Hölder estimates for the $\bar{\partial}$ problem for (p,q) forms on product domains

Yifei Pan and Yuan Zhang*

Abstract

The purpose of this paper is to study Hölder estimates for the $\bar{\partial}$ problem for (p,q) forms on products of general planar domains. As indicated by an example of Stein and Kerzman, solutions to the $\bar{\partial}$ problem on product domains in $\mathbb{C}^n (n \geq 2)$ does not gain regularity in Hölder spaces. Making use of an integral representation of Nijenhuis and Woolf, we show that given a $\bar{\partial}$-closed (p,q) form with $C^{k,\alpha}$ components, $0 \leq p \leq n, 1 \leq q \leq n$, $k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \leq 1$, there is a $C^{k,\alpha'}$ solution to the $\bar{\partial}$ problem on product domains for any $0 < \alpha' < \alpha$ with the desired Hölder estimate.

1 Introduction and the main theorems

The existence and regularity of the Cauchy-Riemann equations have been thoroughly studied in literature along the line of Hörmander's L^2 theory. An alternative approach is to express solutions in integral representations. Through a series of work including Grauert-Lieb [10], Henkin [13], Kerzman [16], Henkin-Romanov [15] and Diederich-Fischer-Fornæss [5], sup-norm and Hölder estimates of solutions were established for smooth bounded domains which are strongly pseudoconvex or convex of finite type. Higher order regularity of solutions on sufficiently smooth bounded strongly pseudoconvex or strongly $\mathbb{C}$-linearly convex domains were studied by Siu [22], Lieb-Range [17], and more recently Gong [11] and Gong-Lanzani [12] et al.

Let $\Omega \subset \mathbb{C}^n, n \geq 2$ be a product of bounded planer domains. Namely, $\Omega = D_1 \times \cdots \times D_n$, where each $D_j \subset \mathbb{C}, j = 1, \ldots, n$, is a bounded domain in $\mathbb{C}$ such that ∂D_j consists of a finite number of rectifiable Jordan curves which do not intersect one another. Then Ω is a

*partially supported by NSF DMS-1501024

Keywords. $\bar{\partial}$-equation, product domains, Hölder spaces.

bounded pseudoconvex domain (but not convex in general) with at most Lipschitz boundary. A solution operator to $\bar{\partial}$ was first constructed in a seminal work [19] of Nijenhuis and Woolf in an *iterated* Hölder space over polydiscs. The supnorm estimate for C^1 data up to the boundary was proved by Henkin [14] on the bidisc. Recently, Chen-McNeal [3] studied a type of L^p-Sobolev estimates for $(0,1)$ forms on general product domains in $\mathbb{C}^2$. They further showed that Henkin's solution operator is not bounded in $L^p, 1 \le p < 2$. For product domains of arbitrary dimensions, Fassina-Pan [9] constructed a solution operator for $(0,1)$ forms through one-dimensional method, from which they obtained L^∞ estimates for smooth data. See also Bertrams [1], Ehsani [8], Chakrabarti-Shaw [2], Dong-Li-Treuer [6] and the references therein for investigation of the canonical solutions on product domains.

We should point out that unlike strictly pseudoconvex smooth domains, the $\bar{\partial}$ problem on product domains does not gain regularity. Indeed, motivated by an example of Stein and Kerzman [16], one can construct examples to show that the $\bar{\partial}$ problem on product domains in general has no gain of regularity in the (standard) Hölder spaces. The examples are verified at the end of Section 5. Therefore, a natural question is, given a Hölder data on product domains, whether there exists a solution to the $\bar{\partial}$ equation in the same Hölder class. It is our goal to generalize the result of [19] and study the classical Hölder estimate of a $\bar{\partial}$ solution operator for (p,q) forms on general product domains.

Let $C^{k,\alpha}(\Omega)$ be the (standard) Hölder space, $k \in \mathbb{Z}^+ \cup \{0\}$, $0 < \alpha \le 1$, and (p,q) form is said to be in $C^{k,\alpha}_{(p,q)}(\Omega)$, $0 \le p \le n$, $1 \le q \le n$, if all its components are in $C^{k,\alpha}(\Omega)$. (See Section 2 for the definition.) Given a function $f \in C^{k,\alpha}(\Omega)$, define for $z \in \Omega$, the solid and boundary Cauchy type integrals below, respectively.

$$
\begin{aligned}
T_j f(z) &:= -\frac{1}{2\pi i}\int_{D_j} \frac{f(z_1,\dots,z_{j-1},\zeta_j,z_{j+1},\dots,z_n)}{\zeta_j - z_j} d\bar{\zeta}_j \wedge d\zeta_j;\\
S_j f(z) &:= \frac{1}{2\pi i}\int_{\partial D_j} \frac{f(z_1,\dots,z_{j-1},\zeta_j,z_{j+1},\dots,z_n)}{\zeta_j - z_j} d\zeta_j.
\end{aligned}
\tag{1}
$$

The boundedness of these operators was established by Nijenhuis and Woolf in [19] on polydiscs with respect to an iterated Hölder norm, which is stronger than the (standard) Hölder norm. See Section 3 for a revisit of the related work in [19]. Thus the resulting iterated Hölder spaces are subspaces of the corresponding (standard) Hölder spaces. Since their approach relies also largely on rich symmetry of polydiscs, the method no longer works either for the standard Hölder spaces or over general product domains. In this paper, we prove the Hölder regularity for T_j and S_j in the (standard) Hölder spaces on general product domains. Indeed, as demonstrated by examples in Section 4 in contrast to their one dimensional counterparts on planar domains, the following Hölder estimates for T_j and S_j turn out to be optimal.

Theorem 1.1. *a). T_j is a bounded linear operator sending $C^{k,\alpha}(\Omega)$ into $C^{k,\alpha}(\Omega)$, $k \in \mathbb{Z}^+ \cup \{0\}$, $0 < \alpha < 1$. Namely, there exists some constant C dependent only on Ω, k and α, such that for any $f \in C^{k,\alpha}(\Omega)$,*

$$\|T_j f\|_{C^{k,\alpha}(\Omega)} \leq C \|f\|_{C^{k,\alpha}(\Omega)}. \tag{2}$$

b). S_j is a bounded linear operator sending $C^{k,\alpha}(\Omega)$ into $C^{k,\alpha'}(\Omega)$, $k \in \mathbb{Z}^+ \cup \{0\}$, $0 < \alpha' < \alpha \leq 1$. Namely, there exists some C dependent only on Ω, k, α and α', such that for any $f \in C^{k,\alpha}(\Omega)$,

$$\|S_j f\|_{C^{k,\alpha'}(\Omega)} \leq C \|f\|_{C^{k,\alpha}(\Omega)}. \tag{3}$$

As an application of the boundedness of these operators in Hölder spaces, an estimate of a $\bar{\partial}$ solution in Hölder spaces is obtained with a loss of regularity that can be made arbitrarily small as follows.

Theorem 1.2. *Let $D_j \subset \mathbb{C}$, $j = 1, \ldots, n$, be bounded domains with $C^{k+1,\alpha}$ boundary, $n \geq 2, k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \leq 1$, and let $\Omega := D_1 \times \cdots \times D_n$. Assume that $\mathbf{f} \in C^{k,\alpha}_{(p,q)}(\Omega)$ is a $\bar{\partial}$-closed (p,q) form on Ω, $0 \leq p \leq n, 1 \leq q \leq n$. There exists a solution $\mathbf{u} \in C^{k,\alpha'}_{(p,q-1)}(\Omega)$ to $\bar{\partial}\mathbf{u} = \mathbf{f}$ such that for any $0 < \alpha' < \alpha$, $\|\mathbf{u}\|_{C^{k,\alpha'}(\Omega)} \leq C\|\mathbf{f}\|_{C^{k,\alpha}(\Omega)}$, where C depends only on Ω, k, α and α'. Here when $k = 0$, all equations are understood in the sense of distributions.*

It is desirable to know whether there exists a solution operator that can achieve the same regularity as that of the data in Hölder spaces. However, we do not have answers at this point. We also mention that another type of an iterated Hölder space was studied in [4] where estimates of the solutions depend on higher order derivatives of the data. See Remark 3.3 d) for a brief comparison of these spaces and the corresponding estimates.

