# Derivative formula for the solid Cauchy integral operator and its applications

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**Abstract**

In this paper, we obtain higher order derivative formula of the solid Cauchy integral operator on smooth bounded domains in $\mathbb{C}$. The formula allows us to develop some Calderón-Zygmund type theorem for higher order singular integrals. We also obtain a criterion for the solvability of the $\bar\partial$ problem in the flat category.

**Keywords** Cauchy integral, Calderón-Zygmund theory, flatness, minimal solutions.
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## 1 Introduction

Let $\Omega \subset \mathbb{C}$ be a bounded domain with smooth boundary. Given an integrable complex-valued function $f$ in $\Omega$, consider the operator associated with the Cauchy kernel $\Gamma(z) := \frac{1}{z}$ as follows.

$$Tf(z) := -\frac{1}{2\pi i}\int_\Omega \Gamma(\zeta - z) f(\zeta) d\bar\zeta \wedge d\zeta.$$

It is well known that $T$ is a solution operator to $\bar\partial$. In terms of operator theory, $T$ is a bounded linear operator sending $C^{k+\alpha}(\Omega)$ into $C^{k+1+\alpha}(\Omega)$, $0<\alpha<1$ and $k \in \mathbb{Z}^+ \cup \{0\}$. In Section 2, we obtain the following explicit higher order derivative formula for $T$.

**Theorem 1.1.** *Let $f \in C^{k+\alpha}(\Omega)$ with $0<\alpha<1$ and $k\in \mathbb{Z}^+\cup\{0\}$. Then for any $z \in \Omega$,*

$$\begin{aligned}\partial^{k+1}T(f)(z) = &-\frac{1}{2\pi i}\int_\Omega \partial_z^{k+1}\Gamma(\zeta - z)(f(\zeta) - P_k(\zeta, z))d\bar\zeta\wedge d\zeta \\ &-\frac{1}{2\pi i}\sum_{j=2}^{k+2}\partial^{k+2-j}\left(\sum_{\mu_1+\mu_2=j-2}\frac{\partial^\mu f(z)}{\mu!}\int_{\partial\Omega}\partial_z^{j-2}\Gamma(\zeta-z)(\zeta-z)^{\mu_1}\overline{(\zeta-z)}^{\mu_2}d\bar\zeta\right),\end{aligned} \tag{1}$$

*where $P_k(\zeta, z) = \sum_{\mu_1+\mu_2\le k}\frac{1}{\mu_1!\mu_2!}\partial^{\mu_1}\bar\partial^{\mu_2}f(z)(\zeta-z)^{\mu_1}\overline{(\zeta-z)}^{\mu_2}$, the Taylor expansion of $f$ at $z$ of degree $k$. In particular, if $\Omega$ is a disk in $\mathbb{C}$, then*

$$\partial^{k+1}T(f)(z) = -\frac{1}{2\pi i}\int_\Omega \partial_z^{k+1}\Gamma(\zeta - z)(f(\zeta) - P_k(\zeta, z))d\bar\zeta\wedge d\zeta.$$

Following the notation in the seminal work of Nijenhuis-Woolf [14], we define for $f \in C^{k+\alpha}(\Omega)$, the operator

$$^{k+2}Tf(z) := \frac{-1}{2\pi i}\int_\Omega \partial_z^{k+1}\Gamma(\zeta - z)(f(\zeta) - P_k(\zeta, z))d\bar\zeta\wedge d\zeta.$$

${}^{k+2}T$ is well defined in $C^{k+\alpha}(\Omega)$. In fact, as a consequence of Theorem 1.1, we obtain the Hölder estimate of ${}^{k+2}Tf$ in terms of the semi-norm $||f||_\alpha^{(k)} := \max_{i+j=k}\{||\partial^i\bar\partial^j f||_\alpha\}$ below. It is worthwhile to point out that, unlike the standard Schauder estimates for the elliptic operators, our estimate neither requires to shrink the domain nor involves the boundary value.

**Corollary 1.2.** *For any $f \in C^{k+\alpha}(\Omega)$, we have ${}^{k+2}Tf \in C^\alpha(\Omega)$ with*

$$\|{}^{k+2}Tf\|_\alpha \le C(\Omega,k)\|f\|_\alpha^{(k)},$$

*where $C(\Omega,k)$ is a constant depending only on $\Omega$ and $k$.*

We next study the following principal value of the singular integral.

**Definition 1.3.** For $k \in \mathbb{Z}^+ \cup \{0\}$ and $f \in C_0^\infty(\mathbb{C})$,

$$\mathcal{P}_k(f) := p.v.\frac{1}{2\pi i}\int_{\mathbb{C}} \partial^k\Gamma(\zeta - z)f(\zeta)d\bar\zeta\wedge d\zeta = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\mathbb{C}\setminus D_\epsilon(z)} \partial^k\Gamma(\zeta - z)f(\zeta)d\bar\zeta\wedge d\zeta,$$

if the limit exists.

According to the classical Calderon-Zygmund theorem (see [1, 2, 4, 8, 11, 18] etc.), $\mathcal{P}_1$ is a bounded operator between $L^p(\mathbb{C})$ for $1 < p < \infty$. Making use of the higher order derivative formula for $T$ in Theorem 1.1, we prove the boundedness of $\mathcal{P}_{k+1}$ from $W^{k,p}(\mathbb{C})$ into $L^p(\mathbb{C})$ in Section 3.

**Theorem 1.4.** *$\mathcal{P}_{k+1}(f)$ exists for $f \in C_0^\infty(\mathbb{C})$ with $k \in \mathbb{Z}^+ \cup \{0\}$. Furthermore,*

$$\|\mathcal{P}_{k+1}(f)\|_{L^p(\mathbb{C})} \le C_p\|f\|_{W^{k,p}(\mathbb{C})},$$

*where $1 < p < \infty$ and $C_p$ is a constant depending only on $p$.*

Another application is the investigation on the existence of flat solutions to the $\bar\partial$ problem equipped with flat germs data. Here a germ of a smooth function at 0 is called flat if the derivatives of its representative at 0 vanish at all levels. Flat smooth functions are one of the central subjects in the unique continuation property (UCP) problem originated from the work of Carleman [5]. See [7, 15, 16, 20] and references therein. Motivated by a UCP problem, a natural question asks whether there exists a flat $u$ such that $\bar\partial u = f$ in the sense of germs given a flat germ of a $(0,1)$ form $f$. See recent work [9] of Fassina and the second author concerning nonsolvability of general elliptic operators with real analytic coefficients in the flat category. In Section 4, the following criterion for the existence of local flat solutions is obtained.

**Theorem 1.5.** *Let $f$ be a flat germ at the origin $\mathbb{C}$. The following two statements are equivalent:*

*1) The Cauchy-Riemann equation $\bar\partial u = f d\bar z$ has a flat solution at the origin in the sense of germs.*

*2) There exists some neighborhood $U \subset \mathbb{C}$ of the origin such that*

$$\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_U \frac{f(\zeta)}{\zeta^{n+1}}d\bar\zeta\wedge d\zeta\right)z^n \tag{2}$$

*is holomorphic near the origin.*

Since $f$ is flat in Theorem 1.5, the expression in (2) is well defined. The criterion can be used to construct examples where $\bar{\partial} u = f$ is solvable in the flat category. More interestingly, we are able to construct in Section 5 a family of smooth examples with no flat local solutions in $\mathbb{C}^n, n \geq 1$. We should mention that one can get nonsolvable examples in the flat category applying Borel's theorem as well (see Remark 5.6a)). However, our construction allows to investigate certain behavior of the minimal solutions to the $\bar{\partial}$ problem in terms of Hörmander [12]. Recall that a minimal solution is the unique solution to $\bar{\partial}$ that is orthogonal to the space of holomorphic functions with respect to the weighted $L^2$ norms. Given two smoothly bounded pseudoconvex domains $\Omega_1 \subsetneq \Omega_2$ in $\mathbb{C}^n$, a plurisubharmonic weight function $\phi$ and a $\bar{\partial}$ closed (0,1) form $f$ on $\Omega_2$, let $u_1$ and $u_2$ be the corresponding minimal solutions to $\bar{\partial} u = f|_{\Omega_1}$ on $\Omega_1$ and $\bar{\partial} u = f$ on $\Omega_2$ with respect to $L^2(\Omega_1, e^{-\phi}|_{\Omega_1})$ and $L^2(\Omega_2, e^{-\phi})$ norms. It is generally understood that $u_1$ is not simply the restriction of $u_2$ on $\Omega_1$, yet such examples were rarely seen in the literature. Let $B_r$ be the ball centered at 0 with radius $r > 0$ in $\mathbb{C}^n, n \geq 1$. We provide a family of examples in Section 6 as follows.

**Theorem 1.6.** *There exist a family of smooth $\bar{\partial}$-closed (0,1) form $f$ on $B_1$, such that for any given bounded plurisubharmonic weight function $\phi$ and any positive decreasing sequence $r_n (< 1) \to 0$, the minimal solution $u_n$ to $\bar{\partial} u = f|_{B_{r_n}}$ on $B_{r_n}$ with respect to $L^2(B_{r_n}, e^{-\phi}|_{B_{r_n}})$ is not $u_1|_{B_{r_n}}$.*

On the other hand, in comparison to Theorem 1.6, we show in Example 6.1 (see Błocki [3] when $n = 1$) that there exists a smooth $\bar{\partial}$-closed (0,1) form $f$ on $B_R$, such that for any bounded and radially symmetric plurisubharmonic weight $\phi$ on $B_R$ and $r \leq R$, the minimal solution $u_r$ to $\bar{\partial} u = f|_{B_r}$ on $B_r$ with respect to $L^2(B_r, e^{-\phi}|_{B_r})$ is always $u_R|_{B_r}$. Namely, the restriction of a minimal solution on subdomains could still be minimal in some special cases.

**Notation:** Throughout the rest of the paper, unless otherwise indicated, $0 < \alpha < 1$, $k \in \mathbb{Z}^+ \cup \{0\}$, $1 < p < \infty$. Given $\mu = (\mu_1, \mu_2)$ with both entries nonnegative integers, write $z^\mu := z^{\mu_1} \bar{z}^{\mu_2}$ for $z \in \mathbb{C}$.

