $\left(\frac{\partial \psi_k}{\partial w_j}\right)^{-1}$ is the inverse matrix of matrix $\left(\frac{\partial \psi_k}{\partial w_j}\right)$. By the assumption of ψ, $\mathrm{J}_{\mathbb{C}}(\psi)\left(\frac{\partial \psi_k}{\partial w_j}\right)^{-1}$ extends smoothly to an open neighborhood of $\overline{\Delta^n}$. Therefore, (17) follows from induction. (18) and (19) follows directly from the standard Faà di Bruno's formula [21].

∎

Lemma 5.9. *Let Ω be a bounded domain covered regularly by Δ^n. Then for $l \in \mathbb{Z}^+ \cup \{0\}$,*

- *ψ^* maps the space of $(0,1)$-forms $W^{k,p}_{(0,1)}(\Omega, \delta^l)$ continuously and injectively into $W^{k,p}_{(0,1)}((\Delta^n)^*, \mu^{l+1})$ for each $k \in \mathbb{Z}^+ \cup \{0\}$.*
- *ψ_* maps the space of functions $L^p((\Delta^n)^*, \mu^{l+1}))$ continuously and injectively into $L^p(\Omega, \delta^l)$ and $W^{k,p}((\Delta^n)^*, \mu^{l+1}))$ continuously and injectively into $W^{k,p}\left(\Omega, \delta^{\frac{(2k-1)p}{2}+l}\right)$ for each $k \in \mathbb{Z}^+$.*

Proof. For any $k \geq 0$, given $g = \sum_{j=1}^n g_j d\bar{z}^j \in W^{k,p}_{(0,1)}(\Omega, \delta^l)$, we have $\psi^* g = \sum_{i,j=1}^n g_j \circ \psi \frac{\partial \bar{\psi}_j}{\partial \bar{w}_i} d\bar{w}_i$. Thus by the change of variables and (19),

$$
\begin{aligned}
\|\psi^* g\|^p_{W^{k,p}_{(0,1)}((\Delta^n)^*, \mu^{l+1})} &= \sum_{|\beta| \leq k} \sum_{i,j=1}^n \int_{(\Delta^n)^*} \left| D^\beta_{w,\bar{w}} \left(g_j \circ \psi \frac{\partial \bar{\psi}_j}{\partial \bar{w}_i} \right) \right|^p |\det \mathrm{J}_{\mathbb{C}}(\psi)|^{2l+2} \mathrm{dV}(w) \\
&\lesssim \sum_{|\alpha| \leq k} \sum_{j=1}^n \int_\Omega \left| D^\alpha_{z,\bar{z}}(g_j) \right|^p |\delta|^{2l} \mathrm{dV}(z) = \|g\|^p_{W^{k,p}_{(0,1)}(\Omega, \delta^l)}.
\end{aligned} \tag{20}
$$

On the other hand, given $f \in W^{k,p}\left((\Delta^n)^*, \mu^{l+1}\right)$, it follows from (18) that for $k \geq 1$,

$$
\begin{aligned}
\|\psi_* f\|_{W^{k,p}\left(\Omega, \delta^{\frac{(2k-1)p}{2}+l}\right)} &= \sum_{|\alpha| \leq k} \int_\Omega \left| D^\alpha_{z,\bar{z}} (\psi_* f) \right|^p |\delta|^{\frac{(2k-1)p}{2}+l} \mathrm{dV}(z) \\
&\lesssim \sum_{|\beta| \leq k} \int_{(\Delta^n)^*} \left| D^\beta_{w,\bar{w}} (\psi^* \psi_* f) \right|^p |\mu|^{l+1} \mathrm{dV}(w) \\
&\lesssim \sum_{\sigma \in G} \sum_{|\beta| \leq k} \int_{(\Delta^n)^*} \left| D^\beta_{w,\bar{w}} (f \circ \sigma) \right|^p |\mu|^{l+1} \mathrm{dV}(w) \lesssim \|f\|_{W^{k,p}((\Delta^n)^*, \mu^{l+1})};
\end{aligned}
$$

similarly for $k = 0$,

$$
\|\psi_* f\|_{L^p\left(\Omega, \delta^l\right)} = \int_\Omega |\psi_* f|^p |\delta|^l \mathrm{dV}(z) \lesssim \sum_{\sigma \in G} \int_{(\Delta^n)^*} |f \circ \sigma|^p |\mu|^{l+1} \mathrm{dV}(w) \lesssim \|f\|_{L^p((\Delta^n)^*, \mu^{l+1})}.
$$

Here the last inequality in both cases follows from (14) and the fact that σ extends smoothly to $\overline{\Delta^n}$.