For smooth data up to the boundary of the product domains, the existence of smooth solutions for $(p, 1)$ forms has already been obtained in [2] with Sobolev estimates. As a direct consequence of Theorem 1.2, we obtain the following corollary for (p, q) forms smooth up to the boundary in terms of Hölder estimates.

Corollary 1.3. *Let $D_j \subset \mathbb{C}$, $j = 1, \ldots, n$, be bounded domains with C^∞ boundary, $n \geq 2$, and $\Omega := D_1 \times \cdots \times D_n$. Assume $\mathbf{f} \in C^\infty_{(p,q)}(\overline{\Omega})$ is a $\bar{\partial}$-closed (p,q) form on Ω, $0 \leq p \leq n, 1 \leq q \leq n$. There exists a solution $\mathbf{u} \in C^\infty_{(p,q-1)}(\overline{\Omega})$ to $\bar{\partial}\mathbf{u} = \mathbf{f}$ in Ω. Moreover, for all $k \in \mathbb{Z}^+ \cup \{0\}$, $0 < \alpha' < \alpha \leq 1$, $\|\mathbf{u}\|_{C^{k,\alpha'}(\Omega)} \leq C_{k,\alpha,\alpha'}\|\mathbf{f}\|_{C^{k,\alpha}(\Omega)}$, where $C_{k,\alpha,\alpha'}$ depends only on Ω, k, α and α'.*

The rest of the paper is organized as follows. Section 2 addresses preliminaries about solid and boundary Cauchy integrals on the complex plane. Section 3 is a revisit of the fundamental work of Nijenhuis and Woolf [19] on the $\bar{\partial}$ problem. Theorem 1.1 is proved in Section 4, along with examples demonstrating those estimates are optimal in Hölder category. The last section is devoted to the proof of Theorem 1.2 and Corollary 1.3. In the Appendix, a convergence result of the mollifier method in Hölder spaces is proved.

Acknowledgement: Both authors thank Liding Yao for providing an example in the Appendix. Part of the work was done while the second author was visiting American Institute of Mathematics (AIM). She also appreciates AIM and Association for Women in Mathematics (AWM) for hospitality during her visit.

2 Notations and Preliminaries

As a common notice, we use u and f to represent complex-valued functions, and boldface $\mathbf{u}$ and $\mathbf{f}$ to represent forms. Unless otherwise specified, C represents a constant dependent only on Ω, k, α and α', which may be of different values in different places.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain, the standard Hölder space $C^{k,\alpha}(\Omega), k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \leq 1$ is defined by

$$\{f \in C^k(\Omega) : \|f\|_{C^{k,\alpha}(\Omega)} := \|f\|_{C^k(\Omega)} + \sum_{|\gamma|=k} H^\alpha[D^\gamma f] < \infty\}.$$

Here D^γ represents any $|\gamma|$-th derivative operator,

$$\|f\|_{C^k(\Omega)} := \sum_{|\gamma|=0}^{k} \sup_{z\in\Omega} |D^\gamma f(z)|$$

and the Hölder semi-norm is

$$H^\alpha[f] := \sup_{z,z'\in\Omega, z\neq z'} \frac{|f(z) - f(z')|}{|z - z'|^\alpha}.$$

When $k = 0, 0 < \alpha < 1$, we write $C^{0,\alpha}(\Omega) = C^\alpha(\Omega)$. For a (p, q) form $\mathbf{f} \in C^{k,\alpha}_{(p,q)}(\Omega)$, define $\|\mathbf{f}\|_{C^{k,\alpha}(\Omega)}$ to be the sum of the $C^{k,\alpha}(\Omega)$ norms of all its components.

When $\Omega = D_1 \times \cdots \times D_n$ is a product of planar domains, for each $j \in \{1, \ldots, n\}$, the Hölder semi-norm with respect to j-th variable for each fixed $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in$

$D_1 \times \cdots \times D_{j-1} \times D_{j+1} \times \cdots \times D_n$ is defined by

$$
\begin{aligned}
&H_j^\alpha[f(z_1, \ldots, z_{j-1}, \cdot, z_{j+1}, \ldots, z_n)] : \\
= &\sup_{\zeta, \zeta' \in D_j, \zeta \neq \zeta'} \frac{|f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n) - f(z_1, \ldots, z_{j-1}, \zeta', z_{j+1}, \ldots, z_n)|}{|\zeta - \zeta'|^\alpha}.
\end{aligned}
$$

Clearly,

$$
\sup_{\substack{z_k \in D_k, \\ 1 \leq k (\neq j) \leq n}} H_j^\alpha[f(z_1, \ldots, z_{j-1}, \cdot, z_{j+1}, \ldots, z_n)] \leq H^\alpha[f], \quad j = 1. \ldots, n.
$$

On the other hand, the following elementary lemma for Hölder functions is observed for product domains.

Lemma 2.1. *Let $\Omega = D_1 \times \cdots \times D_n$ be a product of planar domains. Then*

$$
H^\alpha[f] \leq \sum_{1 \leq j \leq n} \sup_{\substack{z_k \in D_k, \\ 1 \leq k (\neq j) \leq n}} H_j^\alpha[f(z_1, \ldots, z_{j-1}, \cdot, z_{j+1}, \ldots, z_n)]. \tag{4}
$$

Proof. For simplicity of exposition, assume $n = 2$ with $\Omega = D_1 \times D_2$. Let C be the right hand side of (4). For any $z = (z_1, z_2) \in D_1 \times D_2, z' = (z_1', z_2') \in D_1 \times D_2$, then $(z_1', z_2) \in D_1 \times D_2$. Hence $|f(z_1, z_2) - f(z_1', z_2')| \leq |f(z_1, z_2) - f(z_1', z_2)| + |f(z_1', z_2) - f(z_1', z_2')| \leq C|z - z'|^\alpha$. ∎

The rest of the section is devoted to classical theory in complex analysis. Let D be a bounded domain in $\mathbb{C}$ with $C^{k+1,\alpha}$ boundary, $k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \leq 1$. Given a complex-valued continuous function $f \in C(\bar{D})$, we define the following two operators related to the Cauchy kernel for $z \in D$:

$$
\begin{aligned}
Tf(z) &:= \frac{-1}{2\pi i} \int_D \frac{f(\zeta)}{\zeta - z} d\bar{\zeta} \wedge d\zeta; \\
Sf(z) &:= \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.
\end{aligned}
$$

Here the positive orientation of ∂D is adopted for the contour integral such that D is always to the left while traversing along the contour(s). As is well known, T is the universal solution operator for the $\bar{\partial}$ operator on D, while S turns integrable functions on ∂D to holomorphic functions in D. In the following, we state some properties of the two operators that will be used in later sections.

Theorem 2.2. *(cf. [23]) Let D be a bounded domain with $C^{1,\alpha}$ boundary, $f \in C(\bar{D})$ and $f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}} \in L^p(D), p > 2$. Then*

$$f = Sf + T(f_{\bar{\zeta}}) \quad in \quad D.$$

Theorem 2.3. *(cf. [23]) Let D be a bounded domain with $C^{k+1,\alpha}$ boundary, and $f \in C^{k,\alpha}(D), k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha < 1$. Then $Tf \in C^{k+1,\alpha}(D)$ and $Sf \in C^{k,\alpha}(D)$. Moreover, there exists a constant C dependent only on D, k and α, such that*

$$\|Tf\|_{C^{k+1,\alpha}(D)} \leq C\|f\|_{C^{k,\alpha}(D)};$$
$$\|Sf\|_{C^{k,\alpha}(D)} \leq C\|f\|_{C^{k,\alpha}(D)}.$$

Theorem 2.4. *(cf. [23]) Let D be a bounded domain. Then $Tf \in C^{\alpha}(D)$ if $f \in L^p(D), p > 2, \alpha = \frac{p-2}{p}$, and there exists a constant C dependent only on D and p, such that*

$$\|Tf\|_{C^{\alpha}(D)} \leq C\|f\|_{L^p}.$$

Moreover, $\bar{\partial}T = id$ on $L^p(D), 1 \leq p < \infty$ in the sense of distributions.