# 2 Higher order derivative formula of the Cauchy operator

## 2.1 First order derivative formula

For any $f \in C^\alpha(\Omega)$, denote its Hölder norm by

$$\|f\|_\alpha := \sup_\Omega |f| + H_\alpha[f],$$

where

$$H_\alpha[f] := \sup \left\{ \frac{|f(z) - f(z')|}{|z - z'|^\alpha} : z, z' \in \Omega \right\}.$$

Letting

$$Sf(z) : = \frac{1}{2\pi i} \int_{\partial\Omega} \Gamma(\zeta - z) f(\zeta) d\zeta, z \in \Omega,$$

The following formulas and estimates are well known concerning the operators $T$, $S$ and $^2Tf$.

**Lemma 2.1.** *[13] If $f \in C^1(\bar{\Omega})$, then $T\bar{\partial} f = f - Sf$ on $\Omega$.*

**Lemma 2.2.** *[14, 21] If $f \in C^\alpha(\Omega)$, then ${}^2Tf \in C^\alpha(\Omega)$ and $Tf \in C^{1+\alpha}(\Omega)$, with*

$$\begin{aligned}\partial Tf(z) &=^2 Tf(z) - \frac{f(z)}{2\pi i}\int_{\partial\Omega}\Gamma(\zeta - z)d\bar{\zeta};\\ \bar{\partial}Tf(z) &= f(z).\end{aligned} \tag{3}$$

*on $\Omega$. Moreover,*

$$H_\alpha[{}^2Tf] \le C(\Omega, \alpha)H_\alpha[f].$$

For convenience of the reader, we provide a different proof of the formula in Lemma 2.2 along the line of [10, 17]. The proof will be generalized to derive the higher order derivative formula for $T$ in the subsequent subsection.

**Proof of Lemma 2.2:** Let $\eta \in C^1(\mathbb{R})$ such that $\eta \equiv 0$ for $t \le 1$, $\eta \equiv 1$ for $t \ge 2$ and $0 \le \eta' \le 2$ for $1 \le t \le 2$. For each $z \in \Omega$, define for $\epsilon > 0$ small enough that

$$w_\epsilon(z) := \frac{-1}{2\pi i}\int_\Omega \Gamma(\zeta - z)\eta_\epsilon(\zeta, z)f(\zeta)d\bar{\zeta}\wedge d\zeta,\ \ \eta_\epsilon(\zeta, z) = \eta(\frac{|\zeta - z|}{\epsilon}).$$

Then we have $w_\epsilon \to Tf$ pointwisely in $\Omega$ and

$$\begin{aligned}\partial w_\epsilon(z) =& \frac{-1}{2\pi i}\int_\Omega \partial_z\left(\eta_\epsilon(\zeta, z)\Gamma(\zeta - z)\right)f(\zeta)d\bar{\zeta}\wedge d\zeta\\ =&\frac{-1}{2\pi i}\int_\Omega \partial_z(\eta_\epsilon(\zeta, z)\Gamma(\zeta - z))\left(f(\zeta) - f(z)\right)d\bar{\zeta}\wedge d\zeta - \frac{f(z)}{2\pi i}\int_\Omega \partial_z\left(\eta_\epsilon(\zeta, z)\Gamma(\zeta - z)\right)d\bar{\zeta}\wedge d\zeta\\ =&: I_1 + I_2\end{aligned}$$

in $\Omega$. Consequently, $I_1$ converges uniformly in compact subsets of $\mathbb{C}$ to ${}^2Tf(z)$ as $\epsilon \to 0$ for $z \in \Omega$. By Stokes' theorem,

$$I_2 = \frac{f(z)}{2\pi i}\int_\Omega \partial_\zeta(\eta_\epsilon(\zeta, z)\Gamma(\zeta - z))d\bar{\zeta}\wedge d\zeta = -\frac{f(z)}{2\pi i}\int_{\partial\Omega}\eta_\epsilon(\zeta, z)\Gamma(\zeta - z)d\bar{\zeta} \to -\frac{f(z)}{2\pi i}\int_{\partial\Omega}\Gamma(\zeta - z)d\bar{\zeta}$$

in $\Omega$. Therefore,

$$\partial Tf(z) = {}^2Tf(z) - \frac{f(z)}{2\pi i}\int_{\partial\Omega}\Gamma(\zeta - z)d\bar{\zeta}\ \text{ in }\Omega.$$

The first equation of (3) is proved.

For $\bar{\partial}Tf$, we similarly compute

$$\begin{aligned}\bar{\partial}w_\epsilon(z) =& \frac{-1}{2\pi i}\int_\Omega \bar{\partial}_z(\eta_\epsilon(\zeta, z)\Gamma(\zeta - z))\left(f(\zeta) - f(z)\right)d\bar{\zeta}\wedge d\zeta - \frac{f(z)}{2\pi i}\bar{\partial}\left(\int_\Omega \eta_\epsilon(\zeta, z)\Gamma(\zeta - z)d\bar{\zeta}\wedge d\zeta\right)\\ =&: I_1' + I_2'.\end{aligned}$$

By definition of $\eta_\epsilon$ and holomorphy of $\Gamma$ away from 0, $I_1'$ can be rewritten as

$$I_1' = \frac{-1}{2\pi i}\int_{\epsilon\le|\zeta - z|\le 2\epsilon,\ \zeta\in\Omega}\bar{\partial}_z\left(\eta_\epsilon(\zeta, z)\right)\Gamma(\zeta - z)\left(f(\zeta) - f(z)\right)d\bar{\zeta}\wedge d\zeta.$$

Since

$$|I_1'| \leq \frac{1}{2\pi}\int_{|\zeta-z|\leq 2\epsilon,\ \zeta\in\Omega}\frac{2}{\epsilon}\cdot|\Gamma(\zeta-z)|\cdot|f(\zeta)-f(z)|d\bar{\zeta}\wedge d\zeta$$
$$\leq \frac{H_\alpha[f]}{2\pi}\left(\frac{2}{\epsilon(\alpha+1)}(2\epsilon)^{\alpha+1}\right),$$

$I_1'$ converges uniformly to 0 on $\bar{\Omega}$ as $\epsilon \to 0$.

On the other hand, $I_2' \to -\frac{f(z)}{2\pi i}\bar{\partial}\left(\int_\Omega \Gamma(\zeta-z)d\bar{\zeta}\wedge d\zeta\right)$ on $\Omega$. We further claim that

$$\bar{\partial}\left(\int_\Omega \Gamma(\zeta-z)d\bar{\zeta}\wedge d\zeta\right) = -2\pi i.$$

Indeed, letting $f(z) = \bar{z}$ in Lemma 2.1, the claim follows from

$$\frac{-1}{2\pi i}\int_\Omega \Gamma(\zeta-z)d\bar{\zeta}\wedge d\zeta = \bar{z} + \frac{1}{2\pi i}\int_{\partial\Omega}\frac{\bar{\zeta}d\zeta}{\zeta-z}$$

and the holomorphy of $\int_{\partial\Omega}\frac{\bar{\zeta}d\zeta}{\zeta-z}$ with respect to $z\in\Omega$. Hence $I_2' \to f(z)$ on $\Omega$. Altogether, we have $\bar{\partial}w_\epsilon \to f(z)$ on $\Omega$. The second equation of (3) is thus proved. ∎

## 2.2 Higher order derivative formula

We first recall a classical result in [21] concerning the Hölder estimates of the operator $S$ as follows.

**Theorem 2.3.** *[21] Let $\partial\Omega \in C^{k+1+\alpha}$ and $f\in C^{k+\alpha}(\partial\Omega)$. Then $S(f)\in C^{k+\alpha}(\Omega)$ and $\|Sf\|_\alpha^{(k)} \leq M(\Omega,\alpha,k)\|f\|_\alpha^{(k)}$ for some constant $M$ dependent only on $\Omega$, $\alpha$ and $k$.*

For simplicity of notations, given $\mu = (\mu_1,\mu_2)$ and $k$, we define for $z\in\Omega$,

$$\mathcal{I}_\Omega(k,\mu)(z) := \frac{1}{2\pi i}\int_{\partial\Omega}\partial_z^k\Gamma(\zeta-z)(\zeta-z)^\mu d\bar{\zeta} = \frac{k!}{2\pi i}\int_{\partial\Omega}\frac{(\zeta-z)^\mu}{(\zeta-z)^{k+1}}d\bar{\zeta},$$

where $(\zeta-z)^\mu = (\zeta-z)^{\mu_1}(\bar{\zeta}-\bar{z})^{\mu_2}$. We have the following lemma.

**Lemma 2.4.** *The following statements hold for $\mathcal{I}_\Omega(k,\mu)(z)$.*
*a) $\mathcal{I}_\Omega(k,\mu)(z) = \frac{-(k+1)!(\mu_1-k-1)}{2\pi i(\mu_2+1)}\int_{\partial\Omega}\frac{(\overline{\zeta-z})^{\mu_2+1}}{(\zeta-z)^{k+2-\mu_1}}d\zeta$.*
*b) $\mathcal{I}_\Omega(k,\mu)(z)\in C^\alpha(\Omega)$ and $\|\mathcal{I}_\Omega(k,\mu)(z)\|_\alpha \leq M(\Omega,k,\mu)$ where $M(\Omega,k,\mu)$ is a constant depending only on $\Omega, k$ and $\mu$.*
*c) When $\Omega = D$, an arbitrary disk in $\mathbb{C}$, then $\mathcal{I}_D(k,\mu)(z)\equiv 0$ for $k-\mu_1\geq 0$, $\mu_2\geq 0$.*