∎

In order to apply the weighted Sobolev theory of $\bar{\partial}$ on $\triangle^n$ for the quotient domain Ω, we also need to verify that, if the datum $g \in W^{1,p}_{(0,1)}(\Omega)$ is $\bar{\partial}$-closed, then $\psi^* g$ is $\bar{\partial}$-closed on Δ^n. This is not immediately clear by definition of Ω since $\psi^* g$ is only pulled back onto $(\Delta^n)^*$. The following proposition justifies this to be true when $\mu \in A_p$.

Proposition 5.10. *For each $k \in \mathbb{Z}^+$, let $f = \sum_{j=1}^n f_j d\bar{w}^j \in W^{k,p}_{(0,1)}((\Delta^n)^*, \mu)$ be $\bar{\partial}$-closed on $(\Delta^n)^*$. If $(\Delta^n)^*$ is a uniform domain and $\mu \in A_p, p > 1$. Then f extends as a $\bar{\partial}$-closed $(0,1)$-form on Δ^n.*

Proof. According to Theorem 1.1 [20], one can obtain an extension of f, still denoted by f with $f \in W^{1,p}(\Delta^n, \mu)$. In particular, for all $1 \le j, k \le n$, the weak derivatives $\frac{\partial f_j}{\partial \bar{w}_k} \in L^p(\Delta^n, \mu)$. We only need to show that f is $\bar{\partial}$-closed on Δ^n.

Let χ be smooth $(n, n-2)$-form in Δ^n with compact support. Then

$$\int_{\Delta^n} \bar{\partial} f \wedge \chi = \sum_{1 \le j < k \le n} \int_{\Delta^n} \left(\frac{\partial f_k}{\partial \bar{z}_j} - \frac{\partial f_j}{\partial \bar{z}_k} \right) \tilde{\chi}_{jk} = \sum_{1 \le j < k \le n} \int_{\Delta^n} \left(\frac{\partial f_k}{\partial \bar{z}_j} - \frac{\partial f_j}{\partial \bar{z}_k} \right) \mu^{\frac{1}{p}} \cdot \mu^{-\frac{1}{p}} \cdot \tilde{\chi}_{jk},$$

where $\tilde{\chi}_{jk}$ is the coefficient of χ with respect to $dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$ with $d\bar{z}^j \wedge d\bar{z}^k$ omitted. Here $\left(\frac{\partial f_k}{\partial \bar{z}_j} - \frac{\partial f_j}{\partial \bar{z}_k} \right) \mu^{\frac{1}{p}} \in L^p(\Delta^n)$, as $f \in W^{1,p}(\Delta^n, \mu)$. Since $\mu \in A_p$, we have $\mu \in A_{\tilde{p}}$ for some $1 < \tilde{p} < p$. Thus $\mu^{-\frac{1}{\tilde{p}-1}} \in L^1(\Delta^n)$, or equivalently, $\mu^{-\frac{1}{p}} \in L^{\frac{p}{\tilde{p}-1}}(\Delta^n)$.

Let $q > 1$ be such that $\frac{1}{p} + \frac{\tilde{p}-1}{p} + \frac{1}{q} = 1$. Let χ_ϵ be a sequence of smooth $(n, n-2)$-forms in Δ^n with compact support in $(\Delta^n)^*$, such that $(\tilde{\chi}_\epsilon)_{jk} \to \chi_{jk}$ in $L^q(\Delta^n)$ as $\epsilon \to 0$. By Hölder inequality,

