Cauchy singular integral operator with parameters in Log-Hölder spaces

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Abstract

This paper is motivated by a claim in the classical textbook of Muskhelishvili concerning the Cauchy singular integral operator S on Hölder functions with parameters. To the contrary of the claim, a counter example was constructed by Tumanov which shows that S with parameters fails to maintain the same Hölder regularity with respect to the parameters. In view of the example, the behavior of the Cauchy singular integral operator with parameters between a type of Log-Hölder spaces is investigated to obtain the sharp norm estimates. At the end of the paper, we discuss its application to the $\bar{\partial}$ problem on product domains.

1 Introductions

Let D be a bounded domain in $\mathbb{C}$, Λ be (the closure of) an open set in $\mathbb{R}$ or $\mathbb{C}$ and $\Omega := D\times\Lambda$. In particular, ∂D consists of a finite number of $C^{1,\alpha}$ Jordan curves possessing no points in common. Given a complex-valued function $f \in C^{\alpha}(\Omega)$, define the Cauchy singular integral along the slice D as follows. For any $(z, \lambda) \in \Omega$,

$$Sf(z,\lambda) := \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta,\lambda)}{\zeta - z}d\zeta. \tag{1}$$

Classical singular integral operators theory in one complex variable states that, there exists a constant C dependent only on Ω and α, such that $Sf(\cdot, \lambda) \in C^{\alpha}(D)$ for each $\lambda \in \Lambda$, and

$$\|Sf(\cdot,\lambda)\|_{C^{\alpha}(D)} \le C\|f(\cdot,\lambda)\|_{C^{\alpha}(D)}.$$

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(See for instance [7][11] et al.) It is plausible to ask whether S in (1) is a bounded linear operator in $C^{\alpha}(\Omega)$. The question was claimed to be true by Muskhelishvili (see [7] p. 49-50). In fact, Muskhelishvili's proof only shows that given any arbitrarily small ϵ with $0<\epsilon<\alpha$, S is bounded sending $C^{\alpha}(\Omega)$ into $C^{\alpha-\epsilon}(\Omega)$.

To the contrary of Muskhelishvili's claim, Tumanov [8] (p. 486) constructed a concrete example showing that S with parameters fails to maintain the same Hölder regularity with respect to the parameters. In order to study the optimal parameter dependence of S in (1) on λ, we introduce the following Log-Hölder spaces, which are considered as refined Hölder spaces and would naturally capture the boundedness of the Cauchy singular integral operator.

Definition 1.1. Let Ω be a domain in $\mathbb{R}^n$, $k\in\mathbb{Z}^+\cup\{0\}$, $0<\alpha\le 1$ and $\nu\in\mathbb{R}$. A function $f\in C^k(\Omega)$ is said to be in $C^{k,L^{\alpha}Log^{\nu}L}(\Omega)$ if

$$\|f\|_{C^{k,L^{\alpha}Log^{\nu}L}(\Omega)} := \sum_{|\gamma|=0}^{k}\sup_{w\in\Omega}|D^{\gamma}f(w)| + \sum_{|\gamma|=k}\sup_{w,w+h\in\Omega,,0<|h|\le\frac{1}{2}}\frac{|D^{\gamma}f(w+h)-D^{\gamma}f(w)|}{|h|^{\alpha}|\ln|h||^{\nu}} < \infty.$$

Note that when $\alpha=1$ and $\nu<0$, $C^{k,L^{\alpha}Log^{\nu}L}(\Omega)$ consists of constant functions only and thus becomes trivial. Without loss of generality, we always assume $\nu\ge 0$ if $\alpha=1$ in the rest of the paper. It can be verified that $C^{k,L^{\alpha}Log^{\nu}L}(\Omega)$ is a Banach space. Moreover, for any $\mu,\nu\in\mathbb{R}^+, k\in\mathbb{Z}^+\cup\{0\}, 0<\epsilon<\alpha<1$, $C^{k,\alpha+\epsilon}(\Omega)\overset{i}{\hookrightarrow}C^{k,L^{\alpha}Log^{-\nu-\mu}L}(\Omega)\overset{i}{\hookrightarrow}C^{k,L^{\alpha}Log^{-\nu}L}(\Omega)\overset{i}{\hookrightarrow}C^{k,\alpha}(\Omega)\overset{i}{\hookrightarrow}C^{k,L^{\alpha}Log^{\nu}L}(\Omega)\overset{i}{\hookrightarrow}C^{k,L^{\alpha}Log^{\nu+\mu}L}(\Omega)\overset{i}{\hookrightarrow}C^{k,\alpha-\epsilon}(\Omega)$, where the inclusion map i at each level is a continuous embedding. The Log-Hölder space $C^{k,L^{\alpha}Log^{\nu}L}(\Omega)$ reduces to the well-understood Log-Lipschitz space $C^{k,L^1LogL}(\Omega)$ when $k=0$ and $\nu=\alpha=1$, and to Hölder space $C^{k,\alpha}(\Omega)$ when $\nu=0$. Our main theorem stated below shows that S is a bounded operator from $C^{k,L^{\alpha}Log^{\nu}L}(\Omega)$ into $C^{k,L^{\alpha}Log^{\nu+1}L}(\Omega), k\in\mathbb{Z}^+\cup\{0\}, 0<\alpha\le 1, \nu\in\mathbb{R}$.

Theorem 1.2. *Let D be a bounded domain in $\mathbb{C}$ with $C^{k,\alpha}$ boundary, $k\in\mathbb{Z}^+\cup\{0\}, 0<\alpha\le 1$, Λ be an open set in $\mathbb{R}$ or $\mathbb{C}$, and $\Omega:=D\times\Lambda$. Then S defined in (1) sends $C^{k,L^{\alpha}Log^{\nu}L}(\Omega)$ into $C^{k,L^{\alpha}Log^{\nu+1}L}(\Omega)$, $\nu\in\mathbb{R}$. Moreover, for any $f\in C^{k,L^{\alpha}Log^{\nu}L}(\Omega)$,*

$$\|Sf\|_{C^{k,L^{\alpha}Log^{\nu+1}L}(\Omega)} \le C\|f\|_{C^{k,L^{\alpha}Log^{\nu}L}(\Omega)},$$

where C is some constant dependent only on Ω, k, α and ν.

In view of Tumanov's example, Theorem 1.2 is optimal in the sense that the target space $C^{k,L^{\alpha}Log^{\nu+1}L}(\Omega)$ can not be replaced by $C^{k,L^{\alpha}Log^{\nu+\mu}L}(\Omega)$ for any $\mu<1$. As an application of the theorem, we study solutions in Log-Hölder spaces to the $\bar{\partial}$ problem on product domains, improving the regularity result of [9].

Theorem 1.3. *Let $D_j \subset \mathbb{C}, j = 1, \ldots, n$, be bounded domains with $C^{k+1,\alpha}$ boundary, $n \geq 2, k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \leq 1$, and $\Omega := D_1 \times \cdots \times D_n$. Assume $\mathbf{f} = \sum_{j=1}^n f_j d\bar{z}_j \in C^{k,L^\alpha Log^\nu L}(\Omega), \nu \in \mathbb{R}$, is a $\bar{\partial}$-closed (0,1) form on Ω (in the sense of distributions if $k = 0$). There exists a solution operator T to $\bar{\partial}u = \mathbf{f}$ such that $T\mathbf{f} \in C^{k,L^\alpha Log^{\nu+n-1} L}(\Omega)$, $\bar{\partial}T\mathbf{f} = \mathbf{f}$ (in the sense of distributions if $k = 0$) and $\|T\mathbf{f}\|_{C^{k,L^\alpha Log^{\nu+n-1} L}(\Omega)} \leq C\|\mathbf{f}\|_{C^{k,L^\alpha Log^\nu L}(\Omega)}$, where C depends only on Ω, k, α and ν.*

We would like to point out, unlike smooth domains, there is no gain of regularity phenomenon for the $\bar{\partial}$ problem on product domains, as indicated by an example of Stein and Kerzman [4] in L^∞ space (See also [9] for examples in Hölder spaces). One can similarly construct examples to show that the $\bar{\partial}$ problem on product domains does not gain regularity in Log-Hölder spaces as follows. On the other hand, the well-known uniform estimates of solutions on product domains (see [2][10] etc.) suggest that the same regularity as that of the data could be expected. This may be an interesting problem to look for optimal solutions to the $\bar{\partial}$ problem on product domains in Hölder (Log-Hölder) spaces.

Example 1.4. Let $\triangle^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ be the bidisc. For each $k \in \mathbb{Z}^+ \cup \{0\}$, $0 < \alpha < 1$ and $\nu \in \mathbb{R}$, consider $\bar{\partial}u = \mathbf{f} := \bar{\partial}((z_1 - 1)^{k+\alpha}\bar{z}_2 \log^\nu(z_1 - 1))$ on $\triangle^2$, $\frac{1}{2}\pi < \arg(z_1 - 1) < \frac{3}{2}\pi$. Then $\mathbf{f} = (z_1 - 1)^{k+\alpha} \log^\nu(z_1 - 1)d\bar{z}_2 \in C^{k,L^\alpha Log^\nu L}(\triangle^2)$ is a $\bar{\partial}$-closed $(0, 1)$ form. However, there does not exist a solution $u \in C^{k,L^\beta Log^\nu L}(\triangle^2)$ to $\bar{\partial}u = \mathbf{f}$ for any β with $\beta > \alpha$.

The rest of the paper is organized as follows. In Section 2, preliminaries about the function spaces and (semi-)norms are defined, as well as the classical theory about the Cauchy type integrals. The example of Tumanov is discussed in Section 3 to show that S does not send $C^\alpha(\triangle^2)$ into itself, $0 < \alpha < 1$. Section 4 is devoted to the boundedness of the Cauchy singular integral operator between Log-Hölder spaces on the complex plane. In Section 5 and Section 6, Theorem 1.2 and Theorem 1.3 are proved respectively, along with the verification of Example 1.4.

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2 Preliminaries and Notations

Throughout the rest of the paper, k, μ, ν and α are always referred to (part of) the indices of the Log-Hölder spaces. γ may represent either a positive integer or an n-tuple, determined by the context. C represents a constant that is dependent only on Ω, k, ν and α, which may be of different values in different places.