**Proof of Lemma 2.4:** Let $\epsilon > 0$ be small enough and $D_\epsilon(z)$ be the disk centered at $z\in\mathbb{C}$ with radius $\epsilon$. Applying Stokes' theorem gives

$$\begin{aligned}
\mathcal{I}_\Omega(k,\mu)(z) =&\frac{k!}{2\pi i}\int_{\partial\Omega}\frac{(\overline{\zeta-z})^{\mu_2}}{(\zeta-z)^{k+1-\mu_1}}d\bar{\zeta}\\
=&\frac{k!}{2\pi i}\int_{\Omega\setminus D_\epsilon(z)}d_\zeta\left(\frac{(\overline{\zeta-z})^{\mu_2}}{(\zeta-z)^{k+1-\mu_1}}d\bar{\zeta}\right)+\frac{k!}{2\pi i}\int_{\partial D_\epsilon(z)}\frac{(\overline{\zeta-z})^{\mu_2}}{(\zeta-z)^{k+1-\mu_1}}d\bar{\zeta}\\
=&\frac{-k!(\mu_1-k-1)}{2\pi i(\mu_2+1)}\int_{\Omega\setminus D_\epsilon(z)}d_\zeta\left(\frac{(\overline{\zeta-z})^{\mu_2+1}}{(\zeta-z)^{k+2-\mu_1}}d\zeta\right)+\frac{k!}{2\pi i}\int_{\partial D_\epsilon(z)}\frac{(\overline{\zeta-z})^{\mu_2}}{(\zeta-z)^{k+1-\mu_1}}d\bar{\zeta}\\
=&\frac{-k!(\mu_1-k-1)}{2\pi i(\mu_2+1)}\Big(\int_{\partial\Omega}\frac{(\overline{\zeta-z})^{\mu_2+1}}{(\zeta-z)^{k+2-\mu_1}}d\zeta-\int_{\partial D_\epsilon(z)}\frac{(\overline{\zeta-z})^{\mu_2+1}}{(\zeta-z)^{k+2-\mu_1}}d\zeta\Big)\\
&+\frac{k!}{2\pi i}\int_{\partial D_\epsilon(z)}\frac{(\overline{\zeta-z})^{\mu_2}}{(\zeta-z)^{k+1-\mu_1}}d\bar{\zeta}.
\end{aligned}$$

Using the polar coordinates, a direct computation gives

$$\begin{aligned}
\int_{\partial D_\epsilon(z)}\frac{(\overline{\zeta-z})^{\mu_2}}{(\zeta-z)^{k+1-\mu_1}}d\bar{\zeta}=&-i\epsilon^{\mu_2+\mu_1-k}\int_0^{2\pi}e^{i\theta(-\mu_2-k-2+\mu_1)}d\theta\\
=&\begin{cases}0, & -\mu_2-k-2+\mu_1\neq 0;\\ -2\pi i\epsilon^{\mu_2+\mu_1-k}, & -\mu_2-k-2+\mu_1=0.\end{cases}
\end{aligned}$$

Similarly,

$$\int_{\partial D_\epsilon(z)}\frac{(\overline{\zeta-z})^{\mu_2+1}}{(\zeta-z)^{k+2-\mu_1}}d\zeta=\begin{cases}0, & -\mu_2-k-2+\mu_1\neq 0;\\ -2\pi i\epsilon^{\mu_2+\mu_1-k}, & -\mu_2-k-2+\mu_1=0.\end{cases}$$

Note that $\mu_2+\mu_1-k>0$ in the case when $-\mu_2-k-2+\mu_1=0$. Letting $\epsilon\to 0$, we have the above two expressions go to 0 and thus a) is proved.

For b), if $k+2-\mu_1\le 0$, then $\mathcal{I}_\Omega(k,\mu)(z)\in C^\infty(\overline{\Omega})$ automatically. When $k+2-\mu_1>0$, we have $\int_{\partial\Omega}\frac{(\overline{\zeta-z})^{\mu_2+1}}{(\zeta-z)^{k+2-\mu_1}}d\zeta=\frac{1}{(k+2-\mu_1)!}\partial^{k+1-\mu_1}\int_{\partial\Omega}\frac{(\overline{\zeta-z})^{\mu_2+1}}{\zeta-z}d\zeta$. Since $(\overline{\zeta-z})^{\mu_2+1}\in C^{k+1-\mu_1,\alpha}(\partial\Omega)$, $\mathcal{I}_\Omega(k,\mu)(z)\in C^\alpha(\overline{\Omega})$ by Theorem 2.3.

For c), we assume $D=D_R(0)$ by translation if necessary. Then

$$\begin{aligned}
\int_{\partial D}\frac{(\overline{\zeta-z})^{\mu_2+1}}{(\zeta-z)^{k+2-\mu_1}}d\zeta=&\sum_{j=0}^{\mu_2+1}\frac{(\mu_2+1)!}{j!(\mu_2+1-j)!}\bar{z}^{\mu_2+1-j}\int_{\partial D}\frac{\bar{\zeta}^j}{(\zeta-z)^{k+2-\mu_1}}d\zeta\\
=&\sum_{j=0}^{\mu_2+1}\frac{(\mu_2+1)!}{j!(\mu_2+1-j)!}\bar{z}^{\mu_2+1-j}\int_{\partial D}\frac{|R|^{2j}}{(\zeta-z)^{k+2-\mu_1}\zeta^j}d\zeta.
\end{aligned}$$

Each term in the sum of the above expression is 0 by the Residue theorem. Indeed, for $j=0,\dots,\mu_2+1$,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial D}\frac{1}{(\zeta-z)^{k+2-\mu_1}\zeta^j}d\zeta\\
=&\begin{cases} Res\left(\frac{1}{(\zeta-z)^{k+2-\mu_1}\zeta^j},z\right)+Res\left(\frac{1}{(\zeta-z)^{k+2-\mu_1}\zeta^j},0\right), & j\ge 1\\ Res\left(\frac{1}{(\zeta-z)^{k+2-\mu_1}},z\right), & j=0\end{cases}\\
=&\begin{cases}\frac{1}{(k+1-\mu_1)!}\frac{d^{k+1-\mu_1}}{dz^{k+1-\mu_1}}\left(\frac{1}{z^j}\right)+\frac{(-1)^{k+1-\mu_1+j}}{(j-1)!}\frac{d^{j-1}}{dz^{j-1}}\left(\frac{1}{z^{k+2-\mu_1}}\right), & j\ge 1\\ 0, & j=0\end{cases}\\
=&0.
\end{aligned}$$

∎

**Proof of Theorem 1.1:** We prove the theorem by induction on $k$. The $k=0$ case is Lemma 2.2. Assuming (1) for $k$, we shall show that for $f \in C^{k+1,\alpha}$ and $z \in \Omega$,

$$\begin{aligned}\partial^{k+2}T(f)(z) = &-\frac{1}{2\pi i}\int_\Omega \partial_z^{k+2}\Gamma(\zeta-z)(f(\zeta)-P_{k+1}(\zeta,z))d\bar{\zeta}\wedge d\zeta \\ &-\sum_{j=2}^{k+3}\partial^{k+3-j}\left(\sum_{|\mu|=j-2}\frac{\partial^\mu f(z)}{\mu!}\mathcal{I}_\Omega(j-2,\mu)(z)\right).\end{aligned} \tag{4}$$

Let $\epsilon > 0$ be small enough and consider

$$\begin{aligned}w_\epsilon(z) := &-\frac{1}{2\pi i}\int_\Omega \partial_z^{k+1}\Gamma(\zeta-z)\eta_\epsilon(\zeta,z)(f(\zeta)-P_k(\zeta,z))d\bar{\zeta}\wedge d\zeta \\ &-\sum_{j=2}^{k+2}\partial^{k+2-j}\left(\sum_{|\mu|=j-2}\frac{\partial^\mu f(z)}{\mu!}\mathcal{I}_\Omega(j-2,\mu)(z)\right)\end{aligned}$$

for $z \in \Omega$. When $\epsilon \to 0$,

$$w_\epsilon(z) \to \partial^{k+1}Tf(z) + \frac{1}{2\pi i}\partial^k\left[f(z)\partial\left(\int_\Omega \Gamma(\zeta-z)d\bar{\zeta}\wedge d\zeta\right)\right] \tag{5}$$

for $z \in \Omega$. On the other hand,

$$\begin{aligned}\partial w_\epsilon(z) =&\frac{1}{2\pi i}\int_\Omega \partial_\zeta\left(\partial_z^{k+1}\Gamma(\zeta-z)\eta_\epsilon(\zeta,z)\right)(f(\zeta)-P_k(\zeta,z))d\bar{\zeta}\wedge d\zeta \\ &-\frac{1}{2\pi i}\int_\Omega \left(\partial_z^{k+1}\Gamma(\zeta-z)\eta_\epsilon(\zeta,z)\right)\partial_z(f(\zeta)-P_k(\zeta,z))d\bar{\zeta}\wedge d\zeta \\ &-\partial\left[\sum_{j=2}^{k+2}\partial^{k+2-j}\left(\sum_{|\mu|=j-2}\frac{\partial^\mu f(z)}{\mu!}\mathcal{I}_\Omega(j-2,\mu)(z)\right)\right] \\ =&:I_1+I_2-\sum_{j=2}^{k+2}\partial^{k+3-j}\left(\sum_{|\mu|=j-2}\frac{\partial^\mu f(z)}{\mu!}\mathcal{I}_\Omega(j-2,\mu)(z)\right).\end{aligned} \tag{6}$$

We shall show as $\epsilon \to 0$,

$$\begin{aligned}I_1+I_2 \to &-\frac{1}{2\pi i}\int_\Omega \partial_z^{k+2}\Gamma(\zeta-z)(f(\zeta)-P_{k+1}(\zeta,z))d\bar{\zeta}\wedge d\zeta \\ &-\sum_{|\mu|=k+1}\frac{\partial^\mu f(z)}{\mu!}\mathcal{I}_\Omega(k+1,\mu)(z)\end{aligned} \tag{7}$$

pointwisely in $\Omega$. Then (6) will imply that

$$\begin{aligned}\partial w_\epsilon(z) \to &-\frac{1}{2\pi i}\int_\Omega \partial_z^{k+2}\Gamma(\zeta-z)(f(\zeta)-P_{k+1}(\zeta,z))d\bar{\zeta}\wedge d\zeta \\ &-\sum_{j=2}^{k+3}\partial^{k+3-j}\left(\sum_{|\mu|=j-2}\frac{\partial^\mu f(z)}{\mu!}\mathcal{I}_\Omega(j-2,\mu)(z)\right)\end{aligned} \tag{8}$$

in $\Omega$ and hence (4) follows from (5) and (8).

