On solutions to $-\Delta u = Vu$ near infinity

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Abstract

In this note, we investigate the unique continuation property and the sign changing behavior of weak solutions to $-\Delta u = Vu$ near infinity under certain conditions on the blow-up rate of the potential V near infinity.

1 Introduction

Given $V \in L^\infty_{loc}(\mathbb{R}^n), n \geq 1$, we study properties of real-valued weak solutions to

$$-\Delta u = Vu \ \text{ on } \ \mathbb{R}^n$$

near infinity. Landis [10] conjectured that if $V \in L^\infty(\mathbb{R}^n)$, then every bounded weak solution u such that $|u| \leq Ce^{-|x|^{1+\epsilon}}$ near infinity for some $C, \epsilon > 0$ must be identically 0. The conjecture has attracted much attention since the fundamental work [2] of Bourgain and Kenig. See, for instance, [3, 4, 9, 13] and the references therein. Recently, Logunov, Malinnikova, Nadirashvili and Nazarov proved the full Landis conjecture when $n = 2$ in [11].

Motivated by the Landis conjecture, we first investigate the following unique continuation property at infinity. Denote by (r, θ) the polar coordinates, $r > 0, \theta \in S^{n-1}$.

Theorem 1.1. *Let $V \in L^\infty_{loc}(\mathbb{R}^n)$. Assume further that when $|x| >> 1$, V is locally Lipschitz,*

$$V = O(|x|^M) \tag{1.1}$$

for some constant M, and

$$\frac{\partial}{\partial r}(r^2 V) \geq 0 \tag{1.2}$$

in terms of the polar coordinates (r, θ). Let $u \in L^2_{loc}(\mathbb{R}^n)$ be a weak solution to

$$-\Delta u = Vu \ \ \text{on} \ \ \mathbb{R}^n.$$

If $\lim_{r\to\infty} r^m \int_{|x|>r} |u|^2 = 0$ for each $m \geq 0$, then $u \equiv 0$.

In comparison to Landis's case, while imposing an additional condition (1.2), we allow V to blow up in a finite order at infinity. In conclusion, we obtain the unique continuation property by assuming that u vanishes to infinite order in the L^2 sense (see Section 2 for the definition) rather than to an exponential order at infinity. Our main tool of the proof involves a local unique continuation property presented in [12] by Li and Nirenberg, coupled with the application of the Kelvin transform which brings the equation from infinity to a punctured neighborhood of

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the origin. As the Kelvin transform introduces extra singularity at the origin, we will need the assumption (1.1) to resolve the singularity of the new equation at the origin (in Lemma 2.3). This assumption also reduces the regularity assumption of the solution to being merely in $L^2_{loc}(B_1\setminus\{0\})$ in Theorem 3.1. In fact, as demonstrated in Example 1 and Example 2, the unique continuation property fails at infinity in general if (1.1) and/or (1.2) is dropped.

In particular, Theorem 1.1 can be applied to infer the unique continuation property at infinity when the potential is homogeneous as follows. See also Example 3 for the construction of more potentials that satisfy the assumptions of Theorem 1.1.

Corollary 1.2. *Let $V \in L^\infty_{loc}(\mathbb{R}^n \setminus \{0\})$ be a homogeneous function of order α on $\mathbb{R}^n$. Assume either $\alpha = 2$, or $\alpha > -2$ and $V \geq 0$ on $\mathbb{R}^n$, or $\alpha < -2$ and $V \leq 0$ on $\mathbb{R}^n$. Let $u \in L^2_{loc}(\mathbb{R}^n)$ be a weak solution to*

$$-\Delta u = Vu \quad on \quad \mathbb{R}^n.$$

If $\lim_{r\to\infty} r^m \int_{|x|>r} |u|^2 = 0$ for each $m \geq 0$, then $u \equiv 0$.

Corollary 1.3. *Let $u \in L^2_{loc}(\mathbb{R}^n)$ be an eigenfunction of $-\Delta$ on $\mathbb{R}^n$ with*

$$-\Delta u = c^2 u \quad on \quad \mathbb{R}^n$$

for some constant c. Then u does not vanish to infinite order in the L^2 sense at ∞. Namely, there exists some $N \geq 0$ such that $\overline{\lim}_{r\to\infty} r^N \int_{|x|>r} |u|^2 = \infty$.

Next, we study the sign changing behavior of bounded weak solutions near infinity. The following theorem states that when V satisfies certain integrability condition near infinity, then the weak solutions must change sign constantly near infinity.

Theorem 1.4. *Let $V \in L^\infty_{loc}(\mathbb{R}^n)$. Assume further that $V \geq 0$ when $|x| >> 1$, $\int_{|x|>r} \frac{1}{|x|^{n+2}V} < \infty$ and $\int_{|x|>r} |x|^{4-2n} V = \infty$ for some $r >> 1$. Then every non-constant bounded weak solution to*

$$-\Delta u = Vu \quad on \quad \mathbb{R}^n.$$

must change sign near ∞. Namely, if $u \geq 0$ (or $u \leq 0$) when $|x| >> 1$, then $u \equiv 0$.

As indicated by the following corollary, there exist ample examples where the assumptions of the potential in Theorem 1.4 are satisfied.

Corollary 1.5. *Let $V \in L^\infty_{loc}(\mathbb{R}^n \setminus \{0\})$ be a positive homogeneous function of order α on $\mathbb{R}^n \setminus \{0\}$. Let u be a bounded weak solution to*

$$-\Delta u = Vu \quad on \quad \mathbb{R}^n.$$

If $\alpha > -2$ when $n \leq 2$, or $\alpha \geq n - 4$ when $n \geq 3$, then u must change sign near ∞.

According to Liouville's theorem, every bounded eigenfunction of Laplacian with respect to eigenvalue 0 (namely, harmonic function) must be a constant. The following theorem shows that bounded eigenfunctions with respect to nonzero eigenvalues will constantly change sign near infinity. As seen in the statement of Corollary 1.5, the order $\alpha = 0$ case is covered by this corollary only when $n \leq 4$. The remaining cases when $n \geq 5$ has to be dealt with through a different approach. See Section 4 for its proof.

Theorem 1.6. *Every bounded eigenfunction of $-\Delta$ with*

$$-\Delta u = c^2 u \quad on \quad \mathbb{R}^n$$

for a constant $c \neq 0$ must change sign near ∞.

Equivalently, the above theorem states that every bounded nonconstant eigenfunction is neither subharmonic nor superharmonic near infinity.

2 Preliminaries

As mentioned in the introduction, when pulling the equation from infinity to the origin via the Kelvin transform, the procedure results in a new equation that carries singularity at the origin. The purpose of this section is to extend the new equation across the origin, and meanwhile resolve the singularity of weak solutions there. We shall also discuss properties of flat functions.