For convenience of notations, given $f \in C^{k, L^\alpha Log^\nu L}(\Omega)$, denote by

$$\|f\|_{C^k(\Omega)} := \sum_{|\gamma|=0}^{k} \sup_{w\in\Omega} |D^\gamma f(w)|$$

and the semi-norm

$$H^\nu[f] := \sup_{w, w+h\in\Omega, 0<|h|\le\frac{1}{2}} \frac{|f(w+h)-f(w)|}{|h|^\alpha |\ln|h||^\nu}.$$

Here α is suppressed from the above notation due to a fixed value of α throughout the paper. When $\nu = 0$, we also suppress ν and write $H[\cdot]$ for $H^0[\cdot]$. Consequently, $\|f\|_{C^{k, L^\alpha Log^\nu L}(\Omega)} = \|f\|_{C^k(\Omega)} + \sum_{|\gamma|=k} H^\nu[D^\gamma f]$.

It is worth noting that the upper bound $\frac{1}{2}$ of $|h|$ under the supreme for $H^\nu[f]$ is not essential. It can be replaced by any positive number less than 1 without changing the function space $C^{k, L^\alpha Log^\nu L}(\Omega)$, and the resulting norm is equivalent by some constant dependent only on D, α, ν and the positive number itself.

In particular when $\Omega = D\times\Lambda$, the Hölder semi-norms along z and λ variables for each fixed $\lambda\in\Lambda$ and fixed $z \in D$ respectively can be defined as follows.

$$H_D^\nu[f(\cdot,\lambda)] := \sup_{\zeta, \zeta+h\in D, 0<|h|\le\frac{1}{2}} \frac{|f(\zeta+h,\lambda)-f(\zeta,\lambda)|}{|h|^\alpha|\ln|h||^\nu};$$

$$H_\Lambda^\nu[f(z,\cdot)] := \sup_{\zeta, \zeta+h\in \Lambda, 0<|h|\le\frac{1}{2}} \frac{|f(z,\zeta+h)-f(z,\zeta)|}{|h|^\alpha|\ln|h||^\nu}.$$

The above two expressions are clearly bounded by $H^\nu[f]$ by definition. On the other hand, the following elementary property for Log-Hölder semi-norms can be observed.

Lemma 2.1. *There exists a constant C dependent only on Ω, α and ν, such that for any function $f \in C^{L^\alpha Log^\nu L}(\Omega)$,*

$$\|f\|_{C^{L^\alpha Log^\nu L}(\Omega)} \le C(\|f\|_{C(\Omega)} + \sup_{\lambda\in\Lambda} H_D^\nu[f(\cdot,\lambda)] + \sup_{z\in D} H_\Lambda^\nu[f(z,\cdot)]).$$

Proof. We only need to show $H^\nu[f] \le C(\|f\|_{C(\Omega)} + \sup_{\lambda\in\Lambda} H_D^\nu[f(\cdot,\lambda)] + \sup_{z\in D} H_\Lambda^\nu[f(z,\cdot)])$. Indeed, for any $w = (z,\lambda) \in D\times\Lambda, w+h = (z+h_1, \lambda+h_2) \in D\times\Lambda$ with $|h| \le r_0 := \min\{e^{-\frac{\nu}{\alpha}}, \frac{1}{2}\}$, then $(z+h_1,\lambda) \in D\times\Lambda$. Hence

$$\begin{aligned}|f(w+h)-f(w)| &\le |f(z+h_1,\lambda+h_2) - f(z+h_1,\lambda)| + |f(z+h_1,\lambda)-f(z,\lambda)|\\ &\le |h_2|^\alpha|\ln|h_2||^\nu \sup_{z\in D} H_\Lambda^\nu[f(z,\cdot)] + |h_1|^\alpha|\ln|h_1||^\nu \sup_{\lambda\in\Lambda} H_D^\nu[f(\cdot,\lambda)]\\ &\le |h|^\alpha|\ln|h||^\nu(\sup_{\lambda\in\Lambda} H_D^\nu[f(\cdot,\lambda)] + \sup_{z\in D} H_\Lambda^\nu[f(z,\cdot)]).\end{aligned}$$

Here the last inequality is due to the non-decreasing property of the real-valued function $s^\alpha |\ln s|^\nu$ on the interval $(0, r_0)$. ∎

Let D be a bounded domain in $\mathbb{C}$ with $C^{k+1,\alpha}$ boundary, $k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \le 1$. Given a complex valued function $f \in C(\bar{D})$, the following two operators related to the Cauchy kernel are well defined for $z \in D$.

$$\begin{aligned} Tf(z) &:= \frac{-1}{2\pi i} \int_{\mathrm{D}} \frac{f(\zeta)}{\zeta - z} d\bar{\zeta} \wedge d\zeta; \\ Sf(z) &:= \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta. \end{aligned} \tag{2}$$

Here the positive orientation of ∂D is such that the domain D is always to its left while traversing along the contour(s). We state some classical results concerning the Cauchy type integrals T and S on the complex plane. The reader may check for instance [11] for reference.

Theorem 2.2. *Let D be a bounded domain with $C^{k+1,\alpha}$ boundary.*
1) If $f \in L^p(D), p > 2$, then $Tf \in C^\alpha(D), \alpha = \frac{p-2}{p}$. Moreover,

$$\|Tf\|_{C^\alpha(D)} \le C\|f\|_{L^p},$$

for some constant C dependent only on D and p.
2) If $f \in C^{k,\alpha}(D), k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha < 1$. Then $Tf \in C^{k+1,\alpha}(D)$ and $Sf \in C^{k,\alpha}(D)$. Moreover,

$$\begin{aligned} \|Tf\|_{C^{k+1,\alpha}(D)} &\le C\|f\|_{C^{k,\alpha}(D)}; \\ \|Sf\|_{C^{k,\alpha}(D)} &\le C\|f\|_{C^{k,\alpha}(D)} \end{aligned}$$

for some constant C dependent only on D, k and α.

3 S does not send $C^\alpha(\triangle^2)$ into itself

In this section, we verify in detail Tumanov's example in [8] (See also [6]) that S defined in (1) does not send $C^\alpha(\triangle^2)$ into itself, $0 < \alpha < 1$. Define for $\lambda \in \triangle$,

$$\tilde{f}(e^{i\theta}, \lambda) = \begin{cases} |\lambda|^\alpha, & -\pi \le \theta \le -|\lambda|^{\frac{1}{2}}; \\ \theta^{2\alpha}, & -|\lambda|^{\frac{1}{2}} \le \theta \le 0; \\ \theta^\alpha, & 0 \le \theta \le |\lambda|; \\ |\lambda|^\alpha, & |\lambda| \le \theta \le \pi. \end{cases}$$

Then $\tilde{f} \in C^\alpha(\partial\triangle \times \triangle)$. Extend $\tilde{f}$ onto $\triangle^2$, denoted as f, such that $f \in C^\alpha(\triangle^2)$ and $\|f\|_{C^\alpha(\triangle^2)} = \|\tilde{f}\|_{C^\alpha(\partial\triangle\times\triangle)}$. (For instance, for each $w \in \triangle^2$, let $f(w) := \inf_{\eta\in\partial\triangle\times\triangle}\{\tilde{f}(\eta) + M|w-\eta|^\alpha\}$, where $M = \|\tilde{f}\|_{C^\alpha(\partial\triangle\times\triangle)}$.)

We first show that $Sf(1,\cdot) \notin C^\alpha(\triangle)$. Indeed, a direct computation gives for $\lambda \in \triangle$,

$$\begin{aligned}2\pi i Sf(1,\lambda) =& \int_{\partial\triangle} \frac{\tilde{f}(\zeta,\lambda)}{\zeta-1} d\zeta\\ =& i\int_{-\pi}^{\pi} \frac{\tilde{f}(e^{i\theta},\lambda)e^{i\theta}}{e^{i\theta}-1} d\theta\\ =& \frac{1}{2}\int_{-\pi}^{\pi} \frac{\tilde{f}(e^{i\theta},\lambda)e^{\frac{i\theta}{2}}}{\sin\frac{\theta}{2}} d\theta\\ =& \frac{1}{2}\int_{-\pi}^{\pi} \tilde{f}(e^{i\theta},\lambda)\cot\frac{\theta}{2} d\theta + \frac{i}{2}\int_{-\pi}^{\pi} \tilde{f}(e^{i\theta},\lambda) d\theta =: I + II.\end{aligned}$$

Here the third equality uses the identity that $e^{i\theta} - 1 = \cos\theta - 1 + i\sin\theta = 2i\sin\frac{\theta}{2}e^{\frac{i\theta}{2}}$. Since $\tilde{f} \in C^\alpha(\partial\triangle \times \triangle)$, we have $II \in C^\alpha(\triangle)$.

On the other hand, write

$$I = \frac{1}{2}\int_{-\pi}^{\pi} \tilde{f}(e^{i\theta},\lambda)(\cot\frac{\theta}{2} - \frac{2}{\theta}) d\theta + \int_{-\pi}^{\pi} \frac{\tilde{f}(e^{i\theta},\lambda)}{\theta} d\theta.$$

Notice that $\cot\frac{\theta}{2} - \frac{2}{\theta}$ extends as a continuous function on $[-\pi,\pi]$. Hence $\int_{-\pi}^{\pi} \tilde{f}(e^{i\theta},\lambda)(\cot\frac{\theta}{2} - \frac{2}{\theta}) d\theta \in C^\alpha(\triangle)$ as a function of $\lambda \in \triangle$. For the second term in I, from construction of $\tilde{f}$,

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\tilde{f}(e^{i\theta},\lambda)}{\theta} d\theta =& \int_{-|\lambda|^{\frac{1}{2}}}^{0} \frac{\theta^{2\alpha}}{\theta} d\theta + \int_{0}^{|\lambda|} \frac{\theta^{\alpha}}{\theta} d\theta + \int_{|\lambda|}^{|\lambda|^{\frac{1}{2}}} \frac{|\lambda|^\alpha}{\theta} d\theta\\ =& \frac{|\lambda|^\alpha}{2\alpha} + \frac{1}{2}|\lambda|^\alpha |\ln|\lambda||.\end{aligned}$$

We thus obtain $I \notin C^\alpha(\triangle)$ and hence $Sf(1,\cdot) \notin C^\alpha(\triangle)$.