First consider $I_1$.

$$
\begin{aligned}
I_1 =& \frac{1}{2\pi i}\int_\Omega \partial_\zeta\left(\partial_z^{k+1}\Gamma(\zeta-z)\eta_\epsilon(\zeta,z)\right)(f(\zeta)-P_{k+1}(\zeta,z))d\bar{\zeta}\wedge d\zeta \\
&+\frac{1}{2\pi i}\sum_{|\mu|=k+1}\frac{\partial^\mu f(z)}{\mu!}\int_\Omega \partial_\zeta\left(\partial_z^{k+1}\Gamma(\zeta-z)\eta_\epsilon(\zeta,z)\right)(\zeta-z)^\mu d\bar{\zeta}\wedge d\zeta.
\end{aligned}
$$

Using Stokes' theorem to the second term of the above equation, one obtains that

$$
\begin{aligned}
I_1 =& \frac{1}{2\pi i}\int_\Omega \partial_\zeta\left(\partial_z^{k+1}\Gamma(\zeta-z)\eta_\epsilon(\zeta,z)\right)(f(\zeta)-P_{k+1}(\zeta,z))d\bar{\zeta}\wedge d\zeta \\
&-\frac{1}{2\pi i}\sum_{|\mu|=k+1}\frac{\partial^\mu f(z)}{\mu!}\int_{\partial\Omega} \partial_z^{k+1}\Gamma(\zeta-z)\eta_\epsilon(\zeta,z)(\zeta-z)^\mu d\bar{\zeta} \\
&-\frac{1}{2\pi i}\sum_{|\mu|=k+1}\frac{\partial^\mu f(z)}{\mu!}\int_\Omega \left(\partial^{k+1}\Gamma(\zeta-z)\eta_\epsilon(\zeta,z)\right)\partial_\zeta(\zeta-z)^\mu d\bar{\zeta}\wedge d\zeta.
\end{aligned}
$$

On the other hand,

$$
I_2 = \frac{1}{2\pi i}\int_\Omega \left(\partial_z^{k+1}\Gamma(\zeta-z)\eta_\epsilon(\zeta,z)\right)\partial_z P_k(\zeta,z)d\bar{\zeta}\wedge d\zeta.
$$

Therefore,

$$
\begin{aligned}
I_1+I_2 =& \frac{1}{2\pi i}\int_\Omega \partial_\zeta\left(\partial_z^{k+1}\Gamma(\zeta-z)\eta_\epsilon(\zeta,z)\right)(f(\zeta)-P_{k+1}(\zeta,z))d\bar{\zeta}\wedge d\zeta \\
&-\frac{1}{2\pi i}\sum_{|\mu|=k+1}\frac{\partial^\mu f(z)}{\mu!}\int_{\partial\Omega} \partial_z^{k+1}\Gamma(\zeta-z)\eta_\epsilon(\zeta,z)(\zeta-z)^\mu d\bar{\zeta} \\
&-\frac{1}{2\pi i}\int_\Omega \left(\partial_z^{k+1}\Gamma(\zeta-z)\eta_\epsilon(\zeta,z)\right)\left(\sum_{|\mu|=k+1}\frac{\partial^\mu f(z)}{\mu!}\partial_\zeta(\zeta-z)^\mu-\partial_z P_k(\zeta,z)\right)d\bar{\zeta}\wedge d\zeta \\
=:& I_3+I_4+I_5.
\end{aligned}
$$

When $\epsilon\to 0$, we have for $z\in\Omega$,

$$
\begin{aligned}
&I_3\to -\frac{1}{2\pi i}\int_\Omega \partial_z^{k+2}\Gamma(\zeta-z)(f(\zeta)-P_{k+1}(\zeta,z))d\bar{\zeta}\wedge d\zeta; \\
&I_4\to -\frac{1}{2\pi i}\sum_{|\mu|=k+1}\frac{\partial^\mu f(z)}{\mu!}\int_{\partial\Omega}\partial_z^{k+1}\Gamma(\zeta-z)(\zeta-z)^\mu d\bar{\zeta} = -\sum_{|\mu|=k+1}\frac{\partial^\mu f(z)}{\mu!}\mathcal{I}_\Omega(k+1,\mu)(z).
\end{aligned}
\tag{9}
$$

For $I_5$, since $P_k(\zeta,z)=\sum_{l=0}^k\sum_{|\mu|=l}\frac{1}{\mu!}\partial^\mu f(z)(\zeta-z)^\mu$, we have

$$
\partial_z P_k(\zeta,z) = \sum_{|\mu|\le k}\frac{\partial\partial^\mu f(z)(\zeta-z)^\mu}{\mu!}+\sum_{|\mu|\le k}\frac{\partial^\mu f(z)\partial_z(\zeta-z)^\mu}{\mu!}.
$$

Notice that

$$\sum_{|\mu|\le k}\frac{\partial\partial^\mu f(z)(\zeta-z)^\mu}{\mu!}=\sum_{|\mu|=k+1}\frac{\partial^\mu f(z)\partial_\zeta(\zeta-z)^\mu}{\mu!}+\sum_{|\mu|\le k-1}\frac{\partial\partial^\mu f(z)(\zeta-z)^\mu}{\mu!},$$

$$\sum_{|\mu|\le k}\frac{\partial^\mu f(z)\partial_z(\zeta-z)^\mu}{\mu!}=-\sum_{|\mu|\le k-1}\frac{\partial\partial^\mu f(z)(\zeta-z)^\mu}{\mu!}.$$

Therefore,

$$\sum_{|\mu|=k+1}\frac{\partial^\mu f(z)}{\mu!}\partial_\zeta(\zeta-z)^\mu-\partial_z P_k(\zeta,z)=0,$$

which implies that $I_5=0$. Combining (7) and (9), we complete the induction. The second part of the theorem is a consequence of Lemma 2.4c). ■

**Proof of Corollary 1.2:** Since $f\in C^{k+\alpha}(\Omega)$, $Tf\in C^{k+1+\alpha}(\Omega)$ and $\partial^{k+1}Tf\in C^\alpha(\Omega)$. Together with Lemma 2.4 and (1), we have ${}^{k+2}Tf(z)\in C^\alpha(\Omega)$ and $\|{}^{k+2}Tf(z)\|_\alpha\le C(\Omega,k,\mu)\|f(z)\|_\alpha^{(k)}$ where $C(\Omega,k,\mu)$ is a constant depending on $\Omega,k$ and $\mu$.■

# 3 Higher order Calderón-Zygmund type Theorem

In this section, we restrict the attention on $\Omega=D$, an arbitrary disk in $\mathbb{C}$. We first prove the following induction formula.

**Lemma 3.1.** *Let $f\in C^{k+\alpha}(D)$, then for $z\in D$,*

$$^{k+2}T(f)(z)={}^{k+1}T(\partial f)(z)-S_{k+1}(f)(z),\tag{10}$$

*where $S_{k+1}(f)(z):=\frac{k!}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{(\zeta-z)^{k+1}}d\bar\zeta$. If $f\in C_0^\infty(D)$, $S_{k+1}(f)(z)=0$.*

**Proof of Lemma 3.1:** Using Stokes' theorem, we have for $z\in D$,

$$\begin{aligned}
{}^{k+2}T(f)(z)=&\frac{-(k+1)!}{2\pi i}\int_D\frac{f(\zeta)-P_k(\zeta,z)}{(\zeta-z)^{k+2}}d\bar\zeta\wedge d\zeta\\
=&\lim_{\epsilon\to0}\frac{-(k+1)!}{2\pi i}\int_{D\setminus D_\epsilon(z)}\frac{f(\zeta)-P_k(\zeta,z)}{(\zeta-z)^{k+2}}d\bar\zeta\wedge d\zeta\\
=&\lim_{\epsilon\to0}\frac{-k!}{2\pi i}\int_{D\setminus D_\epsilon(z)}\frac{\partial f(\zeta)-\partial_\zeta P_k(\zeta,z)}{(\zeta-z)^{k+1}}d\bar\zeta\wedge d\zeta\\
&-\frac{k!}{2\pi i}\int_{\partial D}\frac{f(\zeta)-P_k(\zeta,z)}{(\zeta-z)^{k+1}}d\bar\zeta-\lim_{\epsilon\to0}\frac{k!}{2\pi i}\int_{\partial D_\epsilon(z)}\frac{f(\zeta)-P_k(\zeta,z)}{(\zeta-z)^{k+1}}d\bar\zeta\\
=&{}^{k+1}T(\partial f)(z)-\frac{k!}{2\pi i}\int_{\partial D}\frac{f(\zeta)-P_k(\zeta,z)}{(\zeta-z)^{k+1}}d\bar\zeta,
\end{aligned}$$

where the last identity is because $\left|\int_{\partial D_\epsilon(z)}\frac{f(\zeta)-P_k(\zeta,z)}{(\zeta-z)^{k+1}}d\bar\zeta\right|\le C\epsilon^\alpha\to0$. Furthermore, by Lemma 2.4c),

$$\begin{aligned}
\frac{k!}{2\pi i}\int_{\partial D}\frac{P_k(\zeta,z)}{(\zeta-z)^{k+1}}d\bar\zeta=&\frac{k!}{2\pi i}\sum_{l=0}^k\sum_{|\mu|=l}\frac{1}{\mu!}\partial^\mu f(z)\int_{\partial D}\frac{(\zeta-z)^\mu}{(\zeta-z)^{k+1}}d\bar\zeta\\
=&\sum_{l=0}^k\sum_{|\mu|=l}\frac{1}{\mu!}\partial^\mu f(z)\mathcal{I}_D(k,\mu)(z)=0
\end{aligned}$$

in $\Omega$. The lemma is thus proved. ■

**Proof of Theorem 1.4:** Without loss of generality, assume supp$f \subset D$ with the radius $R$ of $D$ to be chosen sufficiently large later. For $z \in D$,

$$\begin{aligned}
\mathcal{P}_{k+1}(f)(z) =&\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{D\setminus D_\epsilon(z)}\partial^{k+1}\Gamma(\zeta-z)f(\zeta)d\bar\zeta\wedge d\zeta\\
=&\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{D\setminus D_\epsilon(z)}\partial^{k+1}\Gamma(\zeta-z)\left(f(\zeta)-P_k(\zeta,z)\right)d\bar\zeta\wedge d\zeta\\
&+\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{D\setminus D_\epsilon(z)}\partial^{k+1}\Gamma(\zeta-z)P_k(\zeta,z)d\bar\zeta\wedge d\zeta\\
=&-^{k+2}T(f)(z)+\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{D\setminus D_\epsilon(z)}\partial^{k+1}\Gamma(\zeta-z)P_k(\zeta,z)d\bar\zeta\wedge d\zeta.
\end{aligned}\tag{11}$$

Let $J(f) := \int_{D\setminus D_\epsilon(z)}\partial^{k+1}\Gamma(\zeta-z)P_k(\zeta,z)d\bar\zeta\wedge d\zeta$. We show $J(f)=0$ in $D$ by letting $\epsilon$ small enough, which by the definition of $P_k$ suffices to show that

$$\int_{D\setminus D_\epsilon(z)}\frac{\bar\zeta^j}{(\zeta-z)^{k+2}}d\bar\zeta\wedge d\zeta = 0 \text{ in } \ D\setminus D_\epsilon(z) \ \text{ for } \ 0\le j\le k.$$