Denote by B_r the ball of radius r centered at 0 in $\mathbb{R}^n$. Given u on $\mathbb{R}^n \setminus B_1$, recall that the Kelvin transform of u is defined by $w(x) := \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right)$ on $B_1 \setminus \{0\}$. One of the properties for the Kelvin transform is that in the distribution sense,

$$\Delta w(x) = \frac{1}{|x|^{n+2}} \Delta u\left(\frac{x}{|x|^2}\right) \quad \text{on} \quad B_1 \setminus \{0\}. \tag{2.1}$$

In particular, the harmonicity is invariant under the Kelvin transform, a property that we will apply repeatedly.

Lemma 2.1. *Let $V \in L^\infty_{loc}(\mathbb{R}^n \setminus B_1)$, and $u \in L^2_{loc}(\mathbb{R}^n \setminus B_1)$. Define*

$$w(x) := \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right) \quad \text{and} \quad W(x) := \frac{1}{|x|^4} V\left(\frac{x}{|x|^2}\right) \quad \text{on} \quad B_1 \setminus \{0\}. \tag{2.2}$$

Then the following properties hold.
(1) $W \in L^\infty_{loc}(B_1 \setminus \{0\})$, and $w \in L^2_{loc}(B_1 \setminus \{0\})$.
(2) If $V = O(|x|^M)$ near ∞ for some constant M, then $W = O\left(\frac{1}{|x|^{M+4}}\right)$ near 0.
(3) For every $r > 0$, one has

$$\int_{|x|>\frac{1}{r}} \frac{1}{|x|^{n+2}V} = \int_{|x|<r} \frac{1}{|x|^{n+2}W} \quad \text{and} \quad \int_{|x|>\frac{1}{r}} |x|^{4-2n} V = \int_{|x|<r} W.$$

(4) If

$$\lim_{r\to\infty} r^m \int_{|x|>r} |u|^2 = 0$$

for some constant $m \geq 0$, then $w \in L^2_{loc}(B_1)$ with

$$\lim_{r\to 0} r^{-(m+4)} \int_{|x|<r} |w|^2 = 0.$$

(5) If u is a weak solution to

$$-\Delta u = Vu \quad \text{on} \quad \mathbb{R}^n \setminus B_1,$$

then w is a weak solution to

$$-\Delta w = Ww \quad \text{on} \quad B_1 \setminus \{0\}.$$

Proof. (1) and (2) are true by definition. For (3), we apply change of coordinates $x \to \frac{x}{|x|^2}$ to verify the equalities directly as follows.

$$\int_{|x|<r} \frac{1}{|x|^{n+2}W} = \int_{|x|<r} \frac{1}{|x|^{n-2}V\left(\frac{x}{|x|^2}\right)} = \int_{|x|>\frac{1}{r}} \frac{|x|^{n-2}}{V(x)}|x|^{-2n} = \int_{|x|>\frac{1}{r}} \frac{1}{|x|^{n+2}V};$$

$$\int_{|x|<r} W = \int_{|x|<r} \frac{1}{|x|^4} V\left(\frac{x}{|x|^2}\right) = \int_{|x|>\frac{1}{r}} |x|^4 V(x)|x|^{-2n} = \int_{|x|>\frac{1}{r}} |x|^{4-2n} V.$$

Here we used the fact that the Jacobian of the coordinates change is $|x|^{-2n}$.

For the decay property of w in (4), by the same change of coordinates as above,

$$\int_{|x|<r} |w|^2 = \int_{|x|<r} \frac{1}{|x|^{2n-4}} \left| u\left(\frac{x}{|x|^2}\right)\right|^2 = \int_{|x|>\frac{1}{r}} |x|^{2n-4} |u(x)|^2 |x|^{-2n}$$
$$= \int_{|x|>\frac{1}{r}} |x|^{-4} |u(x)|^2 \le r^4 \int_{|x|>\frac{1}{r}} |u|^2.$$

Note that by assumption, $\lim_{r\to 0} r^{-m} \int_{|x|>\frac{1}{r}} |u|^2 = \lim_{r\to\infty} r^m \int_{|x|>r} |u|^2 = 0$. Hence

$$\lim_{r\to 0} r^{-(m+4)} \int_{|x|<r} |w|^2 \le \lim_{r\to 0} r^{-m-4} r^4 \int_{|x|>\frac{1}{r}} |u|^2 = 0.$$

(5) follows from (2.1) by a straightforward computation: on $B_1 \setminus \{0\}$,

$$-\Delta w(x) = -\frac{1}{|x|^{n+2}} \Delta u\left(\frac{x}{|x|^2}\right) = \frac{1}{|x|^{n+2}} V\left(\frac{x}{|x|^2}\right) u\left(\frac{x}{|x|^2}\right) = \frac{1}{|x|^4} V\left(\frac{x}{|x|^2}\right) w(x) = W(x) w(x)$$

in the distribution sense.

□

Next we extend the weak solutions across the singularity at the origin, with the help of a Harvey-Polking type (see [6]) lemma below for an isolated singularity. Throughout the rest of the paper we use the following notation: two quantities A and B are said to satisfy $A \lesssim B$, if $A \le CB$ for some constant $C > 0$ that depends only possibly on n.

Lemma 2.2. *Let $P(x, D)$ be a linear differential operator of order l and $f \in L^1(B_1)$. If $u \in L^1(B_1)$ satisfies $P(x, D)u = f$ on $B_1 \setminus \{0\}$ and*

$$\lim_{r\to 0} r^{-l} \int_{|x|<r} |u| = 0, \tag{2.3}$$

then $P(x, D)u = f$ on B_1.

Proof. Let ϕ^r be a smooth function on B_1 such that $\phi^r = 1$ on $B_{\frac{r}{2}}$, $\phi^r = 0$ outside B_r and $|\nabla^k \phi^r| \lesssim r^{-k}$ on B_r, $k \le l$. Then for any testing function ϕ on B_1, $(1-\phi^r)\phi$ is a testing function on $B_1 \setminus \{0\}$. Thus

$$\langle P(x, D)u - f, (1-\phi^r)\phi\rangle = 0,$$

and so

$$\langle P(x, D)u - f, \phi\rangle = \langle P(x, D)u - f, \phi^r\phi\rangle = \langle u, {}^tP(x, D)(\phi^r\phi)\rangle - \langle f, \phi^r\phi\rangle.$$

Passing r to 0, since $f \in L^1(B_1)$,

$$\langle f, \phi^r\phi\rangle \lesssim \int_{B_r} |f| \to 0.$$

On the other hand, by assumption (2.3),

$$\langle u, {}^tP(x, D)(\phi^r\phi)\rangle \lesssim r^{-l} \int_{|x|<r} |u| \to 0.$$

We thus have the desired identity $\langle P(x, D)u - f, \phi\rangle = 0$.