$$\begin{aligned} &\left\| \left(\frac{\partial f_k}{\partial \bar{z}_j} - \frac{\partial f_j}{\partial \bar{z}_k} \right) \mu^{\frac{1}{p}} \cdot \mu^{-\frac{1}{p}} \cdot ((\tilde{\chi}_\epsilon)_{jk} - \chi_{jk}) \right\|_{L^1(\Delta^n)} \\ \le &\left\| \left(\frac{\partial f_k}{\partial \bar{z}_j} - \frac{\partial f_j}{\partial \bar{z}_k} \right) \mu^{\frac{1}{p}} \right\|_{L^p(\Delta^n)} \left\| \mu^{-\frac{1}{p}} \right\|_{L^{\frac{p}{\tilde{p}-1}}(\Delta^n)} \|(\tilde{\chi}_\epsilon)_{jk} - \chi_{jk}\|_{L^q(\Delta^n)} \to 0 \end{aligned}$$

as $\epsilon \to 0$. Hence

$$\int_{\Delta^n} \bar{\partial} f \wedge \chi = \lim_{\epsilon \to 0} \int_{\Delta^n} \bar{\partial} f \wedge \chi_\epsilon,$$

which is 0 by the $\bar{\partial}$-closedness of f on $(\Delta^n)^*$. Therefore, $\bar{\partial} f = 0$ in Δ^n in the sense of currents. ∎

The next proposition allows us to transfer the solution of $\bar{\partial}$ on Δ^n to that on Ω.

Proposition 5.11. *Let $(\Delta^n)^*$ be a uniform domain and $\mu \in A_p, p > 1$. For each $k \in \mathbb{Z}^+$, let $g \in W^{k,p}_{(0,1)}(\Omega)$ be a $\bar{\partial}$-closed $(0,1)$ form. If T is a solution operator of $\bar{\partial}$ on Δ^n, then $\frac{1}{m} \bar{\partial} \psi_*(T(\psi^* g)) = g$ on Ω.*

Proof. Suppose that χ is a smooth $(n, n-1)$-form with compact support in Ω.

$$\begin{aligned}\int_\Omega \bar\partial\psi_*(T(\psi^*g))\wedge\chi &= -\int_\Omega \psi_*(T(\psi^*g))\wedge\bar\partial\chi = -\int_{(\Delta^n)^*} T(\psi^*g)\wedge\psi^*(\bar\partial\chi)\\ &= -\int_{(\Delta^n)^*} T(\psi^*g)\wedge\bar\partial\psi^*(\chi) = \int_{(\Delta^n)^*}\bar\partial T(\psi^*g)\wedge\psi^*(\chi)\\ &= \int_{(\Delta^n)^*}\psi^*g\wedge\psi^*(\chi) = \int_{(\Delta^n)^*}\psi^*(g\wedge\chi) = m\int_\Omega g\wedge\chi,\end{aligned}$$

where the second equality in line 2 follows from that ψ is proper (so $\psi^*(\chi)$ has compact support on $(\Delta^n)^*$), and the first equality in line 3 holds due to Proposition 5.10. ∎

Now we are ready to prove the main theorem.

Proof of Theorem 1.1. It follows from Lemma 5.9 and Proposition 5.10 that $\psi^*g \in W^{k,p}_{(0,1)}(\Delta^n,\mu)$ is $\bar\partial$-closed and $\|\psi^*g\|_{W^{k,p}_{(0,1)}(\Delta^n,\mu)} \lesssim \|g\|_{W^{k,p}_{(0,1)}(\Omega)}$. By Theorem 4.1 and Lemma 5.9,

$$\|T(\psi^*g)\|_{W^{k-n+1,p}(\Delta^n,\mu)} \lesssim \|\psi^*g\|_{W^{k,p}_{(0,1)}(\Delta^n,\mu)} \lesssim \|g\|_{W^{k,p}_{(0,1)}(\Omega)}.$$

Combining Lemma 5.9 and Proposition 5.11, $v = \frac{1}{m}\psi_*(T(\psi^*g))$ solves $\bar\partial v = g$ on Ω and satisfies the weighted Sobolev estimates

$$\|v\|_{W^{k-n+1,p}(\Omega,\delta^l)} \lesssim \|T(\psi^*g)\|_{W^{k-n+1,p}\left(\Delta^n,\mu^{l+1-\frac{(2(k-n+1)-1)p}{2}}\right)} \lesssim \|T(\psi^*g)\|_{W^{k-n+1,p}(\Delta^n,\mu)} \lesssim \|g\|_{W^{k,p}_{(0,1)}(\Omega)}.$$

Here we used the assumption that $l \geq \frac{(2k-2n+1)p}{2}$ in the second inequality.