Indeed,

$$\begin{aligned}
\int_{D\setminus D_\epsilon(z)}\frac{\bar\zeta^j}{(\zeta-z)^{k+2}}d\bar\zeta\wedge d\zeta &= \frac{1}{j+1}\int_{D\setminus D_\epsilon(z)}\bar\partial\left(\frac{\bar\zeta^{j+1}}{(\zeta-z)^{k+2}}d\zeta\right)\\
&=\frac{1}{j+1}\left(\int_{\partial D}\frac{\bar\zeta^{j+1}}{(\zeta-z)^{k+2}}d\zeta-\int_{\partial D_\epsilon(z)}\frac{\bar\zeta^{j+1}}{(\zeta-z)^{k+2}}d\zeta\right)\\
&=\frac{1}{j+1}\left(\int_{\partial D}\frac{1}{(\zeta-z)^{k+2}\zeta^{j+1}}d\zeta-\int_{\partial D_\epsilon}\frac{(\overline{\zeta+z})^{j+1}}{\zeta^{k+2}}d\zeta\right)\\
&=0.
\end{aligned}$$

Here in the second to the last identity, the first term is 0 due to the Residue theorem; the second term is 0 since $\int_{\partial D_\epsilon}\frac{\bar\zeta^\ell}{\zeta^{k+2}}d\zeta = \epsilon^{2\ell}\int_{\partial D_\epsilon}\frac{1}{\zeta^{k+2+\ell}}d\zeta = 0$ for any $0\le \ell\le k+1, k\ge 0$ by the Residue theorem again.

Hence we have shown from (11) that

$$-\mathcal{P}_{k+1}(f) = ^{k+2}T(f)$$

in $D$. Combining it with (10), one inductively infers that

$$-\mathcal{P}_{k+1}(f) =^{k+2} T(f)(z) =^{k+1} T(\partial f)(z) = \cdots =^2 T(\partial^k f)(z),$$

for $f\in C_0^\infty(D)$. Applying Calderon-Zygmund theorem, we obtain

$$\|\mathcal{P}_{k+1}(f)\|_{L^p(D)} = \|^2T(\partial^k f)\|_{L^p(D)} = \|\mathcal{P}_1(\partial^k f)\|_{L^p(D)} \le C_p\|\partial^k f\|_{L^p(D)} \le C_p\|f\|_{W^{k,p}(D)}. \tag{12}$$

On the other hand, let $\chi$ be the step function such that $\chi = 0$ on $D_{2R}$, and 1 elsewhere. Then by Minkowski inequality,

$$\begin{aligned}
\left\|\chi(z)\int_{D_R}\partial^k\Gamma(\zeta-z)f(\zeta)d\bar\zeta\wedge d\zeta\right\|_{L^p(\mathbb{C})} \le& \int_{D_R}|f(\zeta)|\left(\int_{\mathbb{C}\setminus D_{2R}}|\partial^k\Gamma(\zeta-z)|^p d\bar z\wedge dz\right)^{\frac1p}d\bar\zeta\wedge d\zeta\\
=&k!\int_{D_R}|f(\zeta)|\left(\int_{\mathbb{C}\setminus D_{2R}}\frac{1}{|(\zeta-z)|^{(k+1)p}}d\bar z\wedge dz\right)^{\frac1p}d\bar\zeta\wedge d\zeta,
\end{aligned}$$

which converges to 0 as $R \to +\infty$. By selecting $R$ large enough, $\left\|\chi(z)\int_{D_{2R}}\partial^k\Gamma(\zeta-z)f(\zeta)d\bar{\zeta}\wedge d\zeta\right\|_{L^p(\mathbb{C})}$ can be bounded by $C\|f\|_{W^{k,p}(\mathbb{C})}$ for some universal constant $C$. Therefore,

$$\begin{aligned}\|\mathcal{P}_{k+1}(f)\|_{L^p(\mathbb{C})} \leq & \|\chi(z)\mathcal{P}_{k+1}(f)\|_{L^p(\mathbb{C})} + \|(1-\chi(z))\mathcal{P}_{k+1}(f)\|_{L^p(\mathbb{C})} \\ \leq & \left\|\chi(z)\int_{D_R}\partial^{k+1}\Gamma(\zeta-z)f(\zeta)d\bar{\zeta}\wedge d\zeta\right\|_{L^p(\mathbb{C})} + \|\mathcal{P}_{k+1}(f)\|_{L^p(D_{2R})}.\end{aligned}$$

Applying (12) with $D$ replaced by $D_{2R}$, we conclude that

$$\|\mathcal{P}_{k+1}(f)\|_{L^p(\mathbb{C})} \leq C'_p\|f\|_{W^{k,p}(\mathbb{C})}.$$

The operator $\mathcal{P}_{k+1}$ extends as a bounded operator from $W^{k,p}(\mathbb{C})$ into $L^p(\mathbb{C})$ since $C_0^\infty(\mathbb{C})$ is dense in $W^{k,p}(\mathbb{C})$. The proof of the theorem is complete. ∎

# 4 A criterion for the existence of flat solutions

We first recall some classical UCP results for the Laplacian $\Delta$ and the Cauchy-Riemann operator $\bar{\partial}$ in real and complex domains:

(**1**) Let $V \in L^2(D)$ and $u : D \subset \mathbb{R}^2 \to \mathbb{R}^N$ be smooth. If $|\Delta u| \leq V|\nabla u|$, then UCP holds, i.e., whenever $u$ is flat at 0, then $u \equiv 0$ in $D$.

(**2**) Let $V \in L^2(D)$ and $v : D \subset \mathbb{C} \to \mathbb{C}^M$ be smooth. If $|\bar{\partial}v| \leq V|v|$, then UCP holds, i.e., whenever $v$ is flat at 0, then $v \equiv 0$ in $D$.

Properties **(1)** was proved by Chanillo-Sawyer [6]. An interesting question is to see whether the two statements **(1)** and **(2)** are themselves equivalent. It is not hard to see that (**2**) implies (**1**). Indeed, assume $u(x,y) = (u_1(x,y), ..., u_M(x,y)) \in C^\infty(D, \mathbb{R}^M)$ is flat at 0 and satisfy $|\Delta u| \leq V|\nabla u|$. Write $z = x+iy$, $\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$, then $\Delta = 4\bar{\partial}\partial$. Consider

$$v(z) = (v_1(z), ..., v_M(z)) = (\partial u_1(x,y), ..., \partial u_M(x,y)).$$

Hence $v(z)$ is flat at 0. Moreover, since $|\Delta u| = 4|\bar{\partial}v|$ and $|v(z)| = \sqrt{\sum_{j=1}^M |\partial u_j|^2}$ $= \frac{1}{2}|\nabla u(x,y)|$, we have $|\bar{\partial}v| \leq \frac{1}{2}V|v|$. If **(2)** holds, then $v \equiv 0$, which gives $u \equiv 0$ in $D$.

However, it is not clear to us that (**1**) necessarily implies (**2**). The obstruction lies in the existence of flat solutions to Cauchy-Riemann equation. Indeed, assume $v(z) = (v_1(z), ..., v_M(z))$ is flat at 0 and satisfies $|\bar{\partial}v| \leq V|v|$. Let $u(z) = (u_1(z), ..., u_M(z))$ be a smooth solution to the equation $\partial u = v$. Write $u_j(z) = \tilde{u}_{2j-1}(x,y) + i\tilde{u}_{2j}(x,y)$ with $\tilde{u}_{2j-1}$ and $\tilde{u}_{2j}$ both real-valued functions for $j = 1, ..., M$ and denote $\tilde{u}(x,y) = (\tilde{u}_1, ..., \tilde{u}_{2M})$. Then $|\Delta\tilde{u}| = 4|\bar{\partial}v|$ and

$$|v(z)| = \sqrt{\sum_{j=1}^M |v_j|^2} = \sqrt{\sum_{j=1}^M |\partial(\tilde{u}_{2j-1} + i\tilde{u}_{2j})|^2} \leq \frac{1}{\sqrt{2}}|\nabla\tilde{u}(x,y)|.$$

Consequently, $|\bar{\partial}v| \leq V|v|$ gives $|\Delta\tilde{u}| \leq 2\sqrt{2}V|\nabla\tilde{u}|$. If one would be able to find a flat solution $u$ to $\partial u = v$ locally, then applying (**1**) gives $\tilde{u} \equiv 0$ and hence $v \equiv 0$. This steers us to study the flat solutions of the Cauchy-Riemann equation.

Along a slightly different direction, similar questions were raised in the literature to see if there exists a compactly supported solution to $\bar{\partial}u = f$ in $\mathbb{C}^n$ whenever $f$ is a compactly supported

$\bar\partial$-closed smooth $(0,1)$ form. By a classical result, this is true for $n\geq 2$. It fails when $n=1$, which can be argued as follows. Let $h$ be a nontrivial holomorphic function on $\mathbb{C}$. Given any function $f\in C_0^\infty(\mathbb{C})$, if there is a compactly supported solution $u$ to $\bar\partial u=f$, then

$$\int_{\mathbb{C}} h(\zeta)f(\zeta)d\bar\zeta\wedge d\zeta=\int_{\mathbb{C}} h(\zeta)\bar\partial u(\zeta)d\bar\zeta\wedge d\zeta=-\int_{\mathbb{C}}\bar\partial h(\zeta)u(\zeta)d\bar\zeta\wedge d\zeta\equiv 0.$$

This implies $h\equiv 0$, a contradiction.