For proofs of the above theorems, see p. 41 [23] for Theorem 2.2; p. 56 [23] and p. 21 [23] for Theorem 2.3; p. 38 [23] and p. 29 [23] for Theorem 2.4.

3 Revisit of Nijenhuis-Woolf's work on polydiscs

In this section, we present the related results in the fundamental work of Nijenhuis and Woolf [19] for the $\bar{\partial}$ problem on the polydisc $\mathbb{D}^n := \{(z_1, \cdots, z_n) \in \mathbb{C}^n : |z_j| < 1, j = 1, \ldots, n\}$. We shall purposely retain their notation as much as possible for the convenience of readers.

Let f be a complex-valued function on $\mathbb{D}^n$. Define $\triangle_i f$ to be a function on the subset $\mathbb{D}_i$ of $\mathbb{D}^{n+1}$ whose points $Z_i = (z_1, \cdots, z_{i-1}, (z_i, z_i'), z_{i+1}, \cdots, z_n)$ satisfies $z_i \neq z_i'$, such that

$$\triangle_i f(Z_i) = f(z_1, \cdots, z_i, \cdots, z_n) - f(z_1, \cdots, z_{i-1}, z_i', z_{i+1}, \cdots, z_n).$$

Recursively, let $\mathbb{D}_{i_1 \cdots i_k}$ be the subset of $\mathbb{D}^{n+k}$ whose points $Z_{i_1 \cdots i_k} = (z_1, \cdots, (z_{i_1}, z_{i_1}'), \cdots, (z_{i_k}, z_{i_k}'), \cdots, z_n)$ satisfy $z_{i_j} \neq z_{i_j}', j = 1, \cdots, k$. Define on $\mathbb{D}_{i_1 \cdots i_k}$ a function

$$\triangle_{i_1 \cdots i_k} f := \triangle_{i_k} \triangle_{i_1 \cdots i_{k-1}} f.$$

In [19], a naturally defined iterated Hölder space $\mathcal{C}^{\alpha}(\mathbb{D}^n)$ (with the notation slightly different from that of the standard Hölder space) was introduced such that a function $f \in \mathcal{C}^{\alpha}(\mathbb{D}^n)$ if

$$\|f\|_{\mathcal{C}^{\alpha}(\mathbb{D}^n)} := \|f\|_{C(\mathbb{D}^n)} + \sum_{k=1}^{n} H_{\alpha}^{(k)}[f] < \infty. \tag{5}$$

Here

$$H_\alpha^{(k)}[f] := \sup_{\substack{1\le i_1<\ldots<i_k\le n,\\ Z_{i_1\cdots i_k}\in\mathbb{D}_{i_1\cdots i_k}}} \left\{ \frac{|\triangle_{i_1\cdots i_k} f(Z_{i_1\cdots i_k})|}{|z_{i_1}-z'_{i_1}|^\alpha\cdots|z_{i_k}-z'_{i_k}|^\alpha} \right\}.$$

Since $H_\alpha^{(1)}$ is precisely H^α in Section 2, we have $\|\cdot\|_{C^\alpha(\mathbb{D}^n)} \le \|\cdot\|_{\mathcal{C}^\alpha(\mathbb{D}^n)}$. In fact, one further has (p. 485 [19])

$$\mathcal{C}^\alpha(\mathbb{D}^n) \subset C^\alpha(\mathbb{D}^n) \subset \mathcal{C}^{\frac{\alpha}{n}}(\mathbb{D}^n). \tag{6}$$

Let T_j and S_j be the solid and boundary Cauchy integral operators acting on functions over j-th slice of $\mathbb{D}^n$ as in (1). Given a (p,q) form

$$\mathbf{f} = \sum_{\substack{i_1<\cdots<i_p,\\ j_1<\cdots<j_q}} f_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar{z}_{j_1}\wedge\cdots\wedge d\bar{z}_{j_q} \in C^1_{(p,q)}(\bar{\mathbb{D}}^n), \tag{7}$$

define $T_j\mathbf{f}$ and $S_j\mathbf{f}$ to be the action on the corresponding component functions. Namely,

$$T_j\mathbf{f} := \sum_{\substack{1\le i_1<\cdots<i_p\le n,\\ 1\le j_1<\cdots<j_q\le n}} T_j f_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar{z}_{j_1}\wedge\cdots\wedge d\bar{z}_{j_q};$$

$$S_j\mathbf{f} := \sum_{\substack{1\le i_1<\cdots<i_p\le n,\\ 1\le j_1<\cdots<j_q\le n}} S_j f_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar{z}_{j_1}\wedge\cdots\wedge d\bar{z}_{j_q}.$$

To construct a solution operator to the $\bar\partial$ equation for (p,q) forms, [19] introduced a projection operator π_k. Precisely speaking, for the (p,q) form $\mathbf{f}$ given in (7) and each $1\le k\le n$, $\pi_k\mathbf{f}$ is a $(p,q-1)$ form with

$$\pi_k\mathbf{f} := (-1)^p \sum_{\substack{1\le i_1<\cdots<i_p\le n,\\ 1\le k<j_2<\cdots<j_q\le n}} f_{i_1\cdots i_p\bar{k}\bar{j}_2\cdots\bar{j}_q} dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar{z}_{j_2}\wedge\cdots\wedge d\bar{z}_{j_q}.$$

Based on these definitions, a solution operator of the $\bar\partial$ equation for (p,q) forms on polydisc was constructed in [19] (p. 430).

Theorem 3.1. *[19] If $\mathbf{f}\in C^1_{(p,q)}(\bar{\mathbb{D}}^n)$ is $\bar\partial$-closed on $\mathbb{D}^n$, then*

$$T\mathbf{f} := T_1\pi_1\mathbf{f} + T_2S_1\pi_2\mathbf{f} + \cdots + T_nS_1\cdots S_{n-1}\pi_n\mathbf{f} \tag{8}$$

is a solution to $\bar\partial\mathbf{u} = \mathbf{f}$ on $\mathbb{D}^n$.

In terms of the norm estimates of the operators, [19] (p. 435 & p. 487) proved the following fundamental boundedness for both T_j and S_j operators in the iterated Hölder spaces.

Theorem 3.2. *[19] If $\mathbf{f} \in \mathcal{C}^{\alpha}_{(p,q)}(\mathbb{D}^n)$, then there exists a constant C dependent only on n and α such that*

$$\|T_j \mathbf{f}\|_{\mathcal{C}^{\alpha}(\mathbb{D}^n)} \leq C\|\mathbf{f}\|_{\mathcal{C}^{\alpha}(\mathbb{D}^n)};$$
$$\|S_j \mathbf{f}\|_{\mathcal{C}^{\alpha}(\mathbb{D}^n)} \leq C\|\mathbf{f}\|_{\mathcal{C}^{\alpha}(\mathbb{D}^n)}.$$

Consequently, the solution operator T defined in (8) satisfies

$$\|T\mathbf{f}\|_{\mathcal{C}^{\alpha}(\mathbb{D}^n)} \leq C\|\mathbf{f}\|_{\mathcal{C}^{\alpha}(\mathbb{D}^n)}.$$

Remark 3.3. a). Theorem 3.1 was initially constructed for polydiscs in [19]. In fact, in exactly the same way there (p. 430 [19]), one can show that (8) solves the $\bar{\partial}$ problem pointwisely on arbitrary product domains when the datum is $C^1_{(p,q)}$ up to the boundary.

b). We suspect that the approach used in the proof of Theorem 3.2 could be applied to general product domains, since the domain under consideration in [19] was exclusively polydiscs which carry rich symmetry.

c). As will be seen in Example 4.3, the estimate of S_j in Theorem 3.2 fails if we replace $\mathcal{C}^{\alpha}(\mathbb{D}^n)$ by the (standard) Hölder space $C^{\alpha}(\mathbb{D}^n)$.

d). Another type of an iterated Hölder space Λ^{α}_2 was defined by Chen and McNeal [4] for a product of two general bounded domains. In the context of a product of two planar domains D_1 and D_2,

$$\Lambda^{\alpha}_2(D_1 \times D_2) := \{f \in C(D_1 \times D_2) : \|f\|_{\Lambda^{\alpha}_2(D_1\times D_2)} = |f|_{C(D_1\times D_2)} + H^{(2)}_{\alpha}[f] < \infty\}.$$

Λ^{α}_2 is different from $\mathcal{C}^{\alpha}$ in that the sum part in (5) for $\mathcal{C}^{\alpha}$ is replaced by the single term $H^{(2)}_{\alpha}[f]$ for Λ^{α}_2. As a matter of fact, $\|\cdot\|_{\mathcal{C}^{\alpha}(\mathbb{D}^2)} \geq \|\cdot\|_{\Lambda^{\alpha}_2(\mathbb{D}^2)}$ and so $\mathcal{C}^{\alpha}(\mathbb{D}^2) \subset \Lambda^{\alpha}_2(\mathbb{D}^2)$.