□

Lemma 2.3. *Let $W \in L^\infty_{loc}(B_1 \setminus \{0\})$, $W \le O(\frac{1}{|x|^M})$ near 0 for some constant $M \ge 0$, and $w \in L^2_{loc}(B_1)$ be a weak solution to*

$$-\Delta w = Ww \quad on \quad B_1 \setminus \{0\}.$$

If further

$$\lim_{r\to 0} r^{-2M-3} \int_{|x|<r} |w|^2 = 0, \tag{2.4}$$

then $w \in W^{2,2}_{loc}(B_1) \cap W^{2,p}_{loc}(B_1 \setminus \{0\})$ for all $p < \infty$, and w is a weak solution to

$$-\Delta w = Ww \quad on \quad B_1.$$

Proof. First, for all $r << 1$, by (2.4)

$$\int_{|x|<r} |Ww|^2 \lesssim \int_{|x|<r} \frac{|w|^2}{|x|^{2M}} = \sum_{j=0}^{\infty} \int_{2^{-j-1}r<|x|<2^{-j}r} \frac{|w|^2}{|x|^{2M}} \le \sum_{j=0}^{\infty} 2^{2(j+1)M} r^{-2M} \int_{|x|<2^{-j}r} |w|^2 \le Cr^3 \tag{2.5}$$

for some constant C dependent only on M. So $Ww \in L^2_{loc}(B_1)$. On the other hand, by Hölder inequality and (2.4) again,

$$\lim_{r\to 0} r^{-2} \int_{|x|<r} |w| \le \lim_{r\to 0} r^{-2+\frac{n}{2}} \left(\int_{|x|<r} |w|^2\right)^{\frac{1}{2}} \le \lim_{r\to 0} \left(r^{-3} \int_{|x|<r} |w|^2\right)^{\frac{1}{2}} = 0.$$

As a consequence of the Harvey-Polking type Lemma 2.2, $-\Delta w = Ww$ holds true on B_1 in the distribution sense. PDE theory then tells that $w \in W^{2,2}_{loc}(B_1)$. In particular, $w \in L^{p_0}_{loc}(B_1)$ where $p_0 = \frac{2n}{n-4}$ if $n > 4$, or any number less than ∞ if $n \le 4$ by the Sobolev embedding theorem.

Restricted on $B_1 \setminus \{0\}$, since $Ww \in L^{p_0}_{loc}(B_1 \setminus \{0\})$, we further have $w \in W^{2,p_0}_{loc}(B_1 \setminus \{0\})$. With a boot-strap argument, we eventually obtain $w \in W^{2,p}_{loc}(B_1 \setminus \{0\})$ for all $p < \infty$.

□

Next, we discuss properties of flat functions at 0. A function $w \in L^2_{loc}(\mathbb{R}^n)$ is said to vanish to infinite order (or, flat) in the L^2 sense at $x_0 \in \mathbb{R}^n$ if for every $m \ge 0$,

$$\lim_{r\to 0} r^{-m} \int_{|x-x_0|<r} |w|^2 = 0. \tag{2.6}$$

In view of (the proof to) Lemma 2.1 (4), it makes sense to call that a function $u \in L^2_{loc}(\mathbb{R}^n)$ vanishes to infinite order in the L^2 sense at infinity, if for every $m \ge 0$,

$$\lim_{r\to \infty} r^{m} \int_{|x|>r} |u|^2 = 0.$$

Thus, as a consequence of Lemma 2.1 (4), if u vanishes to infinite order in the L^2 sense at infinity, then w defined in (2.2) vanishes to infinite order in the L^2 sense at 0.

In general the flatness of a function does not necessarily imply the flatness of its derivatives. For instance, $w(x) := e^{-\frac{1}{x^2}} \cos(e^{\frac{1}{x^2}}), x \in \mathbb{R}$ is flat at 0. However, the derivative w' is not even L^1_{loc} at 0. On the other hand, the following lemma states that the flatness naturally passes onto its derivatives if w is a weak solution to $-\Delta w = Ww$ when W blows up at most in a finite order at 0. We shall need the following well-known inequality with proof provided below for completeness.

Lemma 2.4. *Let $f \in W^{2,2}_{loc}(B_1)$. Then for any $0 < r < \frac{1}{2}$,*

$$\int_{|x|<r} |\nabla f|^2 \lesssim \frac{1}{r^2} \int_{|x|<2r} |f|^2 + r^2 \int_{|x|<2r} |\Delta f|^2. \tag{2.7}$$

Proof. Let ϕ be a nonnegative smooth function with compact support on B_{2r} such that $\phi = 1$ on B_r, $\phi \le 1$ and $|\nabla^k \phi| \lesssim \frac{1}{r^k}, k \le 2$ on B_{2r}. Since $\phi \nabla f = \nabla(\phi f) - f \nabla \phi$, we have

$$\int_{|x|<2r} \phi^2 |\nabla f|^2 \le 2 \int_{|x|<2r} |f \nabla \phi|^2 + 2 \int_{|x|<2r} |\nabla(\phi f)|^2.$$

Making use of Stokes' theorem to the second term on the right, one further has

$$\int_{|x|<2r} \phi^2 |\nabla f|^2 \le 2 \int_{|x|<2r} |f \nabla \phi|^2 + 2 \int_{|x|<2r} \phi |f| |\triangle(\phi f)|. \tag{2.8}$$

On the other hand, note that $\triangle(\phi f) = f \triangle \phi + \phi \triangle f + 2 \nabla \phi \cdot \nabla f$. By the choice of ϕ and with a repeated application of the Schwartz inequality,

$$\begin{aligned} \int_{|x|<2r} \phi |f| |\triangle(\phi f)| &\le \int_{|x|<2r} \phi |f|^2 |\triangle \phi| + \int_{|x|<2r} \phi^2 |f| |\triangle f| + 2 \int_{|x|<2r} \phi |f| |\nabla \phi| |\nabla f| \\ &\lesssim \frac{1}{r^2} \int_{|x|<2r} |f|^2 + \int_{|x|<2r} \frac{1}{r} |f| \cdot r |\triangle f| + \int_{|x|<2r} \phi |\nabla f| \cdot \frac{1}{r} |f| \\ &\lesssim \frac{1}{r^2} \int_{|x|<2r} |f|^2 + r^2 \int_{|x|<2r} |\triangle f|^2 + \frac{1}{4} \int_{|x|<2r} \phi^2 |\nabla f|^2. \end{aligned} \tag{2.9}$$

Combining (2.8)-(2.9), we have

$$\int_{|x|<2r} \phi^2 |\nabla f|^2 \lesssim \frac{1}{r^2} \int_{|x|<2r} |f|^2 + r^2 \int_{|x|<2r} |\triangle f|^2,$$

from which (2.7) follows.