**Proof of Theorem 1.5:** Without loss of generality, assume that $f$ is defined on $D:=D_r(0)$ for some $r>0$. Any smooth solution $u$ to $\bar\partial u=fd\bar z$ in $D$ can always be written by

$$u=Tf+h \text{ in } D,$$

where $h$ is some holomorphic function on $D$. Hence for any integers $\alpha\geq 0$ and $\beta>0$, we have

$$\begin{aligned}&\partial_z^\alpha u(z)=\partial_z^\alpha Tf(z)+\partial_z^\alpha h(z),\ z\in D;\\ &\bar\partial_z^\beta\partial_z^\alpha u(z)=\bar\partial_z^\beta\partial_z^\alpha Tf(z)=\bar\partial_z^{\beta-1}\partial_z^\alpha f(z),\ z\in D.\end{aligned}$$

Restricting $z$ at 0 in the above expressions and applying the flatness of $f$, one gets

$$\begin{aligned}&\partial_z^\alpha u(0)=\partial_z^\alpha Tf(0)+\partial_z^\alpha h(0);\\ &\bar\partial_z^\beta\partial_z^\alpha u(0)=0.\end{aligned}$$

By Theorem 1.1 and the flatness of $f$ again, $u$ is flat if and only if

$$\partial_z^\alpha h(0)=-\partial_z^\alpha T(f)(0)=\frac{\alpha!}{2\pi i}\int_D\frac{f(\zeta)}{\zeta^{\alpha+1}}d\bar\zeta\wedge d\zeta.$$

Since $h(z)$ is holomorphic on $D$, the above expression is furthermore equivalent to

$$\sum_{\alpha=0}^\infty\left(\frac{1}{2\pi i}\int_D\frac{f(\zeta)}{\zeta^{\alpha+1}}d\bar\zeta\wedge d\zeta\right)z^\alpha$$

is holomorphic near 0. The proof of the theorem is complete. ∎

*Remark* 4.1. The flat solution in Theorem 1.5 is unique. Indeed, if $u_1$ and $u_2$ are two different flat solutions to $\bar\partial u(z)=f(z)$ for $z\in\Omega$, then $\bar\partial(u_1-u_2)=0$, which means that $u_1-u_2$ is holomorphic on $\Omega$ and flat at the origin. Thus $u_1-u_2\equiv 0,\ z\in\Omega$ by the uniqueness of holomorphic functions.

Using Theorem 1.5, one can easily construct nontrivial examples of Cauchy-Riemann equations which have flat solutions at the origin as follows.

**Theorem 4.2.** *Let $\varphi\in C^\infty(\mathbb{R},\mathbb{C})$ be flat at 0 and $g$ be harmonic in $D$. Then $\bar\partial u(z)=f(z)d\bar z:=\varphi(|z|)g(z)d\bar z,\ z\in D$ has a flat solution locally. In particular if $g$ is anti-holomorphic in $D$, then $\frac{-1}{2\pi i}\int_D\frac{f(\zeta)}{\zeta-z}d\bar\zeta\wedge d\zeta$ is the flat solution to $\bar\partial u(z)=f(z)d\bar z$ in $D$ locally.*

**Proof of Theorem 4.2:** By assumption, $f(z)$ is flat at 0. Write the harmonic function $g$ as

$$g(z):=\sum_{n=0}^\infty(a_nz^n+b_n\bar z^n),\ z\in D(=D_r(0)).\tag{13}$$

We have

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_D \frac{f(\zeta)}{\zeta^{\alpha+1}}d\bar{\zeta}\wedge d\zeta = \frac{1}{2\pi i}\int_D \frac{\varphi(|\zeta|)g(\zeta)}{\zeta^{\alpha+1}}d\bar{\zeta}\wedge d\zeta \\
&=\frac{1}{2\pi i}\int_D \frac{\varphi(|\zeta|)\sum_{n=0}^{\infty}a_n\zeta^n}{\zeta^{\alpha+1}}d\bar{\zeta}\wedge d\zeta + \frac{1}{2\pi i}\int_D \frac{\varphi(|\zeta|)\sum_{n=0}^{\infty}b_n\bar{\zeta}^n}{\zeta^{\alpha+1}}d\bar{\zeta}\wedge d\zeta \\
&=\frac{1}{2\pi i}\sum_{n=0}^{\infty}a_n\int_D \frac{\varphi(|\zeta|)\zeta^n}{\zeta^{\alpha+1}}d\bar{\zeta}\wedge d\zeta + \frac{1}{2\pi i}\sum_{n=0}^{\infty}b_n\int_D \frac{\varphi(|\zeta|)\bar{\zeta}^n}{\zeta^{\alpha+1}}d\bar{\zeta}\wedge d\zeta \\
&=\frac{1}{\pi}\sum_{n=0}^{\infty}a_n\int_0^{2\pi}d\theta\int_0^R \varphi(r)r^{n-\alpha}e^{i(n-\alpha-1)}dr + \frac{1}{\pi}\sum_{n=0}^{\infty}b_n\int_0^{2\pi}d\theta\int_0^R \varphi(r)r^{-n-\alpha}e^{i(-n-\alpha-1)}dr \\
&=\frac{1}{\pi}\sum_{n=0}^{\infty}a_n\int_0^{2\pi}e^{i(n-\alpha-1)}d\theta\int_0^R \varphi(r)r^{n-\alpha}dr + \frac{1}{\pi}\sum_{n=0}^{\infty}b_n\int_0^{2\pi}e^{i(-n-\alpha-1)}d\theta\int_0^R \varphi(r)r^{-n-\alpha}dr \\
&=2a_{\alpha+1}\int_0^R \varphi(r)rdr := Ka_{\alpha+1},
\end{aligned}
$$

where $K = 2\int_0^R \varphi(r)rdr$ is a constant. Therefore,

$$
\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_D \frac{f(\zeta)}{\zeta^{n+1}}d\bar{\zeta}\wedge d\zeta\right)z^n = K\sum_{n=0}^{\infty}a_{n+1}z^n,
$$

which is holomorphic in $D$ according to (13). Thus $\bar{\partial}u(z) = f(z)d\bar{z},\ z\in D$ has a flat solution locally by Theorem 1.5.

When $g(z) = \sum_{n=0}^{\infty}b_n\bar{z}^n$, then $\frac{1}{2\pi i}\int_D \frac{f(\zeta)}{\zeta^{\alpha+1}}d\bar{\zeta}\wedge d\zeta = 0$ for all nonnegative integers $\alpha$. Hence the holomorphic function $h\equiv 0$ in the proof of Theorem 1.5 in order that $u$ is flat at 0. ∎

It is worth pointing out that in the proof of Theorem 4.2 the flatness of $\varphi$ is essential for the integrability of the integral $\frac{1}{2\pi i}\int_D \frac{f(\zeta)}{\zeta^{\alpha+1}}d\bar{\zeta}\wedge d\zeta$ for all nonnegative integers $\alpha$.

# 5 Examples of non-solvability in the flat category

In this section, we construct a family of germs of smooth $\bar{\partial}$-closed (0,1) forms $f_n$, flat at $0\in\mathbb{C}^n$, such that there are no flat smooth solutions to $\bar{\partial}u = f_n$ in the sense of germs.

**Theorem 5.1.** *There exists a family of germs of flat $\bar{\partial}$-closed (0,1) forms $f_n$, such that the Cauchy-Riemann equation $\bar{\partial}u = f_n$ has no flat solution in the sense of germs.*

We first give such examples in $n = 1$. Our construction essentially follows from Coffman-Pan [7], which in turn was motivated by Rosay [19]. Let $s$ be a real-valued nondecreasing smooth function on $\overline{\mathbb{R}^+}$ such that $s = 0$ in $[0, \frac{1}{4}]$, $0 < s < 1$ on $(\frac{1}{4}, \frac{3}{4})$ and $s = 1$ on $[\frac{3}{4}, \infty)$. Let $\{r_n\}_{n=1}^{\infty}$ be a decreasing positive sequence such that $r_1 = R$ and $\lim_{n\to\infty} r_n = 0$. Denote $\Delta r_n := r_n - r_{n+1}$, annuli $A_n := \{z\in\mathbb{C} : r_{n+1}\le |z|\le r_n\}$ and smooth functions $\mathcal{X}_n = s(\frac{|\cdot|-r_{n+1}}{\Delta r_n}) : A_n\to\mathbb{R}$. Then $D = (\cup A_n)\cup\{0\}$.

Let $\{p(n)\}_{n=0}^{\infty}$ be an increasing positive integer sequence with $p(0) = 0$, and $\{F(n)\}_{n=0}^{\infty}$ a positive sequence with $F(0) = 1$. Letting $g_n(z) = F(n)z^{p(n)}$, we define the function $f : D\to\mathbb{C}$ by

$$
f(z) = \begin{cases} g_n(z), & z\in A_n \text{ for odd n,}\\ \mathcal{X}_n(z)g_{n-1}(z) + (1-\mathcal{X}_n(z))g_{n+1}(z), & z\in A_n \text{ for even n,}\\ 0, & z = 0. \end{cases} \tag{14}
$$

By construction, the function $f$ is smooth on $D \setminus \{0\}$. Moreover, if the parameters are carefully chosen as in the following lemma, then $f$ can be made to be smooth and flat at 0.

**Lemma 5.2.** *[7] If $\frac{(\Delta r_n/r_n)}{(\Delta r_{n+2})/(r_{n+2})}$ is a bounded sequence and for each integer $k \geq 0$,*

$$\lim_{n\to\infty} \frac{F(n+1)(p(n+1))^k r_n^{p(n+1)-4k}}{(\Delta r_n/r_n)^k} = 0, \tag{15}$$

*then $f$ is smooth and vanishes to infinite order at the origin.*

We denote by $\mathbf{S}$ the set of functions $f$ of the form (14) satisfying (15) as well as either one of the following conditions:

$$\begin{aligned}
&\lim_{n\to\infty} \sqrt[p(n)]{F(n)\Delta r_n r_{n+1}} = \infty, \\
&\lim_{n\to\infty} \sqrt[p(n)]{F(n)(\Delta r_{n-1})^2} = \infty, \\
&\lim_{n\to\infty} \sqrt[p(n)]{F(n)\Delta r_{n-1} r_n} = \infty, \\
&\lim_{n\to\infty} \sqrt[p(n)]{F(n)(\Delta r_{n+1})^2} = \infty, \\
&\lim_{n\to\infty} \sqrt[p(n)]{F(n)\Delta r_{n+1} r_{n+2}} = \infty.
\end{aligned} \tag{16}$$

Hence in particular, every element in $\mathbf{S}$ is a representatives of a flat germ. The following example of Rosay [19] gives some $f \in \mathbf{S}$.