In [4], it was shown that for any $\bar{\partial}$-closed $(0,1)$ form $\mathbf{f} = f_1 d\bar{z}_1 + f_2 d\bar{z}_2$ with $f_1, f_2, \frac{\partial f_1}{\partial \bar{z}_2} \in \Lambda^{\alpha}_2(D_1 \times D_2)$, there exists a solution $T\mathbf{f} \in \Lambda^{\alpha}_2(D_1 \times D_2)$ to $\bar{\partial} u = \mathbf{f}$ such that

$$\|T\mathbf{f}\|_{\Lambda^{\alpha}_2(D_1\times D_2)} \leq C\left(\|\mathbf{f}\|_{\Lambda^{\alpha}_2(D_1\times D_2)} + \left\|\frac{\partial f_1}{\partial \bar{z}_2}\right\|_{\Lambda^{\alpha}_2(D_1\times D_2)}\right).$$

[4] compared Λ^{α}_2 with (the standard) C^{α} by constructing an example in $\Lambda^{\alpha}_2(\mathbb{D}^2)$ but not in $C^{\alpha}(\mathbb{D}^2)$. They also attempted to find a Lipschitz function in $C^{0,1}(\mathbb{D}^2)$ but not in $\Lambda^{\alpha}_2(\mathbb{D}^2)$

for any $0 < \alpha < 1$. However, this would contradict with (6) because for any $0 < \alpha < 1$, one necessarily has

$$C^{0,1}(\mathbb{D}^2) \subset \mathcal{C}^{\frac{\alpha}{2}}(\mathbb{D}^2) \subset \Lambda_2^{\frac{\alpha}{2}}(\mathbb{D}^2).$$

The mistake is due to the fact that the constant C in part (3) of Example 5.6 [4] actually goes to 0. Thus the limit there would not necessarily go to ∞ as they have claimed.

4 Sharp Hölder bounds of the Cauchy type operators on product domains

Let $D_j \subset \mathbb{C}$, $j = 1, \ldots, n$, be a bounded domain with $C^{k+1,\alpha}$ boundary, $n \geq 2$, $k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \leq 1$, and $\Omega := D_1 \times \cdots \times D_n$. Theorem 2.3-2.4 immediately imply the following lemma.

Lemma 4.1. *There exists a constant C dependent only on Ω, k and α, such that for any $j = 1, \ldots, n$, $f \in C^{k,\alpha}(\Omega)$, $0 < \alpha < 1$, $k \in \mathbb{Z}^+ \cup \{0\}$, $\gamma \in \mathbb{Z}^+ \cup \{0\}$ with $\gamma \leq k$,*

$$\sup_{\substack{z_l \in D_l,\\ 1 \leq l (\neq j) \leq n}} \|D_j^\gamma T_j f(z_1, \ldots, z_{j-1}, \cdot, z_{j+1}, \ldots, z_n)\|_{C^\alpha(D_j)} \leq \begin{cases} C\|f\|_{C(\Omega)}, & \gamma = 0 \\ C\|f\|_{C^{\gamma-1,\alpha}(\Omega)}, & \gamma \geq 1 \end{cases} \leq C\|f\|_{C^{\gamma,\alpha}(\Omega)};$$

$$\sup_{\substack{z_l \in D_l,\\ 1 \leq l (\neq j) \leq n}} \|D_j^\gamma S_j f(z_1, \ldots, z_{j-1}, \cdot, z_{j+1}, \ldots, z_n)\|_{C^\alpha(D_j)} \leq C\|f\|_{C^{\gamma,\alpha}(\Omega)}.$$

Here D_j^γ represents any γ-th derivative operator with respect to the j-th variable.

Although the solid Cauchy integral operator T defined in Section 2 is a smoothing operator in dimension one, T_j in (1) does not improve regularity along slice of higher dimensional domains, as demonstrated by the following example.

Example 4.2. Consider $f(z_1, z_2) = |z_2|^\alpha$ on $\mathbb{D}^2$. Then $f \in C^\alpha(\mathbb{D}^2)$. However a straight forward computation shows that $T_1 f(z_1, z_2) = \bar{z}_1 |z_2|^\alpha \notin C^{\alpha+\epsilon}(\mathbb{D}^2)$ for any $\epsilon > 0$.

On the other hand, in contrast to the boundary Cauchy integral operator S in one dimensional case, its counterpart S_j in (1) no longer maintains Hölder regularity in higher dimensions. Indeed, Tumanov ([21] p.486) constructed the following concrete function $\tilde{f} \in C^\alpha(\partial\mathbb{D} \times \mathbb{D})$ but $S_1 \tilde{f} \notin C^\alpha(\partial\mathbb{D} \times \mathbb{D}), 0 < \alpha < 1$.

Example 4.3. [21] Define for $z_2 \in \mathbb{D}$,

$$\tilde{f}(e^{i\theta}, z_2) = \begin{cases} |z_2|^\alpha, & -\pi \le \theta \le -|z_2|^{\frac{1}{2}}; \\ \theta^{2\alpha}, & -|z_2|^{\frac{1}{2}} \le \theta \le 0; \\ \theta^\alpha, & 0 \le \theta \le |z_2|; \\ |z_2|^\alpha, & |z_2| \le \theta \le \pi. \end{cases}$$

Then $\tilde{f} \in C^\alpha(\partial\mathbb{D} \times \mathbb{D})$. However $S_1\tilde{f} \notin C^\alpha(\partial\mathbb{D} \times \mathbb{D})$. Extend $\tilde{f}$ onto $\mathbb{D}^2$, denoted as f, such that $f \in C^\alpha(\mathbb{D}^2)$. One can check that $S_1 f \notin C^\alpha(\mathbb{D}^2)$. See [20] for more details of the verification.

In view of Example 4.2-4.3, Theorem 1.1 characterizes the optimal Hölder bounds of the two Cauchy type operators T_j and S_j.

Proof of Theorem 1.1. a). We only prove (2) when $j = 1$ and $n = 2$ for simplicity. The other cases are proved accordingly.

We first show $\|T_1 f\|_{C^k(\Omega)} \le C\|f\|_{C^{k,\alpha}(\Omega)}$ for some constant C independent of f. Write $D^\gamma = D_1^{\gamma_1} D_2^{\gamma_2}$, $\gamma_1 + \gamma_2 \le k$. Then $D^\gamma T_1 f = D_1^{\gamma_1} T_1(D_2^{\gamma_2} f)$. Hence by Lemma 4.1,

$$\|D^\gamma T_1 f\|_{C(\Omega)} = \sup_{z_2 \in D_2} \|D_1^{\gamma_1} T_1(D_2^{\gamma_2} f)(\cdot, z_2)\|_{C(D_1)} \le C\|D_2^{\gamma_2} f\|_{C^{\gamma_1}(\Omega)} \le C\|f\|_{C^{k,\alpha}(\Omega)}.$$

Next, we show $H^\alpha[D^\gamma T_1 f] \le C\|f\|_{C^{k,\alpha}(\Omega)}$ for some constant C independent of f for all $|\gamma| = k$. By Lemma 4.1, for each $z_2 \in D_2$, $D^\gamma T_1 f(\zeta, z_2)$ as a function of $\zeta \in D_1$ satisfies

$$H_1^\alpha[D^\gamma T_1 f(\cdot, z_2)] \le \|D_1^{\gamma_1} T_1(D_2^{\gamma_2} f)(\cdot, z_2)\|_{C^\alpha(D_1)} \le C\|D_2^{\gamma_2} f\|_{C^{\gamma_1,\alpha}(\Omega)} \le C\|f\|_{C^{k,\alpha}(\Omega)}$$

for some constant C independent of f and z_2.