□

Lemma 2.5. *Suppose $w \in L^2_{loc}(B_1)$ vanishes to infinite order in the L^2 sense at 0. Then the following holds.*
(1) If $f = O\left(\frac{1}{|x|^M}\right)$ near 0 for some constant M, then $fw \in L^2$ near 0, and vanishes to infinite order in the L^2 sense at 0.
(2) If w is a weak solution to

$$-\Delta w = Ww \quad on \quad B_1.$$

for some $W \in L^\infty_{loc}(B_1 \setminus \{0\})$ such that $W = O(\frac{1}{|x|^M})$ near 0 for some constant M. Then ∇w vanishes to infinite order in the L^2 sense at 0. In particular, $\nabla_\theta w$ and w_r with respect to the polar coordinates (r, θ) vanish to infinite order in the L^2 sense at 0.

Proof. (1) can be proved by making use of a similar argument as in (2.5), with the assumption (2.4) replaced by the L^2 flatness (2.6) of w at 0.

For (2), since w vanishes to infinite order in the L^2 sense at 0, w satisfies (2.4). Hence $w \in W^{2,2}_{loc}(B_1)$ by Lemma 2.3. On the other hand, by (1) Ww also vanishes to infinite order in the L^2 sense at 0. Thus we apply (2.7) to w and obtain the flatness of ∇w in the L^2 sense.

That $\nabla_\theta w$ and w_r vanish to infinite order in the L^2 sense at 0 follows directly from the equality

$$|\nabla w|^2 = |w_r|^2 + \frac{1}{r^2}|\nabla_\theta v|^2$$

in terms of the polar coordinates (r, θ).

$\square$

Lemma 2.6. *Suppose that $w \in L^2(B_1) \cap C(B_1 \setminus \{0\})$ satisfies*

$$\lim_{r\to 0} r^{-1} \int_{|x|<r} |w|^2 = 0.$$

Then there exists $r_j \to 0$ such that

$$\int_{|x|=r_j} |w|^2 \to 0.$$

Proof. First note that by assumption, $\int_{|x|=r} |w|^2$ as a function of r is in $C((0,1))$. By the mean-value theorem, for each $j \geq 1$, there exists $r_j \in (\frac{r}{2^j}, \frac{r}{2^{j-1}})$ such that

$$\int_{2^{-j}r<|x|<2^{-j+1}r} |w|^2 = \int_{2^{-j}r}^{2^{-j+1}r} \int_{|x|=s} |w|^2 = 2^{-j}r \int_{|x|=r_j} |w|^2.$$

When $j \to \infty$, we have $r_j \to 0$ and thus by assumption,

$$\int_{|x|=r_j} |w|^2 \leq \frac{2^j}{r} \int_{2^{-j}r<|x|<2^{-j+1}r} |w|^2 \leq 2\left(2^{-j+1}r\right)^{-1} \int_{|x|<2^{-j+1}r} |w|^2 \to 0.$$

$\square$

3 Unique continuation property at infinity

To prove Theorem 1.1, we shall make use of a unique continuation property near the origin that was established by Li and Nirenberg [12, Theorem 10] for smooth solutions without an assumption on the potential's growth near the singularity. However, to adapt its proof to our context of L^2 weak solutions, considerable modifications are needed in order for its application. It is also crucial to point out that a specific boundary term involving V in [12] can not be discarded, due to insufficient information on the blow-up rate of the potential near the origin in [12]. By imposing the additional assumption (1.1), we have the desired control of singularity for the potential near the origin when pulling to the origin by the Kelvin transform, so that the aforementioned boundary term can be managed. For clarification we are compelled to provide the detailed proof below.

Theorem 3.1. *Let $W \in L^\infty_{loc}(B_1 \setminus \{0\})$ with $W = O(\frac{1}{|x|^M})$ near 0 for some constant M. Assume further there exists some $0 < r_0 < 1$ such that W is locally Lipschitz in $B_{r_0} \setminus \{0\}$ with $\frac{\partial}{\partial r}(r^2 W) \leq 0$ on $0 < r < r_0$. Let $w \in L^2_{loc}(B_1 \setminus \{0\})$ be a weak solution to*

$$-\Delta w = Ww \quad on \quad B_1 \setminus \{0\}.$$

If w vanishes to infinite order in the L^2 sense at 0, then $w \equiv 0$.

Proof. By Lemma 2.3, we have $w \in W^{2,2}_{loc}(B_1) \cap W^{2,p}_{loc}(B_1 \setminus \{0\})$ for all $p < \infty$, and

$$-\Delta w = Ww \quad \text{on} \quad B_1.$$

In particular, $w \in C^1(B_1 \setminus \{0\})$ by Sobolev embedding theorem. Under the polar coordinates (r, θ), $-\Delta w = Ww$ is written as

$$r^2 w_{rr} + (n-1) r w_r + \Delta_\theta w = -r^2 Ww, \quad 0 < r < 1, \ \theta \in S^{n-1}.$$

Setting $r = e^s$, $s < 0$, then a direct computation gives

$$\begin{aligned} w_s &= w_r e^s = r w_r, \\ w_{ss} &= w_{rr} e^{2s} + w_r e^s = r^2 w_{rr} + r w_r. \end{aligned}$$

Thus we have

$$w_{ss} + (n-2) w_s + \Delta_\theta w = -e^{2s} Ww, \quad s < 0, \ \theta \in S^{n-1}. \tag{3.1}$$

Now let $w = e^{as} v$ with $a = -\frac{n-2}{2}$. One can further verify that

$$\begin{aligned} w_s &= a e^{as} v + e^{as} v_s = e^{as}(av + v_s), \\ w_{ss} &= a e^{as}(av + v_s) + e^{as}(a v_s + v_{ss}) = e^{as}(v_{ss} + 2a v_s + a^2 v). \end{aligned}$$

Plugging the above into (3.1), after simplification we obtain

$$v_{ss} + \Delta_\theta v = \left(-e^{2s} W + \frac{(n-2)^2}{2} \right) v := mv, \tag{3.2}$$

where $m = -e^{2s} W + \frac{(n-2)^2}{2}$. By assumption, there exists some $s_0 < 0$ such that

$$m_s = -(e^{2s} W)_s = -(r^2 W)_r e^s \geq 0, \quad s < s_0. \tag{3.3}$$

On the other hand, by the assumption of the L^2 flatness of w at 0, we have $w_s (= r w_r)$ and $\nabla_\theta w$ vanish to infinite order in the L^2 sense at $r = 0$ by Lemma 2.5. Moreover, since $v = r^{-a} w$, one infers that $v_s = r^{-a} w_s - av$ and $\nabla_\theta v = r^a \nabla_\theta w$. Thus v_s and $\nabla_\theta v$ vanish to infinite order in the L^2 sense at $r = 0$ by Lemma 2.5 again. Similarly, as $m = O\left(\frac{1}{|x|^N}\right)$ near $r = 0$ by definition, where $N := \max\{M-2, 0\}$, we also have $|m|^{\frac{1}{2}} v$ vanishes to infinite order in the L^2 sense at $r = 0$. Thus applying Lemma 2.6 to $|v_s| + |\nabla_\theta v| + |m^{\frac{1}{2}} v|$, there exists $s_j \to -\infty$ such that

$$\int_{S^{n-1}} v_s^2 \big|_{s=s_j} + \int_{S^{n-1}} |\nabla_\theta v|^2 \big|_{s=s_j} + \int_{S^{n-1}} |m| v^2 \big|_{s=s_j} \to 0. \tag{3.4}$$