For the estimate of the canonical solution, note that $T(\psi^*g) \in L^p(\Delta^n,\mu)$ and thus $v = \frac{1}{m}\psi_*(T(\psi^*g)) \in L^p(\Omega)$ by Lemma 5.9. Therefore $u = v - \mathcal{B}_\Omega v$ is the canonical solution of $\bar\partial u = g$. Also, since

$$u = \frac{1}{m}\psi_*\left(T\left(\psi^*g\right) - \psi^*\mathcal{B}_\Omega\left(\frac{1}{m}\psi_*T(\psi^*g)\right)\right),$$

it follows from Lemma 5.9 similarly and Corollary 5.6 that

$$\begin{aligned}\|u\|_{W^{k-n+1,p}\left(\Omega,\delta^{\frac{3(k-n+1)p}{2}}\right)} &\lesssim \left\|T\left(\psi^*g\right) - \psi^*\mathcal{B}_\Omega\left(\frac{1}{m}\psi_*T(\psi^*g)\right)\right\|_{W^{k-n+1,p}\left(\Delta^n,\mu^{\frac{(k-n+2)p}{2}+1}\right)}\\ &\lesssim \|T(\psi^*g)\|_{W^{k-n+1,p}(\Delta^n,\mu)}\\ &\lesssim \|g\|_{W^{k,p}_{(0,1)}(\Omega)}.\end{aligned}$$

∎

6 Examples

6.1 Generalized Hartogs triangles

For $l_j \in \mathbb{Z}^+$ with $\gcd(l_1, \cdots, l_n) = 1$, let

$$\Omega = \mathbb{H}_{l_1,\cdots,l_n} = \{z \in \mathbb{C}^n : |z_1|^{l_1} < \cdots < |z_n|^{l_n} < 1\}$$

be the generalized Hartogs triangle introduced in [6] (see also [41] and a special case with $l_1 = \cdots = l_n = 1$ in [14]), where the L^p boundedness of the Bergman projection is investigated in [6]. Let $k_j \in \mathbb{Z}^+$ with $k_1 l_1 = \cdots = k_n l_n$ and $\gcd(k_1, \cdots, k_n) = 1$. There exists a proper, surjective holomorphic map $\psi : \Delta \times (\Delta^*)^{n-1} \to \mathbb{H}_{l_1,\cdots,l_n}$ given by $\psi(w) = \left((w_1 \cdots w_n)^{k_1}, (w_2 \cdots w_n)^{k_2}, \cdots, w_n^{k_n}\right)$. Direct calculation shows $\mathrm{J}_{\mathbb{C}}(\psi)(w) = \prod_{i=1}^n k_i \cdot w_1^{k_1-1} w_2^{k_1+k_2-1} \cdots w_n^{k_1+k_2+\cdots+k_n-1}$ and the degree of ψ is $m := \prod_{j=1}^n k_j$. One may thus verify that Ω is covered regularly by Δ^n since G consists of rotations of the form $\sigma = (e^{\sqrt{-1}\theta_1}, \cdots, e^{\sqrt{-1}\theta_n})$, for $(\theta_1, \cdots, \theta_n) \in \mathbb{R}^n$ depending on $(k_1, \cdots, k_n)$. Note that Δ and Δ^* are uniform domains and it follows that $\Delta \times (\Delta^*)^{n-1}$ is also a uniform domain. Let $\delta = \frac{1}{m}\psi_*\mu$. In the case when Ω is the Hartogs triangle $\mathbb{H}_{1,1}$, $\psi(w) = (w_1 w_2, w_2) : \Delta \times \Delta^* \to \mathbb{H}_{1,1}$ is a biholomorphism and thus $\delta = |z_2|^2$ by a direct calculation. We obtain the following estimates for the canonical solution on $\mathbb{H}_{l_1,\cdots,l_n}$, from which Corollary 1.2 follows.