On the other hand, let $z_2'(\ne z_2) \in D_2$ and consider $F_{z_2,z_2'}(\zeta) := \frac{D_2^{\gamma_2} f(\zeta,z_2) - D_2^{\gamma_2} f(\zeta,z_2')}{|z_2 - z_2'|^\alpha}$ on D_1. Since $f \in C^{k,\alpha}(\Omega)$, it follows $F_{z_2,z_2'} \in C^{\gamma_1}(D_1)$ and $\|F_{z_2,z_2'}\|_{C^{\gamma_1}(D_1)} \le \|f\|_{C^{k,\alpha}}(\Omega)$. If $\gamma_1 = 0$, by Lemma 4.1,

$$\|D_1^{\gamma_1} T_1 F_{z_2,z_2'}\|_{C(D_1)} = \|T_1 F_{z_2,z_2'}\|_{C(D_1)} \le C\|F_{z_2,z_2'}\|_{C^0(D_1)} \le C\|f\|_{C^{k,\alpha}(\Omega)},$$

where C is independent of f, z_2 and z_2'. For $\gamma_1 \ge 1$, we have by Lemma 4.1,

$$\|D_1^{\gamma_1} T_1 F_{z_2,z_2'}\|_{C(D_1)} \le C\|F_{z_2,z_2'}\|_{C^{\gamma_1-1,\alpha}(D_1)} \le C\|F_{z_2,z_2'}\|_{C^{\gamma_1}(D_1)} \le C\|f\|_{C^{k,\alpha}(\Omega)}$$

for some constant C independent of f, z_2 and z_2'. In sum, for each fixed $z_1 \in D_1$,

$$\frac{|D^\gamma T_1 f(z_1, z_2) - D^\gamma T_1 f(z_1, z_2')|}{|z_2 - z_2'|^\alpha} = |D_1^{\gamma_1} T_1 F_{z_2,z_2'}(z_1)| \le \|D_1^{\gamma_1} T_1 F_{z_2,z_2'}\|_{C(D_1)} \le C\|f\|_{C^{k,\alpha}(\Omega)},$$

where C is independent of f, z_1, z_2 and z_2'. We have thus proved $H_2^{\alpha}[D^{\gamma}T_1 f(z_1,\cdot)] \leq C\|f\|_{C^{k,\alpha}(\Omega)}$ with C independent of f and z_1, and (2) as a consequence of Lemma 2.1.

b). As in part a), we only prove (3) for $j = 1$ and $n = 2$. Let $|\gamma| \leq k$. Since $S_1 f$ is holomorphic with respect to z_1 variable, we can further assume $D^{\gamma} = \partial_1^{\gamma_1} D_2^{\gamma_2}$. Write $\partial D_1 = \cup_{j=1}^N \Gamma_j$, where each Jordan curve Γ_j is connected, positively oriented with respect to D_1, and of total arclength s_j. Let $\zeta_1(s)$ be a parameterization of ∂D_1 in terms of the arclength variable s, such that $\zeta_1|_{s\in[\sum_{m=1}^{j-1} s_m, \sum_{m=1}^{j} s_m)}$ is a $C^{k+1,\alpha}$ parametrization of Γ_j. In particular, $\bar{\zeta}_1' = \frac{1}{\zeta_1'} \neq 0$ on ∂D_1. For any $(z_1, z_2) \in \Omega$, it follows by integration by part,

$$
\begin{aligned}
\partial_1 S_1 f(z_1,z_2) =& \frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\sum_{m=1}^{j-1}s_m}^{\sum_{m=1}^{j}s_m} \partial_{z_1}\Big(\frac{1}{\zeta_1(s)-z_1}\Big) f(\zeta_1(s),z_2)\zeta_1'(s)ds\\
=& -\frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\sum_{m=1}^{j-1}s_m}^{\sum_{m=1}^{j}s_m} \partial_{s}\Big(\frac{1}{\zeta_1(s)-z_1}\Big) f(\zeta_1(s),z_2)ds\\
=& \frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\sum_{m=1}^{j-1}s_m}^{\sum_{m=1}^{j}s_m} \frac{\partial_s(f(\zeta_1(s),z_2))}{\zeta_1(s)-z_1}ds\\
=& \frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\sum_{m=1}^{j-1}s_m}^{\sum_{m=1}^{j}s_m} \frac{\partial_{\zeta_1}(f(\zeta_1(s),z_2))\zeta_1'(s)+\partial_{\bar{\zeta}_1}(f(\zeta_1(s),z_2))\bar{\zeta}_1'(s)}{\zeta_1(s)-z_1}ds\\
=& \frac{1}{2\pi i}\int_{\partial D_1} \frac{\partial_{\zeta_1}f(\zeta_1,z_2)+\partial_{\bar{\zeta}_1}(f(\zeta_1,z_2))(\bar{\zeta}_1'(s))^2}{\zeta_1-z_1}d\zeta_1.
\end{aligned}
$$

Applying the integration by part inductively, one shall see $D^{\gamma}S_1 f = S_1\tilde{f}$, for some function $\tilde{f}$ satisfying $\|\tilde{f}\|_{C^{\alpha}(\Omega)} \leq \|f\|_{C^{k,\alpha}(\Omega)}$. Therefore, we only need to prove $\|S_1\tilde{f}\|_{C^{\alpha'}(\Omega)} \leq C\|\tilde{f}\|_{C^{\alpha}(\Omega)}$ for some constant C independent of $\tilde{f}$.

Firstly, by Lemma 4.1, one has

$$\|S_1\tilde{f}\|_{C(\Omega)} = \sup_{z_2\in D_2}\|S_1\tilde{f}(\cdot,z_2)\|_{C(D_1)} \leq C\|\tilde{f}\|_{C^{\alpha}(\Omega)}$$

for some constant C independent of $\tilde{f}$.

Next, we show $H^{\alpha'}[S_1\tilde{f}] \leq C\|\tilde{f}\|_{C^{\alpha}(\Omega)}$ for some constant C independent of $\tilde{f}$. By Lemma 4.1, for each $z_2 \in D_2$, $S_1\tilde{f}(\zeta,z_2)$ as a function of $\zeta \in D_1$ satisfies

$$H_1^{\alpha'}[S_1\tilde{f}(\cdot,z_2)] \leq \|S_1\tilde{f}(\cdot,z_2)\|_{C^{\alpha'}(D_1)} \leq C\|\tilde{f}\|_{C^{\alpha'}(\Omega)} \leq C\|\tilde{f}\|_{C^{\alpha}(\Omega)}$$

for some constant C independent of $\tilde{f}$ and z_2.

We further show there exists a constant C independent of $\tilde{f}$ and z_1, such that for each $z_1 \in D_1$, $H_2^{\alpha'}[S_1\tilde{f}(z_1,\cdot)] \le C\|\tilde{f}\|_{C^\alpha(\Omega)}$. First consider $z_1 = t_1 \in \partial D_1$. Without loss of generality, assume $t_1 \in \Gamma_1$ with $\zeta_1|_{s=0} = t_1$. Since $\partial D_1 \in C^1$, ∂D_1 satisfies the so-called *chord-arc* condition. In other words, for any $\zeta_1(s), \zeta_1(s') \in \Gamma_j, j = 1, \ldots, N$, there exists a constant $C \ge 1$ dependent only on ∂D_1 such that

$$|\zeta_1(s) - \zeta_1(s')| \le \min\{s - s', s' + s_j - s\} \le C|\zeta_1(s) - \zeta_1(s')|.$$

Here s_j is the total arclength of Γ_j. In particular, when $0 \le s \le s_1$,

$$|d\zeta_1| \le C|ds| \quad \text{and} \quad |\zeta_1(s) - t_1| \ge C\min\{s, s_1 - s\} \tag{9}$$

for some constant C dependent only on D_1. By Sokhotski–Plemelj Formula (see [18] for instance), the non-tangential limit of $S_1\tilde{f}$ at $(t_1, z_2) \in \partial D_1 \times D_2$ is

$$\Phi_1\tilde{f}(t_1, z_2) := \frac{1}{2\pi i}\int_{\partial D_1} \frac{\tilde{f}(\zeta_1, z_2)}{\zeta_1 - t_1} d\zeta_1 + \frac{1}{2}\tilde{f}(t_1, z_2).$$

Here the first term is interpreted as the Principal Value. We shall prove that for $z_2, z_2' \in D_2$ with $h := |z_2 - z_2'| \ne 0$,

$$|\Phi_1\tilde{f}(t_1, z_2) - \Phi_1\tilde{f}(t_1, z_2')| \le Ch^{\alpha'}\|\tilde{f}\|_{C^\alpha(\Omega)}$$

for some constant C independent of $\tilde{f}, t_1, z_2$ and z_2', essentially following the idea of Muskhelishvili [18].