Multiply both sides of (3.2) by $2v_s$ and integrate from s_j to s about s and over S^{n-1} about θ. Then

$$\int_{S^{n-1}} \int_{s_j}^s 2 v_s v_{ss} + 2 \int_{S^{n-1}} \int_{s_j}^s \Delta_\theta v v_s = 2 \int_{S^{n-1}} \int_{s_j}^s m v v_s \tag{3.5}$$

For the second term on the left hand, we use Green's theorem and obtain

$$\begin{aligned} 2 \int_{S^{n-1}} \int_{s_j}^s \Delta_\theta v v_s &= -2 \int_{s_j}^s \int_{S^{n-1}} \nabla_\theta v \cdot \nabla_\theta v_s = -\int_{S^{n-1}} \int_{s_j}^s (|\nabla_\theta v|^2)_s \\ &= -\int_{S^{n-1}} |\nabla_\theta v|^2 + \int_{S^{n-1}} |\nabla_\theta v|^2 \big|_{s=s_j} = \int_{S^{n-1}} v \Delta_\theta v + \int_{S^{n-1}} |\nabla_\theta v|^2 \big|_{s=s_j}. \end{aligned}$$

For the rest of the terms in (3.5), we compute directly to have

$$\int_{S^{n-1}}\int_{s_j}^{s} 2v_s v_{ss} = \int_{S^{n-1}}\int_{-\infty}^{s} (v_s^2)_s = \int_{S^{n-1}} v_s^2 - v_s^2\big|_{s=s_j},$$

$$2\int_{S^{n-1}}\int_{s_j}^{s} mvv_s = \int_{S^{n-1}}\int_{s_j}^{s}(mv^2)_s - \int_{S^{n-1}}\int_{s_j}^{s} m_s v^2 = \int_{S^{n-1}} mv^2 - \int_{S^{n-1}}\int_{s_j}^{s} m_s v^2 - \int_{S^{n-1}} mv^2\big|_{s=s_j}$$

Altogether we infer

$$\begin{aligned}\int_{S^{n-1}} v_s^2 =& -\int_{S^{n-1}} v\Delta_\theta v + \int_{S^{n-1}} mv^2 - \int_{S^{n-1}}\int_{s_j}^{s} m_s v^2 \\ &+ \left(\int_{S^{n-1}} v_s^2\big|_{s=s_j} - \int_{S^{n-1}} |\nabla_\theta v|^2\big|_{s=s_j} - \int_{S^{n-1}} mv^2\big|_{s=s_j}\right).\end{aligned}$$

Here the last term $\int_{S^{n-1}} mv^2|_{s=s_j}$ could potentially be troublesome due to the presence of uncontrolled growth of W in m at $r=0$, but seems to have been overlooked in [12]. As seen in (3.4), the additional assumption $W = O(\frac{1}{r^M})$ allows us to eliminate this term as $j\to\infty$. In detail, letting $s_j \to -\infty$, and making use of (3.3)-(3.4), we have

$$\int_{S^{n-1}} v_s^2 \le -\int_{S^{n-1}} v\Delta_\theta v + \int_{S^{n-1}} mv^2. \tag{3.6}$$

Now we consider

$$\rho(s) = \int_{S^{n-1}} v^2(s,\theta)d\theta, \quad s < s_0.$$

We shall prove that

$$\rho_s^2 \le \rho\rho_{ss}, \quad s< s_0. \tag{3.7}$$

In fact,

$$\rho_s = 2\int_{S^{n-1}} vv_s,$$

and by (3.2),

$$\rho_{ss} = 2\int_{S^{n-1}} v_s^2 + vv_{ss} = 2\int_{S^{n-1}} v_s^2 + 2\int_{S^{n-1}} v(-\Delta_\theta v + mv).$$

Making use of Hölder inequality and (3.6), we have

$$\begin{aligned}\rho_s^2 \le& 4\int_{S^{n-1}} v^2 \int_{S^{n-1}} v_s^2 = 4\rho\int_{S^{n-1}} v_s^2 \\ \le&\rho\left(2\int_{S^{n-1}} v_s^2 + 2\int_{S^{n-1}} v(-\Delta_\theta v) + 2\int_{S^{n-1}} mv^2\right) = \rho\rho_{ss}.\end{aligned}$$

(3.7) is proved. In particular, this implies that whenever $\rho > 0$,

$$(\log\rho)_{ss} \ge 0. \tag{3.8}$$

Assume by contradiction that there exists $\bar{s} < s_0$ such that $\rho(\bar{s}) > 0$. Let $s^\sharp$ be the infimum of all $\hat{s}$ such that $\rho > 0$ on the interval $(\hat{s},\bar{s})$. Then by (3.8),

$$\log\rho(s) \ge \log\rho(\bar{s}) + \frac{d}{ds}\log\rho(\bar{s})(s-\bar{s}), s^\sharp < s < \bar{s}. \tag{3.9}$$

In particular, this implies that $\rho(s^\sharp) > 0$ whenever $s^\sharp$ is finite, which would violate the choice of $s^\sharp$. Thus $s^\sharp = -\infty$. Namely, $\rho > 0$ for all $s < \bar{s}$. Consequently, by (3.9) there exists some C_1, C_2 such that $\rho(s) \geq C_1 e^{C_2 s}$ for all $s < \bar{s}$. Equivalently, for all $r < r_0 := e^{\bar{s}}$,

$$\int_{S^{n-1}} v^2 \geq C_1 r^{C_2}.$$

Recalling that $w = r^{-\frac{n-2}{2}} v$, we further have when $r << 1$,

$$\int_{|x|<r} w^2 = \int_0^r t^{n-1} \int_{S^{n-1}} w^2 = \int_0^r t^{n-1} t^{-n+2} \int_{S^{n-1}} v^2 \geq C_1 \int_0^r t^{1+C_2},$$

which is either infinite or $O(r^{2+C_2})$. This contradicts with the L^2 flatness of w at 0. Thus $\rho = 0$ for all $s < s_0$ and so is w on $|x| < e^{s_0}$. We further apply the classical unique continuation property for L^∞ potentials (see, for instance, [7]) to get $w \equiv 0$.

□

Theorem 1.1 is a direct consequence of the above theorem after imposing the Kelvin transform.