Theorem 6.1. *Let $p > \sum_{1\le j\le n} k_j$. For any $\bar\partial$-closed $(0,1)$-form $g \in W^{k,p}_{(0,1)}(\mathbb{H}_{l_1,\cdots,l_n})$ on $\mathbb{H}_{l_1,\cdots,l_n}$ with integer $k \ge n-1$, the canonical solution u of $\bar\partial u = g$ is in* $W^{k-n+1,p}\left(\mathbb{H}_{l_1,\cdots,l_n}, \delta^{\frac{3(k-n+1)p}{2}}\right)$ *and satisfies*

$$\|u\|_{W^{k-n+1,p}\left(\mathbb{H}_{l_1,\cdots,l_n}, \delta^{\frac{3(k-n+1)p}{2}}\right)} \lesssim \|g\|_{W^{k,p}_{(0,1)}(\mathbb{H}_{l_1,\cdots,l_n})}.$$

Proof. Let $\tilde m := \sum_{1\le j\le n} k_j$. Then $\tilde m \ge 2$. By Theorem 1.1, it boils down to show that $\mu = |\mathrm{J}_{\mathbb{C}}(\psi)|^2 \in A_p^*$ if $p > \tilde m$. For any fixed $\hat w_j \in \mathbb{C}^{n-1}$ such that neither of its components is zero, and for all discs $B \subset \mathbb{C}$, we notice

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B \mu(w) dV(w_j)\right)\left(\frac{1}{|B|}\int_B \mu(w)^{\frac{1}{1-p}} dV(w_j)\right)^{p-1}\\ =&\left(\frac{1}{|B|}\int_B |w_j|^{2\tilde m_j - 2} dV(w_j)\right)\left(\frac{1}{|B|}\int_B |w_j|^{\frac{2\tilde m_j-2}{1-p}} dV(w_j)\right)^{p-1},\end{aligned}$$

where $\tilde m_j = \sum_{1\le i\le j} k_i \ge 1$. By Example 2.3 the right hand side is uniformly bounded if and only if $2\tilde m_j - 2 < 2(p-1)$, or equivalently, $p > \tilde m_j$. Since $\tilde m_j \le \tilde m$ for $j = 1, \ldots, n$ and $\tilde m_n = \tilde m$, we have $\mu \in A_p^*$ if $p > \tilde m$.

∎

Remark 6.2. By modifying the argument in the previous section, one may also obtain the weighted Sobolev estimates of the canonical solution of $\bar\partial$ on the bounded monomial polyhedrons in [6], which is a much more general quotient domains of Δ^n than the generalized Hartogs triangles.

6.2 Symmetrized polydiscs

The n-dimensional symmetrized polydisc is defined by

$$\Omega = \mathbb{G}^n = \{z = (p_1(w), p_2(w), \ldots, p_n(w)) \in \mathbb{C}^n \, : \, w \in \Delta^n\},$$

with p_j being symmetric polynomials given by

$$p_1(w) = \sum_{j=1}^{n} w_j, \quad p_2(w) = \sum_{j<k} w_j w_k, \quad \cdots, \quad p_n(w) = w_1 w_2 \cdots w_n.$$

One may verify that $\psi(w) = (p_1(w), p_2(w), \cdots, p_n(w)) : \Delta^n \to \mathbb{G}^n$ is a surjective proper holomorphic map. Moreover, Ω is covered regularly by Δ^n with G being the permutation group S_n with action of $\sigma \in G$ on $\mathbb{G}^n$ given by $\sigma(z_1, \cdots, z_n) = (z_{\sigma(1)}, \cdots, z_{\sigma(n)})$ for any $(z_1, \cdots, z_n) \in \mathbb{G}^n$ and the degree of ψ is $m = n!$. It is obtained in [26] (cf. also [16]) that $\mathrm{J}_{\mathbb{C}}(\psi)(w) = \prod_{1 \le j < k \le n}(w_j - w_k)$. Therefore, $\delta = \frac{1}{n!}\psi_* \left(\prod_{1 \le j < k \le n} |w_j - w_k|^2\right)$.