Suppose by contradiction that $Sf \in C^\alpha(\triangle^2)$. Then the non-tangential limit of Sf on $\partial\triangle \times \triangle$, denoted by Φf, is in C^α as well. In particular, $\Phi f(1,\cdot) \in C^\alpha(\triangle)$. On the other hand, by Sokhotski-Plemelj formula, $\Phi f(1,\cdot) = Sf(1,\cdot) + \frac{1}{2}f(1,\cdot)$. This contradicts with the fact that $Sf(1,\cdot) \notin C^\alpha(\triangle)$.

Remark 3.1. For f constructed above, $Sf \notin C^{L^\alpha Log^\mu L}(\triangle^2)$ for any $\mu < 1$.

4 Cauchy singular integral in Log-Hölder spaces in $\mathbb{C}$

Let D be a bounded domain in $\mathbb{C}$ with $C^{1,\alpha}$ boundary, $k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \le 1$. In this section, we shall prove that S defined in (2) is a bounded linear operator from $C^{L^\alpha Log^\nu L}(D)$ into itself if $0 < \alpha < 1$, and into $C^{L^1 Log^{\nu+1} L}(D)$ if $\alpha = 1$ (and $\nu \ge 0$). Since $C^{L^\alpha Log^\nu L}(D)$ is a subspace of $C^\epsilon(D)$ for $0 < \epsilon < \alpha$, Sf is well defined for $f \in C^{L^\alpha Log^\nu L}(D)$ by the classical theory of S in Hölder spaces.

Write $\partial D = \cup_{j=1}^N \Gamma_j$, where each Jordan curve Γ_j is connected and positively oriented with respect to D, and of total arclength s_j. Since ∂D is Lipschitz in particular, ∂D satisfies the so-called *chord-arc* condition. In other words, for any $t, t' \in \Gamma_j, j = 1, \ldots, N$, let $|t, t'|$ be the smaller length of the two arcs of Γ_j with t and t' as the two end points. There exists a constant $c_0 \ge 1$ dependent only on ∂D such that

$$|t - t'| \le |t, t'| \le c_0 |t - t'|. \tag{3}$$

The following calculus lemma is elementary but will be frequently used in this section.

Lemma 4.1. *Let $0 < \alpha \le 1$ and $\nu \in \mathbb{R}$. There exists a constant C dependent only on α and ν, such that for all $0 < h \le h_0 := \min\{e^{-\frac{2\nu}{\alpha}}, e^{\frac{2\nu}{1-\alpha}}, \frac{1}{2}\}$,*
1) $\int_0^h s^{\alpha-1} |\ln s|^\nu ds \le C h^\alpha |\ln h|^\nu$ when $0 < \alpha \le 1$.
2) $\int_h^{h_0} s^{\alpha-2} |\ln s|^\nu ds \le \begin{cases} C h^{\alpha-1} |\ln h|^\nu, & 0 < \alpha < 1; \\ C |\ln h|^{\nu+1}, & \alpha = 1. \end{cases}$

Proof. 1) Using integration by part directly,

$$\int_0^h s^{\alpha-1} |\ln s|^\nu ds = \frac{1}{\alpha} \int_0^h |\ln s|^\nu ds^\alpha = \frac{1}{\alpha} h^\alpha |\ln h|^\nu + \frac{\nu}{\alpha} \int_0^h s^{\alpha-1} |\ln s|^{\nu-1} ds.$$

If $\nu \le 0$, the lemma follows directly from the above identity by dropping off the last negative term. If $\nu > 0$, since $s \le h_0 \le e^{\frac{-2\nu}{\alpha}}$, $1 - \frac{\nu}{\alpha |\ln s|} \ge \frac{1}{2}$, which implies $\int_0^h s^{\alpha-1} |\ln s|^\nu ds - \frac{\nu}{\alpha} \int_0^h s^{\alpha-1} |\ln s|^{\nu-1} ds = \int_0^h s^{\alpha-1} |\ln s|^\nu (1 - \frac{\nu}{\alpha |\ln s|}) ds \ge \frac{1}{2} \int_0^h s^{\alpha-1} |\ln s|^\nu ds$. Hence

$$\int_0^h s^{\alpha-1} |\ln s|^\nu ds \le \frac{2}{\alpha} h^\alpha |\ln h|^\nu.$$

2) When $0 < \alpha < 1$,

$$\int_h^{h_0} s^{\alpha-2} |\ln s|^\nu ds = \frac{1}{1-\alpha} (h^{\alpha-1} |\ln h|^\nu - h_0^{\alpha-1} |\ln h_0|^\nu) - \frac{\nu}{1-\alpha} \int_h^{h_0} s^{\alpha-2} |\ln s|^{\nu-1} ds.$$

So we have

$$\int_h^{h_0} s^{\alpha-2}|\ln s|^\nu ds \le \frac{1}{1-\alpha}h^{\alpha-1}|\ln h|^\nu - \frac{\nu}{1-\alpha}\int_h^{h_0} s^{\alpha-2}|\ln s|^{\nu-1}ds.$$

If $\nu \ge 0$, the lemma is proved as in 1). If $\nu < 0$, notice $1 + \frac{\nu}{(1-\alpha)|\ln s|} \ge \frac{1}{2}$ when $s \le h_0 \le e^{\frac{2\nu}{1-\alpha}}$, we have $\int_h^{h_0} s^{\alpha-2}|\ln s|^\nu ds + \frac{\nu}{1-\alpha}\int_h^{h_0} s^{\alpha-2}|\ln s|^{\nu-1}ds = \int_h^{h_0} s^{\alpha-2}|\ln s|^\nu(1+\frac{\nu}{(1-\alpha)|\ln s|})ds \ge \frac{1}{2}\int_h^{h_0} s^{\alpha-2}|\ln s|^\nu ds$. Hence $\int_h^{h_0} s^{\alpha-2}|\ln s|^\nu ds \le \frac{2}{1-\alpha}h^{\alpha-1}|\ln h|^\nu$.

When $\alpha = 1$ and $\nu \ge 0$,

$$\int_h^{h_0} \frac{|\ln s|^\nu}{s}ds = \frac{1}{\nu+1}(|\ln h|^{\nu+1} - |\ln h_0|^{\nu+1}) \le \frac{1}{\nu+1}|\ln h|^{\nu+1}.$$

Both desired inequalities are proved. ∎

We first consider points on ∂D. When $t \in \partial D$, by Sokhotski-Plemelj Formula (see [7] for instance), the nontangential limit of Sf at $t \in \partial D$ is

$$\Phi f(t) := Sf(t) + \frac{1}{2}f(t) := \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - t}d\zeta + \frac{1}{2}f(t) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)-f(t)}{\zeta - t}d\zeta + f(t).$$

Here $Sf(t) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta-t}d\zeta$ is interpreted as the Principal Value when $t \in \partial D$ and is well defined if f is in Hölder spaces. In particular, $\frac{1}{2\pi i}\int_{\partial D}\frac{1}{\zeta-t}d\zeta = \frac{1}{2}$ when $t \in \partial D$. Let h_0 and c_0 be defined as in Lemma 4.1 and (3) respectively, $s_0 := \min_{1\le j\le N}\{s_j\} > 0$ and $\delta_0 := \inf_{1\le j\ne m\le N}\{|t-t'| : t \in \Gamma_j, t' \in \Gamma_m\} > 0$.

Lemma 4.2. *Let $0 < \alpha \le 1$. If $f \in C^{L^\alpha Log^\nu L}(D)$, then for $t, t+h \in \partial D$ with $|h| \le \min\{\frac{h_0}{3c_0}, \frac{s_0}{6c_0}, \frac{\delta_0}{2}\}$,*

$$|\Phi f(t+h) - \Phi f(t)| \le \begin{cases} C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha|\ln|h||^\nu, & 0<\alpha<1; \\ C\|f\|_{C^{L^1 Log^\nu L}(D)}|h||\ln|h||^{\nu+1}, & \alpha = 1 \end{cases}$$

for a constant C dependent only on D, α and ν.

Proof. Assume $t \in \Gamma_1$ without loss of generality. Since $|t+h-t| = |h| \le \frac{\delta_0}{2}$, $t+h \in \Gamma_1$ as well. By Sokhotski-Plemelj Formula,

$$\begin{aligned}\Phi f(t+h) - \Phi f(t) =& \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)-f(t+h)}{\zeta - t - h}d\zeta - \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)-f(t)}{\zeta-t}d\zeta + (f(t+h)-f(t)) \\ =& \frac{1}{2\pi i}(\int_{\cup_{j=1}^N\Gamma_j}\frac{f(\zeta)-f(t+h)}{\zeta-t-h}d\zeta - \int_{\cup_{j=1}^N\Gamma_j}\frac{f(\zeta)-f(t)}{\zeta-t}d\zeta) + (f(t+h)-f(t)).\end{aligned}$$

Because $\cup_{j=2}^N\Gamma_j$ does not intersect with Γ_1 and $t, t+h \in \Gamma_1$, we have $|\zeta - t| \geq C$ and $|\zeta - t - h| \geq C$ on $\cup_{j=2}^N\Gamma_j$ for some positive C dependent only on ∂D. It immediately follows that

$$\begin{aligned}
&\left|\int_{\cup_{j=2}^N\Gamma_j} \frac{f(\zeta)-f(t+h)}{\zeta-t-h}d\zeta - \int_{\cup_{j=2}^N\Gamma_j} \frac{f(\zeta)-f(t)}{\zeta-t}d\zeta\right|\\
=&\left|\int_{\cup_{j=2}^N\Gamma_j} \frac{(f(\zeta)-f(t))h+(f(t)-f(t+h))(\zeta-t)}{(\zeta-t-h)(\zeta-t)}d\zeta\right|\\
\leq&\int_{\cup_{j=2}^N\Gamma_j} C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha|\ln|h||^\nu|d\zeta|\\
\leq&C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha|\ln|h||^\nu.
\end{aligned}$$

It thus suffices to show, in view of the chord-arc condition, for $t, t+h \in \Gamma_1$ with $\tilde{h} := |t+h, t| \leq \min\{\frac{h_0}{3}, \frac{s_0}{6}\}$,

$$\left|\int_{\Gamma_1} \frac{f(\zeta)-f(t+h)}{\zeta-t-h}d\zeta - \int_{\Gamma_1} \frac{f(\zeta)-f(t)}{\zeta-t}d\zeta\right| \leq \begin{cases} C\|f\|_{C^{L^\alpha Log^\nu L}(D)}\tilde{h}^\alpha|\ln\tilde{h}|^\nu, & 0<\alpha<1;\\ C\|f\|_{C^{L^1 Log^\nu L}(D)}\tilde{h}|\ln\tilde{h}|^{\nu+1}, & \alpha=1.\end{cases}$$

Due to the $C^{1,\alpha}$ boundary of Γ_1, $|d\zeta| \approx |ds|$. Denote by s the arclength parameter of Γ_1 with $\zeta|_{s=0} = t$, and by l the arc on Γ_1 centered at t of total arclength $4\tilde{h}$. Recall that s_1 is the total arclength of Γ_1. The chord-arc condition implies $|\zeta - t| \approx |\zeta, t| = \min\{s, s_1 - s\}$ on Γ_1.