Proof of Theorem 1.1: Adopt the Kelvin transform to obtain w and W under the setting in Lemma 2.1. In order to apply Theorem 3.1, we only need to verify that $(r^2 W)_r \leq 0$ for $0 < r << 1$. Clearly W is locally Lipschitz on $B_{r_0} \setminus \{0\}$ for some $r_0 > 0$. In terms of the polar coordinates (r, θ),

$$\begin{aligned}(r^2 W)_r(r,\theta) &= \left(\frac{V(\frac{1}{r},\theta)}{r^2}\right)_r = -\frac{2}{r^3} V\left(\frac{1}{r},\theta\right) - \frac{1}{r^4} V_r\left(\frac{1}{r},\theta\right) \\ &= -\frac{1}{r^2}\left(\frac{2}{r} V\left(\frac{1}{r},\theta\right) + \frac{1}{r^2} V_r\left(\frac{1}{r},\theta\right)\right).\end{aligned}$$

On the other hand, when $r << 1$, by assumption

$$0 \leq (r^2 V)_r\left(\frac{1}{r},\theta\right) = \left(2rV + r^2 V_r\right)\left(\frac{1}{r},\theta\right) = \frac{2}{r} V\left(\frac{1}{r},\theta\right) + \frac{1}{r^2} V_r\left(\frac{1}{r},\theta\right).$$

So we have $(r^2 W)_r \leq 0$. Theorem 3.1 thus applies to give $w = 0$ on B_1. Then $u = 0$ outside B_1. That $u = 0$ on B_1 is a direct consequence of the classical unique continuation property with L^∞_{loc} potentials.

□

Remark 3.2. *An inspection of the proof to Theorem 1.1 indicates that the local Lipschitzian assumption of V in Theorem 1.1 can be weakened to merely assuming $V_r \in L^\infty_{loc}$ for $r >> 1$ in the polar coordinates (r, θ).*

Proof of Corollary 1.2 and Corollary 1.3: Note that every homogeneous function in $L^\infty_{loc}(\mathbb{R}^n \setminus \{0\})$ is automatically Lipschitz along the r direction. In view of Theorem 1.1 and Remark 3.2, we just need to show that $(r^2 V)_r \geq 0$ for Corollary 1.2. Since $V(r,\theta) = r^\alpha V(1,\theta)$ for $\theta \in S^{n-1}$, by a straight-forward computation we get

$$(r^2 V)_r(r,\theta) = (r^{2+\alpha} V(1,\theta))_r = (2+\alpha) r^{\alpha+1} V(1,\theta) = (2+\alpha) r V(r,\theta).$$

This is always non-negative in either one of the three cases in the corollary. Corollary 1.2 is thus proved.

Corollary 1.3 is a special case of Corollary 1.2 with $\alpha = 0$ and $V = c^2 \geq 0$. Hence we immediately see that u does not vanish to infinite order in the L^2 sense at 0. In other words, there exists some $N > 0$, such that $\overline{\lim}_{r\to\infty} r^N \int_{|x|>r} |u|^2 > 0$. Thus $\overline{\lim}_{r\to\infty} r^{N+1} \int_{|x|>r} |u|^2 = \infty$.

□

The following two examples show that the unique continuation at infinity in Theorem 1.1 fails if (1.1) and/or (1.2) is dropped.

Example 1. *Given a C^2 function ϕ on $\mathbb{R}^+ \cup \{0\}$ such that $\phi(r) = 1$ when $0 \leq r \leq \frac{1}{2}$ and $\phi(r) = r$ when $r \geq 1$, let $u = e^{-e^{\phi(r)}}$ on $\mathbb{R}^n$. Then u vanishes to infinite order in the L^2 sense at ∞ and is a weak solution to*

$$-\Delta u = Vu \quad on \quad \mathbb{R}^n$$

with

$$V := -e^{2\phi(r)} \left(\phi'(r)\right)^2 + \left(\phi''(r) + \left(\phi'(r)\right)^2 + \frac{n-1}{r}\phi'(r)\right) e^{\phi(r)}.$$

Clearly, $V \in L^\infty_{loc}(\mathbb{R}^n)$. Note that $V = O(e^{2r})$ when $r > 1$.

Example 2. *For every $\epsilon \geq 2$, the function $u = e^{-r^\epsilon}$ vanishes to infinite order in the L^2 sense at ∞, and is a weak solution to*

$$-\Delta u = Vu \quad on \quad \mathbb{R}^n$$

with

$$V := -\epsilon^2 r^{2\epsilon-2} + ((n-2)\epsilon + \epsilon^2) r^{\epsilon-2} \in L^\infty_{loc}(\mathbb{R}^n).$$

Then $V = O(r^{2\epsilon-2})$ when $r >> 1$. Note that a direct computation shows that when $r >> 1$,

$$(r^2 V)' = -r^{\epsilon-1}\epsilon\left(2\epsilon^2 r^\epsilon - (n-2)\epsilon - \epsilon^2)\right) < 0.$$

On the other hand, it is worth pointing out that there are many functions satisfying the assumptions for V in Theorem 1.1. Thus the theorem can be readily applied to obtain the unique continuation property of $-\Delta u = Vu$ at infinity for such potentials.

Example 3. *Given any measurable nonnegative function g on $\mathbb{R}^n$ such that $g = O(|x|^M)$ for some constant M when $x >> 1$, let $V(r, \theta) = \frac{1}{r^2} \int_1^r g(r, \theta) dr$ for a.e. (r, θ) in its polar coordinates. Then V satisfies all the assumptions in Theorem 1.1.*

We end the section by providing a simple case of the Landis conjecture when the potential decays sufficiently near infinity.

Proposition 3.3. *Let $V \in L^\infty(\mathbb{R}^n), n \geq 3$ and $V = O(\frac{1}{|x|^M})$ near ∞ for some constant $M \geq 2$. Let $u \in L^2_{loc}(\mathbb{R}^n)$ be a weak solution to*

$$-\Delta u = Vu \quad on \quad \mathbb{R}^n.$$

If $\lim_{r\to\infty} r^m \int_{|x|>r} |u|^2 = 0$ for each $m \geq 0$, then $u \equiv 0$. In particular, if V has compact support, then $u \equiv 0$ whenever u vanishes to infinite order in the L^2 sense at ∞.

Proof. Applying the Kelvin transform as in Lemma 2.1, we have

$$-\Delta w = Ww \quad \text{on} \quad B_1 \setminus \{0\}$$

for w and W defined in (2.2). By Lemma 2.3, $w \in W^{2,2}_{loc}(B_1)$, vanishes to infinite order in the L^2 sense at 0, and satisfies $-\Delta w = Ww$ on B_1. On the other hand, when $M > 2$, $W = O(|x|^{-4+M}) \in L^{\frac{n}{2}}_{loc}(B_1)$. According to the unique continuation property of [7] for $L^{\frac{n}{2}}$ potentials, we see that $w = 0$ on B_1 and thus $u \equiv 0$. When $M = 2$, we shall use the unique continuation property of [14] for potentials of the form $\frac{C}{|x|^2}$ instead, which completes the proof.

$\square$

The proof of Proposition 3.3 immediately yields the following unique continuation property for harmonic functions near infinity.