Proof of Corollary 1.3: Again we just need to verify that $\mu = |\mathrm{J}_{\mathbb{C}}(\psi)|^2 \in A_p^*$ when $p > n$. Fixing $\hat{w}_1 = (w_2, \ldots, w_n) \in \mathbb{C}^{n-1}$ such that all components are mutually distinct, and a disc $B \subset \mathbb{C}$, μ is reduced to the function $c \prod_{1 < j \le n} |w_1 - w_j|^2$ of w_1 for some nonzero constant $c > 0$. By Hölder inequality

$$\int_B \mu(w) dV(w_1) = c \prod_{1<j\le n} \left(\int_B |w_1 - w_j|^{2(n-1)} dV(w_1) \right)^{\frac{1}{n-1}};$$

$$\int_B \mu(w)^{\frac{1}{1-p}} dV(w_1) = c \prod_{1<j\le n} \left(\int_B |w_1 - w_j|^{\frac{2(n-1)}{1-p}} dV(w_1) \right)^{\frac{1}{n-1}}.$$

Thus

$$\left(\frac{1}{|B|} \int_B \mu(w) dV(w_1) \right) \left(\frac{1}{|B|} \int_B \mu(w)^{\frac{1}{1-p}} dV(w_1) \right)^{p-1}$$
$$= \prod_{1<j\le n} \left(\frac{1}{|B|} \int_B |w_1 - w_j|^{2(n-1)} dV(w_1) \left(\frac{1}{|B|} \int_B |w_1 - w_j|^{\frac{2(n-1)}{1-p}} dV(w_1) \right)^{p-1} \right)^{\frac{1}{n-1}}.$$

By Example 2.3 again, the last term is uniformly bounded if and only if $2(n-1) < 2(p-1)$. Namely, $\mu(\cdot, \hat{w}_1) \in A_p$ in $\mathbb{C}$ with a uniformly A_p constant when $p > n$. The rest of the cases is due to the symmetry of μ.

∎

6.3 L^p boundness of Bergman projection operators

It was shown in [16] that $\mathcal{B}_\Omega$ is bounded in $L^p(\Omega)$ provided that $\mathcal{B}_{\Delta^n}$ is bounded from $L^p(\Delta^n, \mu)$ into itself with $\mu = |\det \mathrm{J}_{\mathbb{C}}(\psi)|^{2-p}$. Combined with this result, our Proposition 5.5 readily applies to obtain the following boundedness result of the Bergman projection operator on Ω.

Corollary 6.3. *Let $\psi : (\Delta^n)^* \to \Omega$ be a surjective proper holomorphic map. Suppose $|\det J_{\mathbb{C}}(\psi)|^{2-p} \in A_p^*$, $p > 1$. Then $\mathcal{B}_\Omega$ is bounded from $L^p(\Omega)$ into itself.*

Since the condition $|\det \mathrm{J}_{\mathbb{C}}(\psi)|^{2-p} \in A_p^*$ can be checked in many cases fairly easily, we immediately obtain boundedness of the Bergman projection operator in the following examples, recovering these interesting known cases. In fact, the L^p boundedness of $\mathcal{B}_{\mathbb{H}_{l_1,\cdots,l_n}}$ is originally due to [6, 41] (with totally different approaches); the L^p boundedness of $\mathcal{B}_{\mathbb{G}^n}$ is mentioned in a remark in [15] without detailed proof. We believe the method also applies to the bounded monomial polyhedrons studied in [6] and leave the detail to interested readers.

Corollary 6.4. *$\mathcal{B}_\Omega$ is bounded from $L^p(\Omega)$ into itself*

- *if $p \in (\frac{2\tilde{m}}{\tilde{m}+1}, \frac{2\tilde{m}}{\tilde{m}-1})$ for $\Omega = \mathbb{H}_{l_1,\cdots,l_n}$ with $\tilde{m} = \sum_{1\le j\le n} k_j$;*
- *if $p \in (\frac{2n}{n+1}, \frac{2n}{n-1})$ for $\Omega = \mathbb{G}^n$.*

Proof. On $\mathbb{H}_{l_1,\cdots,l_n}$, $|\det \mathrm{J}_{\mathbb{C}}(\psi)|^{2-p} \in A_p^*$ if and only if $-2 < (\tilde{m}-1)(2-p) < 2(p-1)$, equivalently, if and only if $p \in (\frac{2\tilde{m}}{\tilde{m}+1}, \frac{2\tilde{m}}{\tilde{m}-1})$. On $\mathbb{G}^n$, $|\det \mathrm{J}_{\mathbb{C}}(\psi)|^{1-\frac{p}{2}} \in A_p^*$ if and only if $-2 < (n-1)(2-p) < 2(p-1)$. The desired boundedness interval for p follows. ∎

References

[1] Agler J.; Lykova Z. A.; Young N. J.: *Algebraic and geometric aspects of rational Γ-inner functions*, Adv. Math. **328** (2018), 133–159.