Let h_0 be a positive number such that $h^{\alpha-\alpha'}\ln\frac{1}{h} \le 1$ for $0 < h \le h_0 < \min\{1, \frac{s_1}{2}\}$. Then h_0 depends only on α and α'. When $h \ge h_0$,

$$|\Phi_1\tilde{f}(t_1, z_2) - \Phi_1\tilde{f}(t_1, z_2')| \le 2\|S_1\tilde{f}\|_{C^0(\Omega)} \le C\|\tilde{f}\|_{C^\alpha(\Omega)} \le \frac{C}{h_0^{\alpha'}}h^{\alpha'}\|\tilde{f}\|_{C^\alpha(\Omega)} \le Ch^{\alpha'}\|\tilde{f}\|_{C^\alpha(\Omega)}$$

for some constant C independent of $\tilde{f}, t_1, z_2$ and z_2'.

When $h < h_0$, write

$$\begin{aligned}
\Phi_1\tilde{f}(t_1, z_2) - \Phi_1\tilde{f}(t_1, z_2') =& \frac{1}{2\pi i}\int_{\partial D_1} \frac{\tilde{f}(\zeta_1, z_2) - \tilde{f}(t_1, z_2) - \tilde{f}(\zeta_1, z_2') + \tilde{f}(t_1, z_2')}{\zeta_1 - t_1} d\zeta_1 \\
&+ \frac{\tilde{f}(t_1, z_2) - \tilde{f}(t_1, z_2')}{2\pi i}\int_{\partial D_1} \frac{1}{\zeta_1 - t_1} d\zeta_1 + \frac{\tilde{f}(t_1, z_2) - \tilde{f}(t_1, z_2')}{2} \\
=& \frac{1}{2\pi i}\int_{\partial D_1} \frac{\tilde{f}(\zeta_1, z_2) - \tilde{f}(t_1, z_2) - \tilde{f}(\zeta_1, z_2') + \tilde{f}(t_1, z_2')}{\zeta_1 - t_1} d\zeta_1 + \\
&+ (\tilde{f}(t_1, z_2) - \tilde{f}(t_1, z_2')) \\
=&: I + II.
\end{aligned}$$

Here the second equality has used the fact that $\int_{\partial D_1} \frac{1}{\zeta_1 - t_1} d\zeta_1 = \pi i$ when interpreted as the Principal Value, due to the positive orientation of ∂D_1. Obviously

$$|II| \leq Ch^\alpha \|\tilde{f}\|_{C^\alpha(\Omega)}$$

for some constant C independent of $\tilde{f}, t_1, z_2$ and z_2'.

Let l be the arc on ∂D_1 that are centered at t_1 with arclength $2h$. Consequently, $l \subset \Gamma_1$ due to the fact that $h \leq \frac{s_1}{2}$. Write I as follows.

$$\begin{aligned}
I =& \frac{1}{2\pi i} \int_{\Gamma_1 \setminus l} \frac{\tilde{f}(\zeta_1, z_2) - \tilde{f}(t_1, z_2) - \tilde{f}(\zeta_1, z_2') + \tilde{f}(t_1, z_2')}{\zeta_1 - t_1} d\zeta_1 \\
&+ \frac{1}{2\pi i} \int_l \frac{(\tilde{f}(\zeta_1, z_2) - \tilde{f}(t_1, z_2)) - (\tilde{f}(\zeta_1, z_2') - \tilde{f}(t_1, z_2'))}{\zeta_1 - t_1} d\zeta_1 \\
&+ \frac{1}{2\pi i} \int_{\cup_{j=2}^N \Gamma_j} \frac{(\tilde{f}(\zeta_1, z_2) - \tilde{f}(\zeta_1, z_2')) - (\tilde{f}(t_1, z_2) - \tilde{f}(t_1, z_2'))}{\zeta_1 - t_1} d\zeta_1 \\
=& : I_1 + I_2 + I_3.
\end{aligned}$$

For I_3, since $\cup_{j=2}^N \Gamma_j$ does not intersect with Γ_1 and $t_1 \in \Gamma_1$, $|\zeta_1 - t_1| \geq C$ on $\cup_{j=2}^N \Gamma_j$ for some positive C dependent only on ∂D_1. On the other hand, the absolute value of the numerator in I_3 is less than $Ch^\alpha \|\tilde{f}\|_{C^\alpha(\Omega)}$. It immediately follows that

$$|I_3| \leq Ch^\alpha \|\tilde{f}\|_{C^\alpha(\Omega)}.$$

For I_2, the absolute value of the numerator of the integrand is less than $C|\zeta_1 - t_1|^\alpha \|\tilde{f}\|_{C^\alpha(\Omega)}$. We infer from (9) that

$$|I_2| \leq C\|\tilde{f}\|_{C^\alpha(\Omega)} \int_l \frac{1}{|\zeta_1 - t_1|^{1-\alpha}} |d\zeta_1| \leq C\|\tilde{f}\|_{C^\alpha(\Omega)} \int_0^h \frac{1}{s^{1-\alpha}} ds \leq Ch^\alpha \|\tilde{f}\|_{C^\alpha(\Omega)}$$

for some constant C independent of $\tilde{f}, t_1, z_2$ and z_2'. Now we treat with the remaining term I_1. Rearrange I_1 so it becomes

$$|I_1| \leq |\frac{1}{2\pi i} \int_{\gamma_1 \setminus l} \frac{\tilde{f}(\zeta_1, z_2) - \tilde{f}(\zeta_1, z_2')}{\zeta_1 - t_1} d\zeta_1| + |\frac{\tilde{f}(t_1, z_2) - \tilde{f}(t_1, z_2')}{2\pi i} \int_{\gamma_1 \setminus l} \frac{1}{\zeta_1 - t_1} d\zeta_1|.$$

The second term of the above inequality is bounded by $Ch^\alpha \|\tilde{f}\|_{C^\alpha(\Omega)}$ for some constant C independent of $\tilde{f}, t_1, z_2$ and z_2', as in the argument for II. The first term when $h < h_0$ is bounded by

$$Ch^\alpha \|\tilde{f}\|_{C^\alpha(\Omega)} \int_h^{\frac{s_1}{2}} \frac{1}{s} ds \leq Ch^\alpha \ln \frac{1}{h} \|\tilde{f}\|_{C^\alpha(\Omega)} \leq Ch^{\alpha'} \|\tilde{f}\|_{C^\alpha(\Omega)}.$$