Corollary 3.4. *Suppose u is harmonic on $\mathbb{R}^n \setminus B_{r_0}$ for some $r_0 >> 1$ and $\lim_{r\to\infty} r^m \int_{|x|>r} |u|^2 = 0$ for each $m \geq 0$. Then $u \equiv 0$ on $\mathbb{R}^n \setminus B_{r_0}$. Namely, if $u \neq 0$ somewhere on $\mathbb{R}^n \setminus B_{r_0}$, then there exists some $N \geq 0$ such that $\overline{\lim}_{r\to\infty} r^N \int_{|x|>r} |u|^2 = \infty$.*

It is not clear to us but would be desirable to know if the flatness assumption of u at infinity in Proposition 3.3 could be relaxed to a finite order vanishing at infinity in particular when V has compact support. On the other hand, we point out that for the truncated domain $\mathbb{R}^n \setminus B_{r_0}$ in Corollary 3.4, the L^2 flatness assumption of u at infinity can not be weakened. In fact, the following simple example gives harmonic functions on $\mathbb{R}^n \setminus B_1$ that vanishes at infinity at any given order in the L^2 sense.

Example 4. *Let P_k be a homogeneous harmonic polynomial of degree $k \geq 0$ on $\mathbb{R}^n$. Then the function $u = \frac{1}{|x|^{n-2}} P_k\left(\frac{x}{|x|^2}\right)$ is harmonic on $\mathbb{R}^n \setminus B_1$. Note that u is only of $O\left(\frac{1}{|x|^{k+n-2}}\right)$ at ∞. In particular, for each $m \geq 2k - 4 + n$,*

$$\lim_{r\to\infty} r^m \int_{|x|>r} |u|^2 \neq 0$$

Proposition 3.3 can be compared with a uniqueness result of Chirka and Rosay [5] for $\bar{\partial}$, which states that for $V \in L^\infty(\mathbb{R}^2)$ with compact support, any solution u to

$$\bar{\partial} u = Vu \quad \text{on} \quad \mathbb{R}^2$$

with $\lim_{z\to\infty} u = 0$ must vanish identically. We also mention an example of Meshkov in [13]: there exists a nontrivial function which decays superexponentially at infinity and is a weak solution to $-\Delta u = Vu$ for some complex-valued $V \in L^\infty(\mathbb{R}^2)$. Interestingly, in that example V vanishes on a family of concentric annuli with radii going to infinity.

4 Sign changing property near infinity

Once again, we will use the Kelvin transform to convert the solution behavior at infinity to that near the origin. As seen in the previous sections, the order of the potential singularity is changed as a result of the transform, which needs be taken care of in discretion.

Theorem 4.1. *Let $W \in L^{\infty}_{loc}(B_1 \setminus \{0\}), W \geq 0$ on B_1,*

$$W \notin L^1 \quad and \quad \frac{1}{|x|^{n+2}W} \in L^1 \tag{4.1}$$

near 0. Let $w \in L^1_{loc}(B_1 \setminus \{0\})$ be a weak solution to

$$-\Delta w = Ww \quad on \quad B_1 \setminus \{0\}, \tag{4.2}$$

with

$$w = O\left(\frac{1}{|x|^{n-2}}\right) \tag{4.3}$$

near 0. If $w \geq 0$ (or $w \leq 0$) on B_1, then $w \equiv 0$.

We first need the following preparation lemmas to resolute the singularity at 0.

Lemma 4.2. *For W and w in Theorem 4.1, one has*

$$Ww \in L^1_{loc}(B_1).$$

Proof. Since $Ww \in L^1_{loc}(B_1 \setminus \{0\})$, we only need to show that $Ww \in L^1$ near 0. Let ζ_ϵ be a cut-off function on B_1 such that $\zeta_\epsilon = 0$ when $|x| \leq \epsilon$ and $|x| \geq \frac{3}{4}$; $\zeta_\epsilon = 1$ when $2\epsilon \leq |x| \leq \frac{1}{2}$; $|\nabla^k \zeta_\epsilon| \lesssim \epsilon^{-k}$ on $\epsilon < |x| < 2\epsilon$, and $|\nabla^k \zeta_\epsilon| \lesssim 1$ on $\frac{1}{2} < |x| < \frac{3}{4}, k \leq 2$.

Since ζ_ϵ^4 is a a testing function for $B_1 \setminus \{0\}$, we have by (4.2)

$$\int_{B_1} -\zeta_\epsilon^4 \Delta w = \int_{B_1} Ww\zeta_\epsilon^4. \tag{4.4}$$

The right hand side is nonnegative since $Ww \geq 0$ by assumption. Using Stokes' theorem, the left hand side

$$\int_{B_1} -\zeta_\epsilon^4 \Delta w = \int_{B_1} -w\Delta\zeta_\epsilon^4 = \int_{\epsilon<|x|<2\epsilon} -w\Delta\zeta_\epsilon^4 + \int_{\frac{1}{2}<|x|<1} -w\Delta\zeta_\epsilon^4. \tag{4.5}$$

Firstly,

$$\left|\int_{\frac{1}{2}<|x|<1} -w\Delta\zeta_\epsilon^4\right| \leq C_1 \tag{4.6}$$

for some constant $C_1 > 0$. By the fact that $\Delta\zeta_\epsilon^4 = 4\zeta_\epsilon^2\left(3|\nabla\zeta_\epsilon|^2 + \zeta_\epsilon\Delta\zeta_\epsilon\right)$ and Hölder inequality,

$$\left|\int_{\epsilon<|x|<2\epsilon} w\Delta\zeta_\epsilon^4\right| \lesssim \int_{\epsilon<|x|<2\epsilon} \epsilon^{-2}w\zeta_\epsilon^2 \lesssim \int_{\epsilon<|x|<2\epsilon} \frac{w\zeta_\epsilon^2}{|x|^2} \leq \left(\int_{\epsilon<|x|<2\epsilon} Ww\zeta_\epsilon^4\right)^{\frac{1}{2}} \left(\int_{\epsilon<|x|<2\epsilon} \frac{w}{|x|^4W}\right)^{\frac{1}{2}}.$$

Note that

$$\frac{w}{|x|^4W} \in L^1(B_{\frac{1}{2}}) \tag{4.7}$$

by (4.1) and (4.3). Then $\left(\int_{\epsilon<|x|<2\epsilon} \frac{w}{|x|^4W}\right)^{\frac{1}{2}} \leq C_2$ for a constant $C_2 > 0$ and thus

$$\left|\int_{\epsilon<|x|<2\epsilon} w\Delta\zeta_\epsilon^4\right| \leq C_2\left(\int_{\epsilon<|x|<2\epsilon} Ww\zeta_\epsilon^4\right)^{\frac{1}{2}} \leq C_2\left(\int_{B_1} Ww\zeta_\epsilon^4\right)^{\frac{1}{2}}. \tag{4.8}$$

Combining (4.4)-(4.8), we have for $A := \left(\int_{B_1} Ww\zeta_\epsilon^4\right)^{\frac{1}{2}}$,

$$A^2 \lesssim C_2 A + C_1.$$

This implies A is bounded by a constant dependent only on C_1 and C_2. Passing ϵ to 0, we have the desired integrability property for Ww near 0.