[2] Agler J.; Young N. J.: *Operators having the symmetrized bidisc as a spectral set*, Proc. Edinburgh Math. Soc. (2) **43** (2000), no. 1, 195–210.

[3] J. Agler J.; Young N. J.: *The hyperbolic geometry of the symmetrized bidisc*, J. Geom. Anal. **14** (2004), no. 3, 375–403.

[4] Bekollé, D. and Bonami, A.: *Inégalités à poids pour le noyau de Bergman*, C. R. Acad. Sci. Paris Sér. A-B, **286** (18) 1978, A775–A778.

[5] Bell, S. R.: *The Bergman kernel function and proper holomorphic mappings*, Trans. Amer. Math. Soc. **270** (1982), no. 2, 685–691.

[6] Bender, C.; Chakrabarti, D.; Edholm, L. and Mainkar, M.: *L^p regularity of the Bergman projection on quotient domains*, Canadian Journal of Mathematics, 1–41, 2021.

[7] Boas, H. P.: *Holomorphic reproducing kernels in Reinhardt domains*, Pacific J. Math. **112** (1984), no. 2, 273–292.

[8] Chakrabarti, D.; Gorai, S.: *Function theory and holomorphic maps on symmetric products of planar domains*, J. Geom. Anal. **25** (2015), no. 4, 2196–2225.

[9] Chakrabarti, D.; Shaw, M.: *The Cauchy-Riemann equations on product domains.* Math. Ann. **349** (2011), no. 4, 977–998.

[10] Chakrabarti, D.; Shaw, M.: *Sobolev regularity of the $\bar{\partial}$-equation on the Hartogs triangle*, Math. Ann. **356** (2013), no. 1, 241–258.

[11] Chakrabarti, D.; Zeytuncu, Y. E.: *L^p mapping properties of the Bergman projection on the Hartogs triangle*, Proc. Amer. Math. Soc. **144** (2016), no. 4, 1643–1653.

[12] Chaumat, J.; Chollet, A.-M.: *Régularité höldérienne de l'opérateur $\bar{\partial}$ sur le triangle de Hartogs*, Ann. Inst. Fourier (Grenoble) **41** (1991), no. 4, 867–882.

[13] Chen, L.: *Weighted Sobolev regularity of the Bergman projection on the Hartogs triangle*, Pacific J. Math. **288** (2017), no. 2, 307–318.

[14] Chen, L.: *The L^p boundedness of the Bergman projection for a class of bounded Hartogs domains*, J. Math. Anal. Appl., **448** (1) 2017, 598–610.

[15] Chen, L.; Jin, M.; Yuan, Y.: *Bergman projection on the symmetrized bidisc*, arXiv:2004.02785.

[16] Chen, L.; Krantz, S. G.; Yuan, Y.: *L^p regularity of the Bergman projection on domains covered by the polydisc*, J. Funct. Anal. **279** (2020), no. 2, 108522.

[17] Chen, L.; McNeal, J.: *A solution operator for $\bar{\partial}$ on the Hartogs triangle and L^p estimates.* Math. Ann. **376** (2020), no. 1-2, 407–430.

[18] Chen, L.; McNeal, J.: *Product domains, multi-Cauchy transforms, and the $\bar{\partial}$ equation.* Adv. Math. **360** (2020), 106930, 42 pp.

[19] Chen, S.-C.; Shaw, M.-C.: *Partial differential equations in several complex variables*, AMS/IP Studies in Advanced Mathematics, 19. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001. xii+380 pp. ISBN: 0-8218-1062-6

[20] Chua, S.: *Extension theorems on weighted Sobolev spaces.* Indiana Univ. Math. J. **41** (1992), no. 4, 1027–1076.