On l, notice that

$$|\zeta-t-h| \geq C|\zeta, t+h| \geq C||\zeta,t| - |t+h,t|| = \begin{cases} C|s-\tilde{h}|, & s \leq \frac{s_1}{2};\\ C|s_1-s-\tilde{h}|, & s \geq \frac{s_1}{2}.\end{cases}$$

Together with the fact that $|f(\zeta) - f(t+h)| \leq \|f\|_{C^{L^\alpha Log^\nu L}(D)}|\zeta - t - h|^\alpha|\ln|\zeta - t - h||^\nu$ and

$|f(\zeta)-f(t)|\leq \|f\|_{C^{L^\alpha Log^\nu L}(D)}|\zeta-t|^\alpha|\ln|\zeta-t||^\nu$ on l, one obtains from Lemma 4.1,

$$\begin{aligned}&\left|\int_l \frac{f(\zeta)-f(t+h)}{\zeta-t-h}d\zeta-\int_l\frac{f(\zeta)-f(t)}{\zeta-t}d\zeta\right|\\ \leq& C\|f\|_{C^{L^\alpha Log^\nu L}(D)}(\int_l|\zeta-t-h|^{\alpha-1}|\ln|\zeta-t-h||^\nu|d\zeta|+\int_l|\zeta-t|^{\alpha-1}|\ln|\zeta-t||^\nu|d\zeta|)\\ \leq& C\|f\|_{C^{L^\alpha Log^\nu L}(D)}(\int_0^{2\tilde h}|s-\tilde h|^{\alpha-1}|\ln|s-\tilde h||^\nu ds+\int_0^{2\tilde h}|s|^{\alpha-1}|\ln s|^\nu ds)\\ \leq& C\|f\|_{C^{L^\alpha Log^\nu L}(D)}(\int_0^{3\tilde h}s^{\alpha-1}|\ln s|^\nu ds+\int_0^{2\tilde h}|s|^{\alpha-1}|\ln s|^\nu ds)\\ \leq& C\|f\|_{C^{L^\alpha Log^\nu L}(D)}\tilde h^\alpha|\ln\tilde h|^\nu.\end{aligned}$$

Next we estimate

$$\begin{aligned}&\left|\int_{\Gamma_1\setminus l}\frac{f(\zeta)-f(t+h)}{\zeta-t-h}d\zeta-\int_{\Gamma_1\setminus l}\frac{f(\zeta)-f(t)}{\zeta-t}d\zeta\right|\\ \leq&\left|\int_{\Gamma_1\setminus l}(f(\zeta)-f(t+h))(\frac{1}{\zeta-t-h}-\frac{1}{\zeta-t})d\zeta\right|+\left|\int_{\Gamma_1\setminus l}\frac{f(t+h)-f(t)}{\zeta-t}d\zeta\right|=:I+II.\end{aligned}$$

Since $II=|f(t+h)-f(t)||\frac{1}{2\pi i}\int_{\Gamma_1-l}\frac{1}{\zeta-t}d\zeta|\leq C|f(t+h)-f(t)|$, II is bounded by $C\|f\|_{C^{L^\alpha Log^\nu L}(D)}\tilde h^\alpha|\ln\tilde h|^\nu$. Now we treat $I=|\frac{h}{2\pi}\int_{\Gamma_1\setminus l}\frac{f(\zeta)-f(t+h)}{(\zeta-t-h)(\zeta-t)}d\zeta|$. Due to the chord-arc condition, $|\zeta,t+h|\geq|\zeta,t|-|t,t+h|=\min\{s-\tilde h,s_1-s-\tilde h\}\geq\tilde h$ on $\Gamma_1\setminus l$. Hence

$$|\zeta-t|\leq|\zeta,t|\leq|\zeta,t+h|+|t+h,t|=|\zeta,t+h|+\tilde h\leq 2|\zeta,t+h|\leq C|\zeta-t-h|,$$

or equivalently,

$$|\zeta-t-h|>C|\zeta-t|\approx\min\{s,s_1-s\}$$

on $\Gamma_1\setminus l$. Let l' be the arc on Γ_1 centered at t with total arclength $\min\{2h_0,s_1\}$ so $l\subset l'\subset\Gamma_1$. Therefore

$$\begin{aligned}I\leq& C\|f\|_{C^{L^\alpha Log^\nu L}(D)}\tilde h\int_{l'\setminus l}\frac{|\zeta-t-h|^{\alpha-1}|\ln|\zeta-t-h||^\nu}{|\zeta-t|}|d\zeta|+\\ &+C\|f\|_{C(D)}\tilde h\int_{\Gamma_1\setminus l'}\frac{1}{|\zeta-t-h||\zeta-t|}|d\zeta|\\ \leq& C\|f\|_{C^{L^\alpha Log^\nu L}(D)}\tilde h\int_{2\tilde h}^{\min\{h_0,\frac{s_1}{2}\}}s^{\alpha-2}|\ln s|^\nu ds+C\|f\|_{C(D)}\tilde h\int_{\min\{h_0,\frac{s_1}{2}\}}^{\frac{s_1}{2}}\frac{1}{s^2}ds\\ \leq& C\|f\|_{C^{L^\alpha Log^\nu L}(D)}\tilde h(\int_{2\tilde h}^{h_0}s^{\alpha-2}|\ln s|^\nu ds+1).\end{aligned}$$

It follows immediately from Lemma 4.1,

$$I \le \begin{cases} C\|f\|_{C^{L^\alpha Log^\nu L}(D)}\tilde{h}^\alpha |\ln \tilde{h}|^\nu, & 0<\alpha<1; \\ C\|f\|_{C^{L^1 Log^\nu L}(D)}\tilde{h} |\ln \tilde{h}|^{\nu+1}, & \alpha=1. \end{cases}$$

∎

For Hölder semi-norm of S at interior points of the domain, classical singular integral operators theory utilizes a generalized version of the Maximum Modulus Theorem of holomorphic functions to a branch of $\frac{Sf(z)-Sf(z')}{(z-z')^\alpha}$ to achieve the boundedness. We adopt here a different approach introduced in [5].

Given $t \in \partial D$, define $\mathcal{N}(t)$, a nontangential approach region (cf. [3] [5]) as follows.

$$\mathcal{N}(t) = \{z \in D : |z-t| \le \min\{4\text{dist}(z,\partial D), \frac{\delta_0}{4}\}\}.$$

If $z \in \mathcal{N}(t)$, then $|\zeta - z| \ge \text{dist}(z,\partial D) \ge \frac{1}{4}|z-t|$ for all $\zeta \in \partial D$. Hence $|\zeta - z| \ge \frac{1}{4}(|\zeta - t| - |\zeta - z|)$, implying $|\zeta - z| \ge \frac{1}{5}|\zeta - t|$ on ∂D. Altogether, for $z \in \mathcal{N}(t)$ and $\zeta \in \partial D$,

$$|\zeta - z| \ge \max\{\frac{1}{4}|z-t|, \frac{1}{5}|\zeta - t|\}. \tag{4}$$

Lemma 4.3. *Let $0 < \alpha \le 1$. If $f \in C^{L^\alpha Log^\nu L}(D)$ and $t \in \partial D$, then for $z \in \mathcal{N}(t)$ with $|z-t| \le \min\{h_0, \frac{s_0}{2}\}$,*

$$|Sf(z) - \Phi f(t)| \le \begin{cases} C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|z-t|^\alpha |\ln|z-t||^\nu, & 0<\alpha<1; \\ C\|f\|_{C^{L^1 Log^\nu L}(D)}|z-t| |\ln|z-t||^{\nu+1}, & \alpha=1 \end{cases}$$

for a constant C dependent only on D, α and ν.