$\square$

Lemma 4.3. *For W and w in Theorem 4.1, we have w is a weak solution to*

$$-\Delta w = Ww \quad \text{on} \quad B_1.$$

Proof. In view of Lemma 2.2, we only need to prove that

$$\lim_{\epsilon\to 0} \epsilon^{-2} \int_{|x|<\epsilon} |w| = 0.$$

Indeed,

$$\epsilon^{-2} \int_{|x|<\epsilon} |w| \le \int_{|x|<\epsilon} \frac{w}{|x|^2} \le \left(\int_{|x|<\epsilon} Ww\right)^{\frac{1}{2}} \left(\int_{|x|<\epsilon} \frac{w}{|x|^4 W}\right)^{\frac{1}{2}},$$

which goes to zero as $\epsilon \to 0$ due to the facts that $Ww \in L^1_{loc}(B_1)$ in Lemma 4.2 and $\frac{w}{|x|^4 W} \in L^1_{loc}(B_1)$ in (4.7).

$\square$

Proof of Theorem 4.1: Assume by contradiction that $w \not\equiv 0$. By Lemma 4.3, we have $-\Delta w = Ww$ on B_1. In particular, $-\Delta w \ge 0$ on B_1. Since $w \ge 0$ on B_1, by mean-value inequality there exists some positive number γ_0 and a small neighborhood of 0, say, B_ϵ, such that $w \ge \gamma_0$ almost everywhere on B_ϵ. Noting that $W \ge 0$, we further have $-\Delta w \ge \gamma_0 W \ge 0$ on B_ϵ. Since $W \notin L^1$ near 0,

$$\int_{B_\epsilon} -\Delta w \ge \gamma_0 \int_{B_\epsilon} W = \infty.$$

However, this would contradict with the fact that $\int_{B_\epsilon} -\Delta w = \int_{B_\epsilon} Ww$, which is finite according to Lemma 4.2. With w replaced by $-w$, one can prove the case when $w \le 0$ on B_1. The proof is completed.

$\square$

Proof of Theorem 1.4 and Corollary 1.5: After performing the Kelvin transform as in Lemma 2.1, we have by Lemma 2.1 *(3)* that W defined by (2.2) satisfy the assumption in Theorem 4.1. The rest of the proof to Theorem 1.4 is then a consequence of Theorem 4.1.

In the setting of Corollary 1.5, one can also directly verify that V satisfies both assumptions in Theorem 1.4 as long as $\alpha > -2$ and $\alpha \ge n-4$. Thus Corollary 1.5 follows from Theorem 1.4.

$\square$

In particular, as a consequence of Corollary 1.5, every non-trivial bounded weak solution to

$$-\Delta u = \frac{c^2}{|x|^\alpha} u \quad \text{on} \quad \mathbb{R}^n$$

must change sign near infinity if $\alpha > -2$ for $n \le 2$, or $\alpha \ge n-4$ for $n \ge 3$.

As will be seen immediately, Theorem 4.1 only allows us to prove Theorem 1.6 when the dimension n is small. To cover the case of higher dimensions, we shall need the following variant of Kato's inequality:

Lemma 4.4. *[1, 8] let $w \in L^1_{loc}(\Omega)$, $f \in L^1_{loc}(\Omega)$ and $-\Delta w \geq f$ on Ω in the distribution sense. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz concave function with $0 \leq \phi' \leq C$ on $\mathbb{R}$. Then $\psi := \phi \circ w \in L^1_{loc}(\Omega)$ and*

$$-\Delta\psi \geq \psi' f \quad \textit{on} \quad \Omega$$

in the distribution sense.

Proof of Theorem 1.6: Repeating a similar procedure as in Lemma 2.1 with $V \equiv c^2$, it boils down to prove for $W := \frac{c^2}{|x|^4}$ on B_1, every weak solution to $-\Delta w = Ww$ on $B_1 \setminus \{0\}$ must change sign in every neighborhood of 0. By replacing w by $-w$, this is further reduced to showing that every nonnegative weak solution to $-\Delta w = Ww$ on $B_1 \setminus \{0\}$ must be identically zero.

If $n \leq 4$, clearly $W \in L^\infty_{loc}(B_1 \setminus \{0\}), W \geq 0, W \notin L^1$ near 0, and $\frac{1}{|x|^{n+2}W} \in L^1(B_1)$. Moreover $w(x) = \frac{1}{|x|^{n-2}} u(\frac{x}{|x|^2}) = O(\frac{1}{|x|^{n-2}})$ near 0 by the boundedness assumption of u. If $w \geq 0$, then one can apply Theorem 4.1 to conclude that $w \equiv 0$.

For $n \geq 5$, assume by contradiction that $w \not\equiv 0$ on B_1. Since $w \geq 0$ and $-\Delta w \geq 0$ on B_1, we have $w \geq \gamma_0$ almost everywhere for some positive γ_0 on a small neighborhood B_ϵ of 0. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the function such that $\phi(s) = 0$ when $s \leq \gamma_0$; $\phi(s) = \log \frac{1}{\gamma_0} - \log \frac{1}{s}$ when $s \geq \gamma_0$. It is straight forward to verify that ϕ satisfies the assumption in Kato's inequality. Let $\psi := \phi \circ w$. Then

$$0 \leq \psi \leq \log \frac{1}{\gamma_0} \quad \text{on} \quad B_1, \tag{4.9}$$

and

$$\psi = \log \frac{1}{\gamma_0} - \log \frac{1}{w} \quad \text{on} \quad B_\epsilon.$$

By Kato's inequality,

$$-\Delta\psi \geq \psi' W w = \frac{c^2}{|x|^4} \quad \text{on} \quad B_\epsilon.$$

On the other hand, note that $\Delta \left(\frac{1}{|x|^2} \right) = \frac{8-2n}{|x|^4}$. So $-\Delta\psi \geq \Delta \left(\frac{c^2}{(8-2n)|x|^2} \right)$ on B_ϵ, or equivalently,

$$-\Delta \left(\psi - \frac{c^2}{(2n-8)|x|^2} \right) \geq 0 \quad \text{on} \quad B_\epsilon.$$

Due to the fact that $\psi - \frac{c^2}{(2n-8)|x|^2} \in L^1(B_\epsilon)$, there exists some constant $\beta > 0$ such that

$$\psi \geq \frac{c^2}{(2n-8)|x|^2} - \beta \quad \text{on} \quad B_{\frac{\epsilon}{2}}.$$

However, $2n-8 > 0$ when $n \geq 5$, which would imply that ψ is unbounded from above, contradicting (4.9).

□

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