We have thus shown there exists a constant C independent of t_1 and $\tilde{f}$, such that for each $z_1 = t_1 \in \partial D_1$, $H_2^{\alpha'}[\Phi_1\tilde{f}(t_1,\cdot)] \le C\|\tilde{f}\|_{C^\alpha(\Omega)}$. Notice that for each fixed $\zeta \in D_2$, $S_1\tilde{f}(z_1,\zeta)$ is holomorphic as a function of $z_1 \in D_1$ and C^α continuous up to the boundary with boundary value equal to $\Phi_1\tilde{f}(z_1,\zeta)$ by Plemelj–Privalov Theorem. For each fixed z_2 and z_2' with $|z_2 - z_2'| \neq 0$, applying Maximum Modulus Theorem to the holomorphic function $\frac{S_1\tilde{f}(z_1,z_2)-S_1\tilde{f}(z_1,z_2')}{|z_2-z_2'|^{\alpha'}}$ of z_1 in D_1, we immediately obtain

$$\begin{aligned}\sup_{z_1\in D_1}|\frac{S_1\tilde{f}(z_1,z_2)-S_1\tilde{f}(z_1,z_2')}{|z_2-z_2'|^{\alpha'}}| \le& \sup_{t_1\in\partial D_1}|\frac{\Phi_1\tilde{f}(t_1,z_2)-\Phi_1\tilde{f}(t_1,z_2')}{|z_2-z_2'|^{\alpha'}}|\\ =& \sup_{t_1\in\partial D_1} H_2^{\alpha'}[\Phi_1\tilde{f}(t_1,\cdot)]\\ \le& C\|\tilde{f}\|_{C^\alpha(\Omega)},\end{aligned}$$

with C independent of f, z_1, z_2 and z_2'. Therefore

$$H_2^{\alpha'}[S_1\tilde{f}(z_1,\cdot)] \le C\|\tilde{f}\|_{C^\alpha(\Omega)}$$

with C independent of $\tilde{f}$ and z_1. The proof of (3) is complete.

■

5 Proof of Theorem 1.2 and Corollary 1.3

As an immediate consequence of Theorem 1.1, we obtain the following theorem.

Theorem 5.1. *Let $D_j \subset \mathbb{C}$, $j = 1,\dots,n$, be bounded domains with $C^{k+1,\alpha}$ boundary, $n \ge 2, k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha \le 1$, and let $\Omega := D_1 \times \cdots \times D_n$. Let $\mathbf{f} \in C^{k,\alpha}_{(p,q)}(\Omega)$, $0 \le p \le n, 1 \le q \le n$. Then for any $0 < \alpha' < \alpha$, $T\mathbf{f}$ defined in (8) belongs to $C^{k,\alpha'}_{(p,q-1)}(\Omega)$ with*

$$\|T\mathbf{f}\|_{C^{k,\alpha'}(\Omega)} \le C\|\mathbf{f}\|_{C^{k,\alpha}(\Omega)}.$$

Proof. The operator T defined by (8) is well defined on $C^{k,\alpha}_{(p,q)}(\Omega)$ due to Theorem 1.1. Choose some positive constant $\epsilon < \frac{\alpha-\alpha'}{n-1}$. Then $\alpha' + (n-1)\epsilon < \alpha \le 1$. Applying Theorem 1.1 repeatedly, it follows for each $j \le n$,

$$\|T_jS_1\cdots S_{j-1}\pi_j\mathbf{f}\|_{C^{k,\alpha'}(\Omega)} \le C\|S_1\cdots S_{j-1}\pi_j\mathbf{f}\|_{C^{k,\alpha'}(\Omega)} \le C\|\pi_j\mathbf{f}\|_{C^{k,\alpha}(\Omega)} \le C\|\mathbf{f}\|_{C^{k,\alpha}(\Omega)}.$$

Therefore, $\|T\mathbf{f}\|_{C^{k,\alpha'}(\Omega)} \le C\|\mathbf{f}\|_{C^{k,\alpha}(\Omega)}$.

■

Remark 5.2. It is worth pointing out that at the top degree $q = n$, under the same assumptions as in Theorem 5.1, $T\mathbf{f}$ will maintain the same regularity as its data to be in $C^{k,\alpha}_{(p,n-1)}(\Omega)$ with

$$\|T\mathbf{f}\|_{C^{k,\alpha}(\Omega)} \le C\|\mathbf{f}\|_{C^{k,\alpha}(\Omega)}.$$

This is because when $q = n$, $\mathbf{f} = \sum_{1\le i_1<\cdots<i_p\le n} f_{i_1\cdots i_p} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$ for some $f_{i_1\cdots i_p} \in C^{k,\alpha}(\Omega)$. Thus $T\mathbf{f} = (-1)^p \sum_{1\le i_1<\cdots<i_p\le n} T_1 f_{i_1\cdots i_p} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n \in C^{k,\alpha}_{(p,n-1)}(\Omega)$ by definition (8). The desired estimate follows from that of T_1 in Theorem 1.1.

As stated in Remark 3.3 a), it was proved in [19] that if $\mathbf{f} \in C^1_{(p,q)}(\bar{\Omega})$ is $\bar{\partial}$ closed, then (8) is a solution to $\bar{\partial}\mathbf{u} = \mathbf{f}$ on Ω. We thus have

Proof of Corollary 1.3. Observe that $C^{\infty}_{(p,q)}(\bar{\Omega}) \subset C^{k,\alpha}_{(p,q)}(\Omega)$ for any integer $k \in \mathbb{Z}^+ \cup \{0\}$ and $0 < \alpha \le 1$. Theorem 1.3 follows directly from the proof of Theorem 5.1 and Remark 3.3 a). ∎

Remark 5.3. When $\mathbf{f} \in C^{n-1,\alpha}_{(p,1)}(\Omega)$, T defined by (8) coincides with the solution operator constructed in [3][9] by repeated application of Theorem 2.2. Therefore the same supnorm estimate in [9] passes onto T if the data is smooth up to the boundary. It would be interesting to know whether the supnorm estimate holds for (p, q) forms smooth up to the boundary.

Assuming $\mathbf{f} \in C^{\alpha}(\Omega), 0 < \alpha < 1$, the $\bar{\partial}$ equation is interpreted in the sense of distributions. The following proposition shows that $T\mathbf{f}$ defined by (8) solves $\bar{\partial}\mathbf{u} = \mathbf{f}$ in this sense.

Proposition 5.4. *Let $D_j \subset \mathbb{C}, j = 1, \ldots, n$, be bounded domains with $C^{1,\alpha}$ boundary, $n \ge 2$, $0 < \alpha \le 1$ and $\Omega := D_1 \times \cdots \times D_n$. Assume $\mathbf{f} \in C^{\alpha}_{(p,q)}(\Omega)$ is $\bar{\partial}$-closed in Ω in the sense of distributions, $0 \le p \le n, 1 \le q \le n$. Then $\mathbf{u} := T\mathbf{f}$ defined in (8) solves $\bar{\partial}\mathbf{u} = \mathbf{f}$ in Ω in the sense of distributions.*

Proof. Given $\mathbf{f} \in C^{\alpha}_{(p,q)}(\Omega)$ for $0 < \alpha \le 1$, $T\mathbf{f} \in C^{\alpha'}_{(p,q-1)}(\Omega)$ with $0 < \alpha' < \alpha$ by Theorem 5.1 with $k = 0$. We use the standard mollifier argument to show that $T\mathbf{f}$ solves $\bar{\partial}\mathbf{u} = \mathbf{f}$ in Ω in the sense of distributions.

For each $j \in \{1, \ldots, n\}$, let $\{D^{(l)}_j\}_{l=1}^{\infty}$ be a family of strictly increasing open subsets of D_j such that

a). for $l \ge N_0 \in \mathbb{N}$, $bD^{(l)}_j$ is $C^{2,\alpha}$, $\frac{1}{l+1} < dist(D^{(l)}_j, D^c_j) < \frac{1}{l}$;

b). $H^{(l)}_j : \bar{D}_j \to \bar{D}^{(j)}_j$ is a C^1 diffeomorphism with $\lim_{l\to\infty} \|H^{(l)}_j - Id\|_{C^1(D_j)} = 0$.

Let $\Omega^{(l)} = D_1^{(l)} \times \cdots \times D_n^{(l)}$ be the product of those planar domains. Denote by $T_j^{(l)}, S_j^{(l)}$ and $T^{(l)}$ the operators defined in (1) and (8) accordingly, with Ω replaced by $\Omega^{(l)}$. Then $T^{(l)}\mathbf{f} \in C^{\alpha'}_{(p,q-1)}(\Omega^{(l)})$ for each $0 < \alpha' < \alpha$. Adopting the mollifier argument to $\mathbf{f} \in C^{\alpha}_{(p,q)}(\Omega)$, we obtain $\mathbf{f}^\epsilon \in C^{1,\alpha}_{(p,q)}(\Omega^{(l)})$ such that for each fixed $0 < \alpha' < \alpha$, $\|\mathbf{f}^\epsilon - \mathbf{f}\|_{C^{\alpha'}(\Omega^{(l)})} \to 0$ (see the Appendix) as $\epsilon \to 0$ and $\bar\partial \mathbf{f}^\epsilon = 0$ on $\Omega^{(l)}$.