Proof. Without loss of generality, assume $t \in \Gamma_1$. By Cauchy's integral formula, $\frac{1}{2\pi i}\int_{\partial D} \frac{1}{\zeta - z} d\zeta = 1$ when $z \in D$. Hence

$$\begin{aligned} Sf(z) - \Phi f(t) =& \left(\frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)-f(t)}{\zeta - z} d\zeta + f(t)\right) - \left(\frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)-f(t)}{\zeta - t} d\zeta + f(t)\right) \\ =& \frac{z-t}{2\pi i}\int_{\partial D} \frac{f(\zeta)-f(t)}{(\zeta-z)(\zeta-t)} d\zeta \\ =& \frac{z-t}{2\pi i}\int_{l} \frac{f(\zeta)-f(t)}{(\zeta-z)(\zeta-t)} d\zeta + \frac{z-t}{2\pi i}\int_{\Gamma_1 \setminus l} \frac{f(\zeta)-f(t)}{(\zeta-z)(\zeta-t)} d\zeta + \\ &+ \frac{z-t}{2\pi i}\int_{\cup_{j=2}^N \Gamma_j} \frac{f(\zeta)-f(t)}{(\zeta-z)(\zeta-t)} d\zeta \\ =&: I + II + III \end{aligned}$$

Here l is the arc on Γ_1 centered at t of total arclength $2|z-t| =: 2|h|$. For III, when $\zeta \in \cup_{j=2}^N \Gamma_j$, $|\zeta - t| \geq \delta_0$, and $|\zeta - z| \geq |\zeta - t| - |t - z| \geq \delta_0 - \frac{\delta_0}{4} = \frac{3\delta_0}{4}$. We thus deduce

$$|III| \leq C|h|\|f\|_{C(D)} \leq C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha |\ln|h||^\nu.$$

Next we estimate I and II. It follows from (4) and Lemma 4.1 that

$$\begin{aligned}
|I| \leq & C|h|\|f\|_{C^{L^\alpha Log^\nu L}(D)} \int_l \frac{|\zeta - t|^{\alpha-1}|\ln|\zeta - t||^\nu}{|\zeta - z|}|d\zeta| \\
\leq & C|h|\|f\|_{C^{L^\alpha Log^\nu L}(D)} \int_l \frac{|\zeta - t|^{\alpha-1}|\ln|\zeta - t||^\nu}{|z - t|}|d\zeta| \\
\leq & C\|f\|_{C^{L^\alpha Log^\nu L}(D)} \int_0^{|h|} s^{\alpha-1}|\ln s|^\nu ds \\
\leq & C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha|\ln|h||^\nu.
\end{aligned}$$

For II, let l' be the arc on Γ_1 centered at t of arclength $\min\{2h_0, s_1\}$ as in the previous lemma.

$$\begin{aligned}
|II| \leq & C|h|\|f\|_{C^{L^\alpha Log^\nu L}(D)} \int_{l' \setminus l} \frac{|\zeta - t|^\alpha |\ln|\zeta - t||^\nu}{|\zeta - t|^2}|d\zeta| + C|h|\|f\|_{C(D)} \int_{\Gamma_1 \setminus l'} \frac{1}{|\zeta - t|^2}|d\zeta|) \\
\leq & C|h|\|f\|_{C^{L^\alpha Log^\nu L}(D)} (\int_{|h|}^{h_0} s^{\alpha-2}|\ln s||^\nu ds + \int_{\min\{h_0, \frac{s_1}{2}\}}^{\frac{s_1}{2}} \frac{1}{s^2} ds) \\
\leq & \begin{cases} C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha |\ln|h||^\nu, & 0 < \alpha < 1; \\ C\|f\|_{C^{L^1 Log^\nu L}(D)}|h||\ln|h||^{\nu+1}, & \alpha = 1. \end{cases}
\end{aligned}$$

∎

Lemma 4.4. *Let $0 < \alpha \leq 1$. If $f \in C^{L^\alpha Log^\nu L}(D)$ and $t \in \partial D$, then for $z, z+h \in \mathcal{N}(t)$ with $|h| \leq \min\{h_0, \frac{\delta_0}{4}, \frac{s_0}{2}\}$,*

$$|Sf(z+h) - Sf(z)| \leq \begin{cases} C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha |\ln|h||^\nu, & 0 < \alpha < 1; \\ C\|f\|_{C^{L^1 Log^\nu L}(D)}|h||\ln|h||^{\nu+1}, & \alpha = 1 \end{cases}$$

for a constant C dependent only on D, α and ν.

Proof. Without loss of generality, assume $t \in \Gamma_1$. Since $z, z+h \in D$, by Cauchy integral formula, we have

$$
\begin{aligned}
Sf(z+h)-Sf(z)=&\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)-f(t)}{\zeta-z-h}-\frac{f(\zeta)-f(t)}{\zeta-z}d\zeta+\\
&+\frac{f(t)}{2\pi i}(\int_{\partial D}\frac{1}{\zeta-z-h}d\zeta-\int_{\partial D}\frac{1}{\zeta-z}d\zeta)\\
=&\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)-f(t)}{\zeta-z-h}-\frac{f(\zeta)-f(t)}{\zeta-z}d\zeta\\
=&\frac{h}{2\pi i}\int_{\partial D}\frac{f(\zeta)-f(t)}{(\zeta-z-h)(\zeta-z)}d\zeta\\
=&\frac{h}{2\pi i}\int_{l}\frac{f(\zeta)-f(t)}{(\zeta-z-h)(\zeta-z)}d\zeta+\frac{h}{2\pi i}\int_{\Gamma_1\setminus l}\frac{f(\zeta)-f(t)}{(\zeta-z-h)(\zeta-z)}d\zeta+\\
&+\frac{h}{2\pi i}\int_{\cup_{j=2}^N\Gamma_j}\frac{f(\zeta)-f(t)}{(\zeta-z-h)(\zeta-z)}d\zeta\\
=&:I+II+III.
\end{aligned}
$$

Here l is the arc on Γ_1 centered at t of total arclength $2|h|$. Note when $\zeta\in\cup_{j=2}^N\Gamma_j$, $|\zeta-z|\geq|\zeta-t|-|t-z|\geq\frac{3\delta_0}{4}$ and $|\zeta-z-h|\geq|\zeta-t|-|t-z|-|h|\geq\frac{\delta_0}{2}$. As in the proof of Lemma 4.3, we immediately obtain

$$|III|\leq C|h|\|f\|_{C(D)}\leq C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha|\ln|h||^\nu.$$

For the remaining two terms I and II, without loss of generality assume $|z-t|\geq|z+h-t|$. Then

$$|z-t|\geq\frac{1}{2}(|z-t|+|z+h-t|)\geq\frac{|h|}{2}.$$

Together with (4), we have

$$|\zeta-z|\geq\max\{C|z-t|,C|\zeta-t|\}\geq\max\{C|h|,C|\zeta-t|\}.\tag{5}$$

Recalling

$$|\zeta-z-h|\geq\max\{C|z+h-t|,C|\zeta-t|\}\geq C|\zeta-t|,$$

and combining it with (5) and Lemma 4.1, one obtains

$$
\begin{aligned}
|I|\leq&C\|f\|_{C^{L^\alpha Log^\nu L}(D)}\int_l|\zeta-t|^{\alpha-1}|\ln|\zeta-t||^\nu|d\zeta|\\
\leq&C\|f\|_{C^{L^\alpha Log^\nu L}(D)}\int_0^{|h|}s^{\alpha-1}|\ln s|^\nu ds\\
\leq&C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha|\ln|h||^\nu.
\end{aligned}
$$

Denote by l' the arc on Γ_1 centered at t of total arclength $\min\{2h_0, s_1\}$. Then

$$\begin{aligned}|II| \leq & C|h|\|f\|_{C^{L^\alpha Log^\nu L}(D)}\int_{l'\setminus l}\frac{|\zeta-t|^\alpha|\ln|\zeta-t||^\nu}{|\zeta-t|^2}|d\zeta| + C|h|\|f\|_{C(D)}\int_{\Gamma_1\setminus l'}\frac{1}{|\zeta-t|^2}|d\zeta|\\ \leq & C|h|\|f\|_{C^{L^\alpha Log^\nu L}(D)}(\int_{|h|}^{h_0} s^{\alpha-2}|\ln s|^\nu ds + \int_{\min\{h_0,\frac{s_1}{2}\}}^{\frac{s_1}{2}}\frac{1}{s^2}ds)\\ \leq & \begin{cases} C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha|\ln|h||^\nu, & 0<\alpha<1;\\ C\|f\|_{C^{L^1 Log^\nu L}(D)}|h||\ln|h||^{\nu+1}, & \alpha=1.\end{cases}\end{aligned}$$

■

We now are in a position to estimate the Log-Hölder semi-norm of Sf in D.

Proposition 4.5. *Let $0<\alpha\leq 1$. If $f\in C^{L^\alpha Log^\nu L}(D)$, then for $z, z+h\in D$ with $|h|\leq\min\{\frac{h_0}{9c_0}, \frac{s_0}{18c_0}, \frac{\delta_0}{16}, \frac{e^{-\nu-1}}{3}\}$,*

$$|Sf(z+h)-Sf(z)|\leq\begin{cases} C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha|\ln|h||^\nu, & 0<\alpha<1;\\ C\|f\|_{C^{L^1 Log^\nu L}(D)}|h||\ln|h||^{\nu+1}, & \alpha=1\end{cases}\tag{6}$$

for a constant C dependent only on D, α and ν.

Proof. Let $t, t'\in\partial D$ such that $|z-t| = \operatorname{dist}(z,\partial D)$ and $|z+h-t'| = \operatorname{dist}(z+h,\partial D)$. Without loss of generality, assume $t\in\Gamma_1$. If both $|z-t|$ and $|z+h-t'|$ are greater than $\frac{\delta_0}{16}$, then $|\zeta-z|\geq|z-t|\geq\frac{\delta_0}{16}$ and $|\zeta-z-h|\geq|t'-z-h|\geq\frac{\delta_0}{16}$ on $\zeta\in\partial D$. Consequently,

$$\begin{aligned}|Sf(z+h)-Sf(z)| = & \left|\frac{h}{2\pi}\int_{\partial D}\frac{f(\zeta)-f(t)}{(\zeta-z-h)(\zeta-z)}d\zeta\right|\\ \leq & C|h|\|f\|_{C(D)}\\ \leq & C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha|\ln|h||^\nu.\end{aligned}$$

Otherwise, suppose one of $|z-t|$ and $|z+h-t'|$ is less than $\frac{\delta_0}{16}$. Say $|z-t|\leq\frac{\delta_0}{16}$, implying $|z+h-t'|\leq|z+h-t|\leq|z-t|+|h|\leq\frac{\delta_0}{8}$. The other case is done similarly. Hence $z\in\mathcal{N}(t)$ and $z+h\in\mathcal{N}(t')$ by definition. Thus if in addition either $z+h\in\mathcal{N}(t)$ or $z\in\mathcal{N}(t')$, (6) follows directly from Lemma 4.4.