[21] Constantine, G. M.; Savits, T. H.: *A multivariate Faà di Bruno formula with applications.* Trans. Amer. Math. Soc. **348** (1996), no. 2, 503–520.

[22] Dong, X.; Pan, Y.; Zhang, Y.: *Uniform estimates for the canonical solution to the $\bar{\partial}$-equation on product domains.* arXiv:2006.14484.

[23] Dufresnoy, A.: *Sur l'opérateur d" et les fonctions différentiables au sens de Whitney*, Ann. Inst. Fourier (Grenoble) **29**, 229–238 (1979)

[24] Edholm, L. D.; McNeal, J. D.: *The Bergman projection on fat Hartogs triangles: L^p boundedness*, Proc. Amer. Math. Soc., **144**(5) 2016, 2185–2196.

[25] Edholm, L. D.; McNeal, J.: *Sobolev Mapping of Some Holomorphic Projections*, J. Geom. Anal. **30** (2020), no. 2, 1293–1311.

[26] Edigarian A., Zwonek, W.: *Geometry of the symmetrized polydisc*, Arch. Math. **84** (2005) 364–374.

[27] Fassina, M.; Pan, Y.: *Supnorm estimates for $\bar{\partial}$ on product domains in $\mathbb{C}^n$.* Preprint. arXiv:1903.10475.

[28] Garcá-Cuerva, J.; Rubio de Francia, J.: *Weighted norm inequalities and related topics.* North-Holland Mathematics Studies, 116. Notas de Matemática [Mathematical Notes], 104. North-Holland Publishing Co., Amsterdam, 1985. x+604 pp.

[29] Heinonen, J.; Kilpeläinen, T.; Martio, O.: *Nonlinear potential theory of degenerate elliptic equations.* Unabridged republication of the 1993 original. Dover Publications, Inc., Mineola, NY, 2006. xii+404 pp.

[30] Henkin, G. M.: *Integral representation of functions in strictly pseudoconvex domains and applications to the $\bar{\partial}$-problem.* Mat. Sbornik. **124** (1970), no. 2, 300–308.

[31] Jin, M.; Yuan, Y.: *On the canonical solution of $\bar{\partial}$ on polydiscs*, C. R. Math. Acad. Sci. Paris. **358** (2020), no. 5, 523–528.

[32] Kurki, E.; Mudarra, C.: *On the extension of Muckenhoupt weights in metric spaces.* Nonlinear Anal. 215 (2022), Paper No. 112671, 20 pp.

[33] Ma, L.; Michel, J.: *C^{k+a}-estimates for the $\bar{\partial}$-equation on the Hartogs triangle*, Math. Ann. **294** (1992), no. 4, 661–675.

[34] Muckenhoupt, B.; Wheeden, R.: *Weighted norm inequalities for fractional integrals.* Trans. Amer. Math. Soc. **192** (1974), 261–274.

[35] Nijenhuis, A.; Woolf, W.: *Some integration problems in almost-complex and complex manifolds.* Ann. of Math. (2) **77** (1963), 424–489.

[36] Prats, M.: *Sobolev regularity of the Beurling transform on planar domains.* Publ. Mat. **61** (2017), no. 2, 291–336.

[37] Rahm, R.; Tchoundja, E.; Wick, B. D.: *Weighted estimates for the Berezin transform and Bergman projection on the unit ball*, Math. Z. **286** (2017), no. 3-4, 1465–1478.

[38] Stein, E.: *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Princeton University Press, Princeton, NJ, 1993. xiv+695 pp.

[39] Straube, E. J.: *Lectures on the L^2-Sobolev theory of the $\bar{\partial}$-Neumann problem*, ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2010. viii+206 pp. ISBN: 978-3-03719-076-0

[40] Vekua, I. N.: *Generalized analytic functions*, vol. 29, Pergamon Press Oxford, 1962.

[41] Zhang, S.: *L^p boundedness for the Bergman projections over n-dimensional generalized Hartogs triangles.* Complex Var. Elliptic Equ. **66** (2021), no. 9, 1591–1608.

Yuan Yuan, yyuan05@syr.edu, Department of Mathematics, Syracuse University, Syracuse NY 13244, USA

Yuan Zhang, zhangyu@pfw.edu, Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, IN 46805-1499, USA