**Example 5.3.** [19] Choose $R = 1,\ p(n) = n,\ r_n = 2^{-n+1},\ \Delta r_n = 2^{-n},\ F(n) = 2^{n^2/2}$. Indeed,

$$\lim_{n\to\infty} \frac{F(n+1)(p(n+1))^k r_n^{p(n+1)-4k}}{(\Delta r_n/r_n)^k} = \lim_{n\to\infty} \frac{2^{(n+1)^2/2}(n+1)^k(2^{-n+1})^{n+1-4k}}{(2^{-1})^k} = 0;$$

$$\lim_{n\to\infty} \sqrt[n]{F(n)(\Delta r_{n-1})^2} = \lim_{n\to\infty} 2^{\frac{n}{2}} 2^{\frac{-2n+2}{n}} = \infty.$$

In fact, one can get ample elements in $\mathbf{S}$ by choosing $r_n$, $F(n)$ and $p(n)$ in the following manner.

**Example 5.4.** Choose $R = 1$, and three polynomial functions $p, t$ and $q$ of degree $d_p$, $d_t$ and $d_q$ on the variable $n$ respectively, such that all leading coefficients of these three polynomials are positive, $t(1) = 0$, $d_q > d_p,\ d_q > d_t$ and $d_q < d_p + d_t$. Let $r_n := 2^{-t(n)}$, $F(n) := 2^{q(n)}$. We have

$$\lim_{n\to\infty} \frac{2^{q(n+1)}(p(n+1))^k(2^{-t(n)})^{p(n+1)-4k}}{(2^{-1})^k} = 0$$

since $d_q < d_p + d_t$, and

$$\Delta r_n = 2^{-t(n)} - 2^{-t(n+1)} = \frac{2^{t(n+1)} - 2^{t(n)}}{2^{t(n)+t(n+1)}} \geq \frac{2^{t(n)}}{2^{t(n)+t(n+1)}} = r_{n+1}$$

for sufficiently large $n$. Then

$$\lim_{n\to\infty} \sqrt[p(n)]{F(n)(\Delta r_{n-1})^2} \geq \lim_{n\to\infty} 2^{\frac{q(n)}{p(n)}} 2^{\frac{-2t(n)}{p(n)}} = \infty,$$

due to the fact that $d_q > d_p$ and $d_q > d_t$. Therefore, this type of functions belong to $\mathbf{S}$ as well.

The proposition below essentially says that the two types of the UCP problems in Section 4 are not equivalent.

**Proposition 5.5.** *For every $f \in \mathbf{S}$, there does not exist a flat smooth $u$ such that $\bar{\partial} u = f d\bar{z}$ near the origin.*

**Proof of Proposition 5.5:** To prove $\bar{\partial} u = f d\bar{z}$ does not have a flat solution locally, it suffices to show that for any neighborhood $U$ near 0, the radius $R_U$ of convergence for $h(z) := \sum_{\alpha=0}^{\infty} \left( \frac{1}{2\pi i} \int_U \frac{f(\zeta)}{\zeta^{\alpha+1}} d\bar{\zeta} \wedge d\zeta \right) z^n$ is 0 by Theorem 1.5. Since the convergence of $h$ is independent of the choice of $U$, we choose $U = D$ and denote $R_U$ by $R_0$. First notice that

$$\frac{\partial^\alpha h(0)}{\alpha!} = \frac{1}{2\pi i} \int_D \frac{f(\zeta)}{\zeta^{\alpha+1}} d\bar{\zeta} \wedge d\zeta = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{A_n} \frac{f(\zeta)}{\zeta^{\alpha+1}} d\bar{\zeta} \wedge d\zeta. \tag{17}$$

When $n$ is odd,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{A_n} \frac{f(\zeta)}{\zeta^\alpha} d\bar{\zeta} \wedge d\zeta =& \frac{1}{2\pi i} \int_{A_n} \frac{g_n(\zeta)}{\zeta^\alpha} d\bar{\zeta} \wedge d\zeta \\
=& \frac{1}{\pi} \int_{\theta=0}^{2\pi} d\theta \int_{r_{n+1}}^{r_n} \frac{F(n) r^{p(n)} e^{ip(n)\theta}}{r^\alpha e^{i\alpha\theta}} r dr \\
=& \frac{F(n)}{\pi} \int_{\theta=0}^{2\pi} e^{i(p(n)-\alpha)\theta} d\theta \int_{r_{n+1}}^{r_n} r^{p(n)-\alpha+1} dr \\
=& \begin{cases} 0, & p(n) \neq \alpha, \\ F(n)(r_n^2 - r_{n+1}^2), & p(n) = \alpha. \end{cases}
\end{aligned} \tag{18}$$

When $n$ is even,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{A_n} \frac{f(\zeta)}{\zeta^\alpha} d\bar{\zeta} \wedge d\zeta =& \frac{1}{2\pi i} \int_{A_n} \frac{\mathcal{X}_n(z) g_{n-1}(z) + (1 - \mathcal{X}_n(z)) g_{n+1}(z)}{\zeta^\alpha} d\bar{\zeta} \wedge d\zeta \\
=& \frac{1}{\pi} \int_0^{2\pi} d\theta \int_{r_{n+1}}^{r_n} \mathcal{X}_n(re^{i\theta}) F(n-1) r^{p(n-1)-\alpha+1} e^{i(p(n-1)-\alpha)\theta} \\
&+ (1 - \mathcal{X}_n(re^{i\theta})) F(n+1) r^{p(n+1)-\alpha+1} e^{i(p(n+1)-\alpha)\theta} dr \\
=& \frac{1}{\pi} \int_0^{2\pi} e^{i(p(n-1)-\alpha)\theta} d\theta \int_{r_{n+1}}^{r_n} s(\frac{r - r_{n+1}}{\Delta r_n}) F(n-1) r^{p(n-1)-\alpha+1} dr \\
&+ \frac{1}{\pi} \int_0^{2\pi} e^{i(p(n+1)-\alpha)\theta} d\theta \int_{r_{n+1}}^{r_n} (1 - s(\frac{r - r_{n+1}}{\Delta r_n})) F(n+1) r^{p(n+1)-\alpha+1} dr \\
=& \begin{cases} 2 \int_{r_{n+1}}^{r_n} (1 - s(\frac{r - r_{n+1}}{\Delta r_n})) F(n+1) r dr, & p(n+1) = \alpha, \\ 2 \int_{r_{n+1}}^{r_n} s(\frac{r - r_{n+1}}{\Delta r_n}) F(n-1) r dr, & p(n-1) = \alpha, \\ 0, & \text{otherwise.} \end{cases} \\
=& \begin{cases} 2K_1 F(n+1)(\Delta r_n)^2 + 2K_2 F(n+1) r_{n+1} \Delta r_n, & p(n-1) = \alpha; \\ (1 - 2K_1) F(n-1)(\Delta r_n)^2 + (2 - 2K_2) F(n-1) r_{n+1} \Delta r_n, & p(n+1) = \alpha; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{19}$$

Here $0 < K_1 := \int_0^1 (1-s(t))tdt < \frac{1}{2}$ and $0 < K_2 := \int_0^1 (1-s(t))dt < 1$ are constants.

Now we consider the subsequence $\alpha(n) := p(n) - 1$ for all odd $n \in \mathbb{N}^+$. By (18) and (19),

$$\begin{aligned}\frac{\partial_z^{\alpha(n)} h(0)}{\alpha(n)!} =& \sum_{k=1}^{\infty} \frac{1}{2\pi i} \int_{A_k} \frac{f(\zeta)}{\zeta^{\alpha(n)+1}} d\bar{\zeta} \wedge d\zeta \\ =& \frac{1}{2\pi i} \int_{A_n} \frac{f(\zeta)}{\zeta^{\alpha(n)+1}} d\bar{\zeta} \wedge d\zeta + \frac{1}{2\pi i} \int_{A_{n-1}} \frac{f(\zeta)}{\zeta^{\alpha(n)+1}} d\bar{\zeta} \wedge d\zeta + \frac{1}{2\pi i} \int_{A_{n+1}} \frac{f(\zeta)}{\zeta^{\alpha(n)+1}} d\bar{\zeta} \wedge d\zeta \\ \geq& 2F(n)\Delta r_n r_{n+1} + 2K_1 F(n)(\Delta r_{n-1})^2 + 2K_2 F(n) r_n \Delta r_{n-1} \\ &+ (1-2K_1)F(n)(\Delta r_{n+1})^2 + (2-2K_2)F(n) r_{n+2}\Delta r_{n+1}.\end{aligned} \tag{20}$$

Since each term in the above equation is positive, from the Cauchy-Hadamard Formula and (16), the radius $R_0$ of convergence for $h$ satisfies

$$\frac{1}{R_0} = \lim_{\alpha\to\infty} \sup\left(\frac{\partial_z^{\alpha} h(0)}{\alpha!}\right)^{\frac{1}{\alpha}} \geq \lim_{n\to\infty}\left(\frac{\partial_z^{\alpha(n)} h(0)}{\alpha(n)!}\right)^{\frac{1}{\alpha(n)}} = \lim_{n\to\infty}\left(\frac{\partial_z^{\alpha(n)} h(0)}{\alpha(n)!}\right)^{\frac{1}{p(n)}} = \infty, \tag{21}$$

which gives rise to $R_0 = 0$. The proof of the proposition is complete. ∎

**Proof of Theorem 5.1**: Without loss of generalization, we assume $n = 2$. Let $f \in \mathbf{S}$ and write $\mathbf{f}(z) = f(z_1)d\bar{z}_1$. Then $\mathbf{f}$ is flat. In particular, $\bar{\partial}\mathbf{f} = 0$. Let $u(z_1) := Tf(z_1)$, then $\bar{\partial}_1 u(z_1) = f(z_1)$ and $u(z_1)$ is a solution to $\bar{\partial}\mathbf{u} = \mathbf{f}$. Hence any solution to $\bar{\partial}\mathbf{u} = \mathbf{f}$ satisfies

$$\mathbf{u}(z) = u(z_1) + \phi(z),$$

where $\phi$ is a holomorphic function, say, on a ball $B \subset \mathbb{C}^2$.