We are only left with the case when both $z+h\in D\setminus\mathcal{N}(t)$ and $z\in D\setminus\mathcal{N}(t')$. Noticing that $|z+h-t|\leq|z-t|+|h|<\frac{\delta_0}{4}$ and $|z-t'|\leq|z-(z+h)|+|z+h-t'|<\frac{\delta_0}{4}$, it implies by definition of $\mathcal{N}(t)$ and $\mathcal{N}(t')$ that $|z+h-t|\geq 4|z+h-t'|$ and $|z-t'|\geq 4|z-t|$, or equivalently,

$$|z+h-t'|\leq\frac{1}{4}|z+h-t| \quad\text{and}\quad |z-t|\leq\frac{1}{4}|z-t|.$$

We claim that

$$|z+h-t'|\le |h|, |z-t|\le |h|, \text{ and } |t-t'|\le 3|h|. \tag{7}$$

Indeed, since $|z+h-t'|\le \frac{1}{4}|z+h-t|\le \frac{1}{4}(|z+h-t'|+|t'-t|)$, we have

$$|z+h-t'|\le \frac{1}{3}|t'-t|.$$

Similarly,

$$|z-t|\le \frac{1}{3}|t'-t|.$$

On the other hand, since $|t'-t|\le |t'-z-h|+|z+h-z|+|z-t|\le \frac{2}{3}|t'-t|+|h|$, one infers

$$|t'-t|\le 3|h|.$$

Hence

$$|z+h-t'|\le |h|, \quad |z-t|\le |h|.$$

The claim is proved.

Now we estimate

$$|Sf(z+h)-Sf(z)|\le |Sf(z+h)-\Phi f(t')|+|Sf(z)-\Phi f(t)|+|\Phi f(t)-\Phi f(t')|$$

for $z, z+h, t$ and t' as previously. Because $z+h\in \mathcal{N}(t')$ and $|z+h-t'|\le |h|\le \min\{h_0, \frac{s_0}{2}, e^{-\nu-1}\}$ by (7), we deduce from Lemma 4.3,

$$\begin{aligned}|Sf(z+h)-\Phi f(t')| &\le \left\{\begin{array}{c} C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|z+h-t'|^\alpha|\ln|z+h-t'||^\nu, 0<\alpha<1\\ C\|f\|_{C^{L^1 Log^\nu L}(D)}|z+h-t'||\ln|z+h-t'||^{\nu+1}, \alpha=1\end{array}\right.\\ &\le \left\{\begin{array}{c} C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha|\ln|h||^\nu, 0<\alpha<1;\\ C\|f\|_{C^{L^1 Log^\nu L}(D)}|h||\ln|h||^{\nu+1}, \alpha=1.\end{array}\right.\end{aligned}$$

Here we have used the non-decreasing property of the real-valued functions $s^\alpha|\ln s|^\nu$ and $s|\ln s|^{\nu+1}$ when s is less than $\min\{h_0, e^{-\nu-1}\}$. Similarly,

$$|Sf(z)-\Phi f(t)||\le \left\{\begin{array}{c} C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha|\ln|h||^\nu, 0<\alpha<1;\\ C\|f\|_{C^{L^1 Log^\nu L}(D)}|h||\ln|h||^{\nu+1}, \alpha=1.\end{array}\right.$$

Lastly, since $|t'-t|\le 3|h|\le \min\{\frac{h_0}{3c_0}, \frac{s_0}{6c_0}, \frac{\delta_0}{2}, e^{-\nu-1}\}$, by Lemma 4.2,

$$\begin{aligned}|\Phi f(t)-\Phi f(t')|| &\le \left\{\begin{array}{c} C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|t-t'|^\alpha|\ln|t-t'||^\nu, 0<\alpha<1\\ C\|f\|_{C^{L^1 Log^\nu L}(D)}|t-t'||\ln|t-t'||^{\nu+1}, \alpha=1\end{array}\right.\\ &\le \left\{\begin{array}{c} C\|f\|_{C^{L^\alpha Log^\nu L}(D)}|h|^\alpha|\ln|h||^\nu, 0<\alpha<1;\\ C\|f\|_{C^{L^1 Log^\nu L}(D)}|h||\ln|h||^{\nu+1}, \alpha=1.\end{array}\right.\end{aligned}$$

The proof of the proposition is complete. ∎

Theorem 4.6. *Let D be a bounded domain in $\mathbb{C}$ with $C^{1,\alpha}$ boundary, $k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \leq 1$. Then S defined in (2) sends $C^{L^\alpha Log^\nu L}(D)$ into itself when $0 < \alpha < 1$, and into $C^{L^1 Log^{\nu+1} L}(D)$ if $\alpha = 1$. Moreover, there exists a constant C dependent only on D, α and ν, such that for any $f \in C^{L^\alpha Log^\nu L}(D)$,*

$$\|Sf\|_{C^{L^\alpha Log^\nu L}(D)} \leq C\|f\|_{C^{L^\alpha Log^\nu L}(D)}$$

if $0 < \alpha < 1$, and

$$\|Sf\|_{C^{L^1 Log^{\nu+1} L}(D)} \leq C\|f\|_{C^{L^1 Log^\nu L}(D)}$$

if $\alpha = 1$.

Proof. Choose ϵ such that $0 < \epsilon < \alpha \leq 1$. We have $\|f\|_{C^\epsilon(D)} \leq C\|f\|_{C^{L^\alpha Log^\nu L}(D)}$ with C dependent only on ν, α, ϵ and D. Hence

$$\|Sf\|_{C(D)} \leq \|Sf\|_{C^\epsilon(D)} \leq C\|f\|_{C^\epsilon(D)} \leq C\|f\|_{C^{L^\alpha Log^\nu L}(D)}.$$

The rest of the theorem follows directly from Proposition 4.5. ∎

5 Proof of Theorem 1.2

We are now in a position to prove Theorem 1.2. Let $\Omega = D \times \Lambda \subset \mathbb{C}^2$, where $D \subset \mathbb{C}$ is a bounded domain with $C^{k+1,\alpha}$ boundary, and Λ is an open set in $\mathbb{R}$ or $\mathbb{C}$. Let S be defined in (1). For $0 < \epsilon < \alpha \leq 1$, there exists a constant C dependent only on ν, α, ϵ and Ω, such that for all $f \in C^{k, L^\alpha Log^\nu L}(\Omega)$,

$$\|Sf\|_{C^k(\Omega)} \leq C\|Sf\|_{C^{k,\epsilon}(\Omega)} \leq C\|f\|_{C^{k,\epsilon}(\Omega)} \leq C\|f\|_{C^{k, L^\alpha Log^\nu L}(\Omega)}.$$

We shall further prove for $|\gamma| = k$, $H^{\nu+1}[D^\gamma Sf] \leq C\|f\|_{C^{k, L^\alpha Log^\nu L}(\Omega)}$. Noticing that Sf is holomorphic with respect to $z \in D$, we assume $D^\gamma = \partial_z^{\gamma_1} D_\lambda^{\gamma_2}$. Making use of integration by part, we obtain for any $(z, \lambda) \in \Omega$,

$$\begin{aligned} D^\gamma Sf(z,\lambda) =& \frac{1}{2\pi i}\partial_z^{\gamma_1} S D_\lambda^{\gamma_2} f(z,\lambda) \\ =& \frac{1}{2\pi i}\partial_z^{\gamma_1 - 1}\int_{\partial D} \partial_z \frac{D_\lambda^{\gamma_2} f(\zeta,\lambda)}{\zeta - z} d\zeta \\ =& \frac{1}{2\pi i}\partial_z^{\gamma_1 - 1}\int_{\partial D} \frac{\partial_\zeta D_\lambda^{\gamma_2} f(\zeta,\lambda)}{\zeta - z} d\zeta \\ & \cdots \\ =:& \ \frac{1}{2\pi i}\int_{\partial D} \frac{\tilde{f}(\zeta,\lambda)}{\zeta - z} d\zeta = S\tilde{f}(z,\lambda) \end{aligned}$$

with $\tilde f := \partial_z^{\gamma_1} D_\lambda^{\gamma_2} f \in C^{L^\alpha Log^\nu L}(\Omega)$ and $\|\tilde f\|_{C^{L^\alpha Log^\nu L}(\Omega)} \le \|f\|_{C^{k,L^\alpha Log^\nu L}(\Omega)}$. (See [9] Proposition 3.3, or [11] p. 21-22 for more details.) Therefore, it will suffice to show $H^{\nu+1}[S\tilde f] \le C\|\tilde f\|_{C^{L^\alpha Log^\nu L}(\Omega)}$. By (the proof of) Proposition 4.5, it is already clear that for each $\lambda \in \Lambda$, $S\tilde f(\zeta, \lambda)$ as a function of $\zeta \in D$ satisfies

$$H_D^{\nu+1}[S\tilde f(\cdot, \lambda)] \le C\|\tilde f\|_{C^{L^\alpha Log^\nu L}(\Omega)}$$

for a constant C independent of $\tilde f$ and λ. In view of Lemma 2.1, we only need to show for each $z \in D$, $S\tilde f(z, \zeta)$ as a function of $\zeta \in \Lambda$ satisfies

$$H_\Lambda^{\nu+1}[S\tilde f(z, \cdot)] \le C\|\tilde f\|_{C^{L^\alpha Log^\nu L}(\Omega)} \tag{8}$$

for a constant C independent of $\tilde f$ and z.

To do so we shall apply the Maximum Modulus Principle of holomorphic functions. First consider $z = t \in \partial D$. Without loss of generality, assume $t \in \Gamma_1$. By Sokhotski–Plemelj Formula, the non-tangential limit of $S\tilde f$ at $(t, \lambda) \in \partial D \times \Lambda$ is

$$\Phi\tilde f(t, \lambda) := \frac{1}{2\pi i}\int_{\partial D} \frac{\tilde f(\zeta, \lambda)}{\zeta - t} d\zeta + \frac{1}{2}\tilde f(t, \lambda).$$

Here the first term is interpreted as the Principal Value. We shall prove that for $\lambda, \lambda + h \in \Lambda$ with $0 < |h| \le \min\{h_0, \frac{s_1}{2}\}$,

$$\left|\int_{\partial D} \frac{\tilde f(\zeta, \lambda + h)}{\zeta - t} d\zeta - \int_{\partial D} \frac{\tilde f(\zeta, \lambda)}{\zeta - t} d\zeta\right| \le C|h|^\alpha |\ln|h||^{\nu+1}\|\tilde f\|_{C^{L^\alpha Log^\nu L}(\Omega)} \tag{9}$$

for a constant C independent of $\tilde f, t, \lambda$ and h.

Indeed, write

$$\begin{aligned}\int_{\partial D} \frac{\tilde f(\zeta, \lambda + h) - \tilde f(\zeta, \lambda)}{\zeta - t} d\zeta &= \int_{\partial D} \frac{\tilde f(\zeta, \lambda + h) - \tilde f(t, \lambda + h) - \tilde f(\zeta, \lambda) + \tilde f(t, \lambda)}{\zeta - t} d\zeta \\ &\quad + (\tilde f(t, \lambda + h) - \tilde f(t, \lambda)) \int_{\partial D} \frac{1}{\zeta - t} d\zeta \\ &=: I + II.\end{aligned}$$

Since $|\int_{\partial D} \frac{1}{\zeta - t} d\zeta|$ is bounded in terms of the Principal Value,

$$|II| \le C|h|^\alpha |\ln|h||^{\nu+1}\|\tilde f\|_{C^{L^\alpha Log^\nu L}(\Omega)}$$

for a constant C independent of $\tilde{f}, t, \lambda$ and h.

For I, let l be the arc on ∂D that is centered at t with total arclength $2|h|$ and s be an arclength parameter of ∂D such that $\zeta|_{s=0} = t$. In particular, $l \subset \Gamma_1$. Then

$$
\begin{aligned}
I = &\int_l \frac{\tilde{f}(\zeta,\lambda+h) - \tilde{f}(t,\lambda+h) - \tilde{f}(\zeta,\lambda) + \tilde{f}(t,\lambda)}{\zeta - t} d\zeta + \\
&+ \int_{\Gamma_1 \setminus l} \frac{(\tilde{f}(\zeta,\lambda+h) - \tilde{f}(t,\lambda+h)) - (\tilde{f}(\zeta,\lambda) - \tilde{f}(t,\lambda))}{\zeta - t} d\zeta + \\
&+ \int_{\cup_{j=2}^N \Gamma_j} \frac{(\tilde{f}(\zeta,\lambda+h) - \tilde{f}(\zeta,\lambda)) - (\tilde{f}(t,\lambda+h) - \tilde{f}(t,\lambda))}{\zeta - t} d\zeta \\
=: &\, I_1 + I_2 + I_3.
\end{aligned}
$$

Because $|\zeta - t| \geq \delta_0$ for $\zeta \in \cup_{j=2}^N \Gamma_j$ and $|\tilde{f}(\zeta,\lambda+h) - \tilde{f}(\zeta,\lambda)) - (\tilde{f}(t,\lambda+h) - \tilde{f}(t,\lambda)| \leq |h|^\alpha |\ln|h||^\nu \|\tilde{f}\|_{C^{L^\alpha Log^\nu L}(\Omega)}$, one has

$$
|I_3| \leq C|h|^\alpha |\ln|h||^\nu \|\tilde{f}\|_{C^{L^\alpha Log^\nu L}(\Omega)}
$$

for a constant C independent of $\tilde{f}, t, \lambda$ and h.

Recall by the chord-arc condition, $|\zeta - t| \approx |\zeta, t| = \min\{s, s_1 - s\}$ on Γ_1. Moreover, the numerator of I_1 is less than $C|\zeta - t|^\alpha |\ln|\zeta - t||^\nu \|\tilde{f}\|_{C^{L^\alpha Log^\nu L}(\Omega)}$. It follows from Lemma 4.1

$$
|I_1| \leq C \int_0^{|h|} s^{\alpha-1} |\ln s|^\nu ds \leq C|h|^\alpha |\ln|h||^\nu \|\tilde{f}\|_{C^{L^\alpha Log^\nu L}(\Omega)}
$$

for a constant C independent of $\tilde{f}, t, \lambda$ and h.

Rearrange I_2 and we obtain

$$
\begin{aligned}
|I_2| &\leq \left| \int_{\Gamma_1 \setminus l} \frac{\tilde{f}(\zeta,\lambda+h) - \tilde{f}(\zeta,\lambda)}{\zeta - t} d\zeta \right| + \left| (\tilde{f}(t,\lambda+h) - \tilde{f}(t,\lambda)) \int_{\Gamma_1 \setminus l} \frac{1}{\zeta - t} d\zeta \right| \\
&\leq C|h|^\alpha |\ln|h||^\nu \|\tilde{f}\|_{C^\alpha(\Omega)} \int_{|h|}^{\frac{s_1}{2}} \frac{1}{s} ds + C|h|^\alpha |\ln|h||^\nu \|\tilde{f}\|_{C^{L^\alpha Log^\nu L}(\Omega)} \\
&\leq C|h|^\alpha |\ln|h||^{\nu+1} \|\tilde{f}\|_{C^{L^\alpha Log^\nu L}(\Omega)}.
\end{aligned}
$$

We have thus shown (9) holds, and hence there exists a constant C such that for each $z = t \in \partial D$, $H_\Lambda^{\nu+1}[\Phi\tilde{f}(t,\cdot)] \leq C\|\tilde{f}\|_{C^{L^\alpha Log^{\nu+1} L}(\Omega)}$ with C independent of $\tilde{f}$ and t. Notice that for each fixed $\zeta \in \Lambda$, $S\tilde{f}(z,\zeta)$ is holomorphic as a function of $z \in D$ and by Plemelj–Privalov Theorem, continuous up to the boundary with boundary value $\Phi\tilde{f}(z,\zeta)$. Applying the Maximum Modulus Theorem to the holomorphic function $\frac{S\tilde{f}(z,\lambda+h)-S\tilde{f}(z,\lambda)}{|h|^\alpha|\ln|h||^{\nu+1}}$ of $z\in D$ for each fixed λ and $\lambda+h$ with $0<|h|\le\min\{h_0,\frac{s_0}{2}\}$, we deduce

$$\begin{aligned}\sup_{z\in D}\frac{|S\tilde{f}(z,\lambda+h)-S\tilde{f}(z,\lambda)|}{|h|^\alpha|\ln|h||^{\nu+1}}\le&\sup_{t\in\partial D}\frac{|\Phi\tilde{f}(t,\lambda+h)-\Phi\tilde{f}(t,\lambda)|}{|h|^\alpha|\ln|h||^{\nu+1}}\\ =&\sup_{t\in\partial D}H_\Lambda^{\nu+1}[\Phi\tilde{f}(t,\cdot)]\\ \le& C\|\tilde{f}\|_{C^{L^\alpha Log^\nu L}(\Omega)},\end{aligned}$$

with C independent of $\tilde{f},z_1,z_2$ and z_2'. (8) is thus verified and the proof of Theorem 1.2 is complete.

We conclude the section by pointing out that the proof of Tumanov's example in Section 3 indicates that for any $\mu<1$, S does not send $C^\alpha(\triangle^2)$ into $C^{L^\alpha Log^\mu L}(\triangle^2), 0<\alpha<1$. Theorem 1.2 thus is sharp in view of the example.

6 The proof of Theorem 1.3

Let $D_j\subset\mathbb{C}$, $j=1,\dots,n$, be bounded domains with $C^{k+1,\alpha}$ boundary, $n\ge 2$, $k\in\mathbb{Z}^+\cup\{0\}, 0<\alpha\le 1$, and $\Omega:=D_1\times\cdots\times D_n$. Given a function $f\in C^{k,L^\alpha Log^\nu L}(\Omega)$, since $C^{k,L^\alpha Log^\nu L}(\Omega)\overset{i}{\hookrightarrow} C^{k,\epsilon}(\Omega)$ for $0<\epsilon<\alpha$, the following two operators are well defined for $z\in\Omega$,

$$\begin{aligned}T_jf(z):=&-\frac{1}{2\pi i}\int_{D_j}\frac{f(z_1,\dots,z_{j-1},\zeta_j,z_{j+1},\dots,z_n)}{\zeta_j-z_j}d\bar{\zeta_j}\wedge\zeta_j;\\ S_jf(z):=&\frac{1}{2\pi i}\int_{\partial D_j}\frac{f(z_1,\dots,z_{j-1},\zeta_j,z_{j+1},\dots,z_n)}{\zeta_j-z_j}d\zeta_j.\end{aligned}\tag{10}$$

By Theorem 1.2, S_j is a bounded operator sending $C^{k,L^\alpha Log^\nu L}(\Omega)$ into $C^{k,L^\alpha Log^{\nu+1} L}(\Omega)$. It was proved in [9] that the operator T_j is bounded between $C^{k,\alpha}(\Omega)$. In the following, we generalize this result and show T_j is bounded sending $C^{k,L^\alpha Log^\nu L}(\Omega)$ into itself.

Proposition 6.1. *For each $j\in\{1,\dots,n\}$, T_j is a bounded operator sending $C^{k,L^\alpha Log^\nu L}(\Omega)$ into $C^{k,L^\alpha Log^\nu L}(\Omega)$, $k\in\mathbb{Z}^+\cup\{0\}, 0<\alpha\le 1,\nu\in\mathbb{R}$. Namely, there exists a constant C dependent only on Ω,k,α and ν, such that for $f\in C^{k,L^\alpha Log^\nu L}(\Omega)$,*

$$\|T_jf\|_{C^{k,L^\alpha Log^\nu L}(\Omega)}\le C\|f\|_{C^{k,L^\alpha Log^\nu L}(\Omega)}.$$

Proof. Without loss of generality, we assume $n = 2$ and $j = 1$. As in [9], $\|T_1 f\|_{C^k(\Omega)} \le C\|f\|_{C^k(\Omega)}$ for a constant C independent of f. We only need to show

$$H^\nu[D^\gamma T_1 f] \le C\|f\|_{C^{k,L^\alpha Log^\nu L}(\Omega)}$$

for some constant independent of f for all $|\gamma| = k$.