Assume by contradiction that there exists a flat solution $\mathbf{u}$ at the origin, then for any $\alpha \geq 0, \beta > 0$,

$$0 = \partial_{z_1}^{\alpha}\mathbf{u}(0,0) = \partial_{z_1}^{\alpha} u(0) + \partial_{z_1}^{\alpha}\phi(0,0), \quad 0 = \partial_{z_1}^{\alpha}\partial_{z_2}^{\beta}\mathbf{u}(0,0) = \partial_{z_1}^{\alpha}\partial_{z_2}^{\beta}\phi(0,0).$$

Therefore, $\partial_{z_1}^{\alpha}\phi(0,0) = -\partial_{z_1}^{\alpha} u(0) = \partial_{z_1}^{\alpha} T(f)(0)$ and $\partial_{z_1}^{\alpha}\partial_{z_2}^{\beta}\phi(0,0) = 0$. Since $f \in \mathbf{S}$, consider the subsequence $\alpha(n) = p(n) - 1$ with odd integers $n$ as we did in the proof of Theorem 5.5. Then

$$\lim_{n\to\infty} \sqrt[\alpha(n)]{\frac{\partial_{z_1}^{\alpha(n)} T(f)(0)}{\alpha(n)!}} = \infty. \tag{22}$$

Hence the radius of convergence for the holomorphic function $\phi(z_1, 0) = \sum_{\alpha=0}^{\infty} \frac{\partial_{z_1}^{\alpha}\phi(0,0)}{\alpha!} z_1^{\alpha}$ is 0, which is impossible. The proof is complete. ∎

*Remark* 5.6. a). One can also make use of Borel's theorem to construct nonsolvable examples to $\bar{\partial}$ in the flat category. In fact, let $\tilde{u}$ be a smooth function near 0 whose Taylor expansion is equal to, say, $\sum_{n=0}^{\infty} n! z^n$ by Borel's theorem. It is easy to verify that $f := \bar{\partial}\tilde{u}$ is flat at 0. Assume there exists a flat solution $u$ solving $\bar{\partial}u = f$ near 0. Then $\bar{\partial}(\tilde{u} - u) = 0$ and thus there exists a holomorphic function $h$ near 0 such that $h = \tilde{u} - u$. However, by the flatness of $u$, this would mean that the jets of $h$ match with those of $\tilde{u}$ at all levels. Contradiction!

b). Bo-Yong Chen suggested a different construction of some function $f$ vanishing in a neighborhood of the origin such that $\bar{\partial}u = f$ has no flat solution. Let $\chi$ be a compactly supported smooth cut-off function satisfying $\chi = 1$ in a neighborhood $B_{1/4}$ of 0 and $f$ be an unbounded holomorphic function on $B_{1/2}(\supset\supset supp\chi)$. Consider $\bar{\partial}u = f\bar{\partial}\chi := v$ in $B_1$. Then $v = 0$ in $B_{1/4}$ and $\bar{\partial}v = 0$ in $B_1$. If there is a flat solution $u \in C^{\infty}(B_1)$ to $\bar{\partial}u = v$, then $u = 0$ in $B_{1/4}$ since $u$ is holomorphic there. It follows that $h := f\chi - u$ is holomorphic in $B_1$ and in particular, $h = f$ on $B_{1/4}$, which implies $f$ has a holomorphic extension $h$ to $B_1$. This is a contradiction! We note this example does not serve our purpose since $v \equiv 0$ in the sense of germs.

# 6 Minimal solutions to the Cauchy-Riemann equations

In this section, we shall make use of the construction in Proposition 5.5 to analyze the behavior of the restriction on subdomains of a minimal solution to $\bar\partial$ with respect to a weight.

**Proof of Theorem 1.6**: We show any $f \in \mathbf{S}$ will suffice the theorem. Indeed, let $f \in \mathbf{S}$ and assume there exist a bounded plurisubharmonic weight function $\phi$ and a positive decreasing sequence $t_n(<1) \to 0$, such that $u_n := u_1|_{B_{t_n}}$ is the minimal smooth solution to $\bar\partial u = f(z_1)d\bar z_1|_{B_{t_n}}$ on $B_{t_n}$ with respect to $\phi|_{B_{t_n}}$. By Hörmander's classical $L^2$ theory and the boundedness of $\phi$, we have for each $n \geq 1$,

$$\int_{B_{t_n}} |u_1|^2 dV \leq C\int_{B_{t_n}} |u_n|^2 e^{-\phi} dV \leq Ct_n^2 \int_{B_{t_n}} |f(z_1)|^2 e^{-\phi} dV \leq Ct_n^2 \int_{B_{t_n}} |f(z_1)|^2 dV. \tag{23}$$

Here and in what follows, $C$ represents a positive constant independent of $n$, which may be different at different places. Since $f \in \mathbf{S}$,

$$\begin{aligned}
\int_{B_{t_n}} |f(z_1)|^2 dV \leq & C\int_0^{2\pi}\int_{r\leq t_n} |f(re^{i\theta})|^2 r dr d\theta \\
\leq & C\sum_{k \text{ odd},\ r_k\leq t_n} F(k)^2 \int_{r_{k+1}}^{r_k} r^{2p(k)+1}dr + C\sum_{k \text{ even},\ r_k\leq t_n} F(k-1)^2 \int_{r_{k+1}}^{r_k} r^{2p(k-1)+1}dr \\
& + C\sum_{k \text{ even},\ r_k\leq t_n} F(k+1)^2 \int_{r_{k+1}}^{r_k} r^{2p(k+1)+1}dr \\
\leq & C\sum_{k \text{ odd},\ r_k\leq t_n} F(k)^2 \int_{r_{k+1}}^{r_k} r^{2p(k)+1}dr + C\sum_{k \text{ odd},\ r_{k+1}\leq t_n} F(k)^2 \int_{r_{k+2}}^{r_{k+1}} r^{2p(k)+1}dr \\
& + C\sum_{k \text{ odd},\ r_{k-1}\leq t_n} F(k)^2 \int_{r_k}^{r_{k-1}} r^{2p(k)+1}dr \\
\leq & C\sum_{r_k\leq t_n} (F(k)r_k^{p(k)})^2 + C\sum_{r_{k+1}\leq t_n} (F(k)r_{k+1}^{p(k)})^2 \\
& + C\sum_{r_{k-1}\leq t_n} (F(k)r_{k-1}^{p(k)})^2.
\end{aligned} \tag{24}$$

For each $N \geq 0$, let $n$ be large enough so that $p(k) \geq N$ for all $k$ with $r_{k+1} \leq t_n$. By (15), we obtain that

$$\begin{aligned}
F(k)r_k^{p(k)} &\leq F(k)r_{k-1}^{p(k)-N} r_k^N \leq \frac{Ct_n^N}{p(k)} \quad \text{when} \quad r_k \leq t_n; \\
F(k)r_{k+1}^{p(k)} &\leq F(k)r_{k-1}^{p(k)-N} r_{k+1}^N \leq \frac{Ct_n^N}{p(k)} \quad \text{when} \quad r_{k+1} \leq t_n; \\
F(k)r_{k-1}^{p(k)} &\leq F(k)r_{k-1}^{p(k)-N} r_{k-1}^N \leq \frac{Ct_n^N}{p(k)} \quad \text{when} \quad r_{k-1} \leq t_n.
\end{aligned} \tag{25}$$

Combining (23), (24), (25) and the fact that $p(k) \geq k$, we have thus shown that for each $N \geq 0$,

$$\int_{B_{t_n}} |u_1|^2 dV \leq Ct_n^N$$

for large $n$. Since $u_1$ is smooth near 0, $u_1$ is flat at 0. This would mean that $\bar{\partial}u = f(z_1)d\bar{z}_1$ with $f \in \mathbf{S}$ has a flat solution, which contradicts with the conclusion in Theorem 5.1. ■

In comparison to Theorem 1.6, Błocki [3] constructed an example where the restriction of a minimal solution of the Cauchy-Riemman equation onto some subdomains can be minimal. The example can be generalized in $\mathbb{C}^n$ as follows.

**Example 6.1.** Let $f_j$ and $g$ be holomorphic in $B_R$ such that $g(0) = 0$ and $\frac{\partial g}{\partial z_j} = f_j$ in $B_R$. Then given any bounded and radially symmetric plurisubharmonic weight $\phi$ on $B_R$, $u(z) = \overline{g(z)}|_{B_r}$ is the minimal solution to $\bar{\partial}u(z) = \overline{f_j(z)}d\bar{z}_j|_{B_r}$ in $B_r$ in $L^2(B_r, e^{-\phi}|_{B_r})$ norm for every $r \leq R$.

**Proof of Example 6.1**: Since $u(z) = \overline{g(z)}|_{B_r}$ is a solution to $\bar{\partial}u(z) = \overline{f_j(z)}d\bar{z}_j|_{B_r}$, we only need to show $\overline{g(z)}|_{B_r}$ is minimal in $L^2(B_r, e^{-\phi}|_{B_r})$ norm, which is equivalent to showing given any nonnegative multi-index $\alpha \neq 0$, $r \leq R$, $\bar{z}^\alpha$ is orthogonal to $z^\gamma$ for all nonnegative multi-index $\gamma$ in $L^2(B_r, e^{-\phi}|_{B_r})$ norm. This is obvious due to the following observation. Without loss of generality, we assume $n = 2$.

$$
\begin{aligned}
&\langle \bar{z}^\alpha, z^\gamma \rangle_{L^2(B_r, e^{-\phi}|_{B_r})} = \int_{B_r} \bar{z}^\alpha \bar{z}^\gamma e^{-\phi(|z|)} dV \\
=&\frac{1}{4} \int_{|z_1| \leq r} \bar{z}_1^{\alpha_1+\gamma_1} \int_{|z_2| \leq \sqrt{1-|z_1|^2}} \bar{z}_2^{\alpha_2+\gamma_2} e^{-\phi(|z|)} dz_2 d\bar{z}_2 dz_1 d\bar{z}_1 \\
=&\int_0^r r_1^{\alpha_1+\gamma_1+1} \int_0^{\sqrt{1-r_1^2}} r_2^{\alpha_2+\gamma_2+1} e^{-\phi(\sqrt{r_1^2+r_2^2})} dr_2 dr_1 \int_0^{2\pi} e^{i(-\alpha_1-\gamma_1)\theta_1} d\theta_1 \int_0^{2\pi} e^{i(-\alpha_2-\gamma_2)\theta_2} d\theta_2 \\
=:&\ C(r, \alpha, \gamma) \int_0^{2\pi} e^{i(-\alpha_1-\gamma_1)\theta_1} d\theta_1 \int_0^{2\pi} e^{i(-\alpha_2-\gamma_2)\theta_2} d\theta_2
\end{aligned}
$$

for some positive smooth function $C(r, \alpha, \gamma)$. Notice the last expression is nonzero only when $\alpha_1 + \gamma_1$ and $\alpha_2 + \gamma_2$ are both zero, which is impossible since $\alpha_1 + \alpha_2 > 0$ and $\gamma_j \geq 0, j = 1, 2$. ■


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