Write $D^\gamma = D_1^{\gamma_1} D_2^{\gamma_2}$. Then $D^\gamma T_1 f = D_1^{\gamma_1} T_1(D_2^{\gamma_2} f)$. If $\alpha < 1$, choose a positive number $0 < \epsilon < 1 - \alpha$. So $\alpha + \epsilon < 1$ and for each $z_2 \in D_2$, $\|D^\gamma T_1 f(\cdot, z_2)\|_{C^{L^\alpha Log^\nu L}(D_1)} \le C\|D_1^{\gamma_1} T_1(D_2^{\gamma_2} f)(\cdot, z_2)\|_{C^{\alpha+\epsilon}(D_1)}$ for some constant C independent of f and z_2. We shall show for each $z_2 \in D_2$, $\|D_1^{\gamma_1} T_1(D_2^{\gamma_2} f)(\cdot, z_2)\|_{C^{\alpha+\epsilon}(D_1)} \le C\|f\|_{C^{|\gamma|}(\Omega)}$. Indeed, by making use of Theorem 2.2, if $\gamma_1 = 0$,

$$\|D_1^{\gamma_1} T_1(D_2^{\gamma_2} f)(\cdot, z_2)\|_{C^{\alpha+\epsilon}(D_1)} = \|T_1(D_2^{\gamma_2} f)(\cdot, z_2)\|_{C^{\alpha+\epsilon}(D_1)} \le C\|D_2^{\gamma_2} f\|_{C(\Omega)} \le C\|f\|_{C^{\gamma_2}(\Omega)};$$

If $\gamma_1 \ge 1$, then

$$\|D_1^{\gamma_1} T_1(D_2^{\gamma_2} f)(\cdot, z_2)\|_{C^{\alpha+\epsilon}(D_1)} \le C\|D_2^{\gamma_2} f\|_{C^{\gamma_1 - 1, \alpha+\epsilon}(\Omega)} \le C\|f\|_{C^{\gamma_1+\gamma_2}(\Omega)}$$

for some constant C independent of f and z_2. Altogether, $D^\gamma T_1 f(\zeta, z_2)$ as a function of $\zeta \in D_1$ satisfies

$$\|D^\gamma T_1 f(\cdot, z_2)\|_{C^{L^\alpha Log^\nu L}(D_1)} \le C\|D_1^{\gamma_1} T_1(D_2^{\gamma_2} f)(\cdot, z_2)\|_{C^{\alpha+\epsilon}(D_1)} \le C\|f\|_{C^{|\gamma|}(\Omega)} \le C\|f\|_{C^{k,L^\alpha Log^\nu L}(\Omega)}$$

for some constant C independent of f and z_2. If $\alpha = 1$ (so $\nu \ge 0$), choose $\epsilon < 1$. Then $\|D^\gamma T_1 f(\cdot, z_2)\|_{C^{L^1 Log^\nu L}(D_1)} \le C\|D_1^{\gamma_1} T_1(D_2^{\gamma_2} f)(\cdot, z_2)\|_{C^1(D_1)} \le C\|T_1(D_2^{\gamma_2} f(\cdot, z_2))\|_{C^{\gamma_1+1,\epsilon}(D_1)}$ and hence by Theorem 2.2,

$$\|D^\gamma T_1 f(\cdot, z_2)\|_{C^{L^1 Log^\nu L}(D_1)} \le C\|D_2^{\gamma_2} f\|_{C^{\gamma_1,\epsilon}(\Omega)} \le C\|f\|_{C^{|\gamma|,\epsilon}(\Omega)} \le C\|f\|_{C^{k,L^1 Log^\nu L}(\Omega)}$$

for some C independent of f and z_2.

Let $z_2'(\ne z_2) \in D_2$ with $|z_2 - z_2'| \le h_0$ and consider $F_{z_2,z_2'}(\zeta) := \frac{D_2^{\gamma_2} f(\zeta, z_2) - D_2^{\gamma_2} f(\zeta, z_2')}{|z_2 - z_2'|^\alpha |\ln|z_2 - z_2'||^\nu}$ on D_1. Since $f \in C^{k,L^\alpha Log^\nu L}(\Omega)$, $F_{z_2,z_2'} \in C^{\gamma_1}(D_1)$ and $\|F_{z_2,z_2'}\|_{C^{\gamma_1}(D_1)} \le \|f\|_{C^{k,L^\alpha Log^\nu L}}(\Omega)$. If $\gamma_1 = 0$,

$$\|D_1^{\gamma_1} T_1 F_{z_2,z_2'}\|_{C(D_1)} = \|T_1 F_{z_2,z_2'}\|_{C(D_1)} \le C\|F_{z_2,z_2'}\|_{C(D_1)} \le C\|f\|_{C^{k,L^\alpha Log^\nu L}(\Omega)}$$

for some constant C independent of f, z_2 and z_2'. For $\gamma_1 \ge 1$, choosing $\epsilon < \alpha$, we have from Theorem 2.2,

$$\|D_1^{\gamma_1} T_1 F_{z_2,z_2'}\|_{C(D_1)} \le C\|F_{z_2,z_2'}\|_{C^{\gamma_1-1,\epsilon}(D_1)} \le C\|F_{z_2,z_2'}\|_{C^{\gamma_1}(D_1)} \le C\|f\|_{C^{k,L^\alpha Log^\nu L}(\Omega)}$$

for some constant C independent of f, z_2 and z_2'. Hence for each $z_1 \in D_1$,

$$\frac{|D^\gamma T_1 f(z_1,z_2) - D^\gamma T_1 f(z_1,z_2')|}{|z_2-z_2'|^\alpha |\ln|z_2-z_2'||^\nu} = |D_1^{\gamma_1} T_1 F_{z_2,z_2'}(z_1)| \le \|D_1^{\gamma_1} T_1 F_{z_2,z_2'}\|_{C(D_1)} \le C\|f\|_{C^{k,L^\alpha Log^\nu L}(\Omega)},$$

where C is independent of f, z_1, z_2 and z_2'. The proof of the proposition is complete in view of Lemma 2.1. ∎

Theorem 6.2. *Let* $\mathbf{f} = \sum_{j=1}^n f_j d\bar{z}_j \in C^{k,L^\alpha Log^\nu L}(\Omega)$, $k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \le 1$ *and* $\nu \in \mathbb{R}$. *Then*

$$T\mathbf{f} := \sum_{j=1}^n \prod_{l=1}^{j-1} T_j S_l f_j = T_1 f_1 + T_2 S_1 f_2 + \cdots + T_n S_1 \cdots S_{n-1} f_n \tag{11}$$

is in $C^{k,L^\alpha Log^{\nu+n-1} L}(\Omega)$ *with* $\|T\mathbf{f}\|_{C^{k,L^\alpha Log^{\nu+n-1}L}(\Omega)} \le C\|\mathbf{f}\|_{C^{k,L^\alpha Log^\nu L}(\Omega)}$ *for some constant* C *dependent only on* Ω, k, α *and* ν.

Proof. The operator T in (11) is well defined on $C^{k,L^\alpha Log^\nu L}(\Omega)$ due to Theorem 4.6 and Proposition 6.1. Moreover, for each $1 \le j \le n$,

$$\begin{aligned}
\|\prod_{l=1}^{j-1} T_j S_l f_j\|_{C^{k,L^\alpha Log^{\nu+n-1}L}(\Omega)} &\le C\|\prod_{l=1}^{j-1} S_l f_j\|_{C^{k,L^\alpha Log^{\nu+n-1}L}(\Omega)} \\
&\le C\|\prod_{l=1}^{j-2} S_l f_j\|_{C^{k,L^\alpha Log^{\nu+n-2}L}(\Omega)} \\
&\le \cdots \\
&\le C\|f_j\|_{C^{k,L^\alpha Log^{\nu+n-j}L}(\Omega)} \\
&\le C\|f_j\|_{C^{k,L^\alpha Log^\nu L}(\Omega)}.
\end{aligned}$$

Therefore, $\|T\mathbf{f}\|_{C^{k,L^\alpha Log^{\nu+n-1}L}(\Omega)} \le C\|\mathbf{f}\|_{C^{k,L^\alpha Log^\nu L}(\Omega)}$. ∎

Proof of Theorem 1.3. When $\mathbf{f}$ is $\bar\partial$-closed, $T\mathbf{f}$ defined by (11) satisfies $\bar\partial T\mathbf{f} = \mathbf{f}$ (in the sense of distributions if $k=0$) by [9]. The rest of the theorem follows from Theorem 6.2. ∎

Proof of Example 1.4. $\mathbf{f}$ is well defined in $\triangle^2$ and $\mathbf{f} = (z_1-1)^{k+\alpha}\log^\nu(z_1-1)d\bar{z}_2 \in C^{k,L^\alpha Log^\nu L}(\triangle^2)$. Assuming $u \in C^{k,L^\beta Log^\nu L}(\triangle^2)$ solves $\bar\partial u = \mathbf{f}$ in $\triangle^2$ for some $\beta > \alpha$, then there exists a holomorphic function h in $\triangle^2$ such that $u = h + (z_1-1)^{k+\alpha}\log^\nu(z_1-1)\bar{z}_2$.

Now consider $w(\xi) := \int_{|z_2|=\frac{1}{2}} u(\xi, z_2)dz_2$ on $\xi \in \triangle = \{z \in \mathbb{C} : |z| < 1\}$. Since $u \in C^{k,L^\beta Log^\nu L}(\triangle^2)$, $w \in C^{k,L^\beta Log^\nu L}(\triangle)$ as well. On the other hand, a direct computation gives

$$
\begin{aligned}
w(\xi) =& \int_{|z_2|=\frac{1}{2}} (\xi-1)^{k+\alpha} \log^\nu(\xi-1)\bar{z}_2 dz_2 \\
=&(\xi-1)^{k+\alpha} \log^\nu(\xi-1) \int_{|z_2|=\frac{1}{2}} \frac{1}{4z_2} dz_2 \\
=&\frac{\pi i(\xi-1)^{k+\alpha} \log^\nu(\xi-1)}{2}.
\end{aligned}
$$

This contradicts with the fact that $(\xi-1)^{k+\alpha}\log^\nu(\xi-1) \notin C^{k,L^\beta Log^\nu L}(\triangle)$ for any $\beta > \alpha$. ∎

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