Fix an $\alpha'(< \alpha)$. For each l, $T^{(l)}\mathbf{f}^\epsilon \in C^{1,\alpha'}_{(p,q-1)}(\Omega^{(l)})$ when ϵ is small and $\bar\partial T^{(l)}\mathbf{f}^\epsilon = \mathbf{f}^\epsilon$ in $\Omega^{(l)}$ by Theorem 5.1. Furthermore, applying Theorem 1.1 at $k = 0$, we have $\|T^{(l)}\mathbf{f}^\epsilon - T^{(l)}\mathbf{f}\|_{C^0(\Omega^{(l)})} \le C\|\mathbf{f}^\epsilon - \mathbf{f}\|_{C^{\alpha'}(\Omega^{(l)})} \to 0$ as $\epsilon \to 0$. We thus have $\lim_{\epsilon\to 0} T^{(l)}\mathbf{f}^\epsilon$ exists in $\Omega^{(l)}$ and is equal to $T^{(l)}\mathbf{f} \in C^{\alpha'}_{(p,q-1)}(\Omega^{(l)})$ pointwisely.

Given a testing $(p, q-1)$ form ϕ with a compact support K, let $l_0 \ge N_0$ be such that $K \subset \Omega^{(l_0-2)}$. Denote by $(\cdot,\cdot)_\Omega$ (and $(\cdot,\cdot)_{\Omega^{(l_0)}}$) the inner product(s) in $L^2_{(p,q-1)}(\Omega)$ (and in $L^2_{(p,q-1)}(\Omega^{(l_0)})$, respectively), and $\bar\partial^*$ the formal adjoint of $\bar\partial$. For $l \ge l_0$, one has

$$(T^{(l)}\mathbf{f}, \bar\partial^*\phi)_{\Omega^{(l_0)}} = \lim_{\epsilon\to 0}(T^{(l)}\mathbf{f}^\epsilon, \bar\partial^*\phi)_{\Omega^{(l_0)}} = \lim_{\epsilon\to 0}(\bar\partial T^{(l)}\mathbf{f}^\epsilon, \phi)_{\Omega^{(l_0)}} = \lim_{\epsilon\to 0}(\mathbf{f}^\epsilon, \phi)_{\Omega^{(l_0)}} = (\mathbf{f}, \phi)_\Omega. \quad (10)$$

We further claim that

$$(T\mathbf{f}, \bar\partial^*\phi)_\Omega = \lim_{l\to\infty}(T^{(l)}\mathbf{f}, \bar\partial^*\phi)_{\Omega^{(l_0)}}. \quad (11)$$

To prove this, for simplicity of notations yet without loss of generality, assume $\pi_j\mathbf{f}$ contains only one component function f_j, so is for ϕ. We will also drop various integral measure, which is clear from context. For each $j \ge 1$,

$$\begin{aligned}
&-(-2i)^n(2\pi i)^j(T_j^{(l)}S_1^{(l)}\cdots S_{j-1}^{(l)}\pi_j\mathbf{f}, \bar\partial^*\phi)_{\Omega^{(l_0)}}\\
&= \int_{z\in K}\int_{\zeta_j\in D_j^{(l)}}\int_{\zeta_1\in\partial D_1^{(l)}}\cdots\int_{\zeta_{j-1}\in\partial D_{j-1}^{(l)}} \frac{f_j(\zeta_1,\cdots,\zeta_j,z_{j+1},\cdots,z_n)\overline{\bar\partial^*\phi(z)}}{(\zeta_1-z_1)\cdots(\zeta_j-z_j)}\\
&= \int_{(z,\zeta_j)\in K\times D_j}\left(\int_{\zeta_1\in\partial D_1^{(l)}}\cdots\int_{\zeta_{j-1}\in\partial D_{j-1}^{(l)}} \frac{f_j(\zeta_1,\cdots,\zeta_j,z_{j+1},\cdots,z_n)\chi_{D_j^{(l)}}(\zeta_j)\overline{\bar\partial^*\phi(z)}}{(\zeta_1-z_1)\cdots(\zeta_j-z_j)}\right).
\end{aligned}$$

Here $\chi_{D_j^{(l)}}$ is the step function on $\mathbb{C}$ such that $\chi_{D_j^{(l)}} = 1$ in $D_j^{(l)}$ and 0 otherwise.

Firstly, as a function of $(z, \xi_j) \in K \times D_j$,

$$\int_{\zeta_1\in\partial D_1^{(l)}}\cdots\int_{\zeta_{j-1}\in\partial D_{j-1}^{(l)}} \frac{f_j(\zeta_1,\cdots,\zeta_j,z_{j+1},\cdots,z_n)\chi_{D_j^{(l)}}(\zeta_j)\overline{\bar\partial^*\phi(z)}}{(\zeta_1-z_1)\cdots(\zeta_j-z_j)} \in L^1(K\times D_j).$$

To see this, notice that if $z \in K(\subset \Omega^{(l_0-2)})$ and $\zeta_k \in \partial D_k^{(l)}, l \geq l_0, k = 1, \ldots, j-1$, then

$$|\zeta_k - z_k| \geq dist((\Omega^{(l)})^c, \Omega^{(l_0-2)}) \geq dist((\Omega^{(l_0)})^c, \Omega^{(l_0-2)}) > \frac{1}{l_0^2} := \delta_0.$$

Hence for each $(z, \zeta_j) \in K \times D_j \setminus \{z_j = \zeta_j\}$,

$$\left| \int_{\zeta_1 \in \partial D_1^{(l)}} \cdots \int_{\zeta_{j-1} \in \partial D_{j-1}^{(l)}} \frac{f_j(\zeta_1, \cdots, \zeta_j, z_{j+1}, \cdots, z_n)\chi_{D_j^{(l)}}(\zeta_j)\overline{\bar{\partial}^*\phi(z)}}{(\zeta_1 - z_1)\cdots(\zeta_j - z_j)} \right| \leq \frac{C}{\delta_0^{j-1}|\zeta_j - z_j|}$$

for some constant $C > 0$, which is integrable in $K \times D_j$.

On the other hand, by continuity of f_j and the construction of $\Omega^{(l)}$,

$$\begin{aligned}
&\lim_{l\to\infty} \int_{\zeta_1 \in \partial D_1^{(l)}} \cdots \int_{\zeta_{j-1} \in \partial D_{j-1}^{(l)}} \frac{f_j(\zeta_1, \cdots, \zeta_j, z_{j+1}, \cdots, z_n)\chi_{D_j^{(l)}}(\zeta_j)\overline{\bar{\partial}^*\phi(z)}}{(\zeta_1 - z_1)\cdots(\zeta_j - z_j)} \\
=& \int_{\zeta_1 \in \partial D_1} \cdots \int_{\zeta_{j-1} \in \partial D_{j-1}} \frac{f_j(\zeta_1, \cdots, \zeta_j, z_{j+1}, \cdots, z_n)\overline{\bar{\partial}^*\phi(z)}}{(\zeta_1 - z_1)\cdots(\zeta_j - z_j)}
\end{aligned}$$

pointwisely in $K \times D_j$. Applying Dominated Convergence Theorem, we obtain

$$\begin{aligned}
&\lim_{l\to\infty} -(-2i)^n(2\pi i)^j(T_j^{(l)}S_1^{(l)}\cdots S_{j-1}^{(l)}\pi_j\mathbf{f}, \bar{\partial}^*\phi)_{\Omega^{(l_0)}} \\
=& \int_{(z,\zeta_j)\in K\times D_j} \int_{\zeta_1 \in \partial D_1} \cdots \int_{\zeta_{j-1} \in \partial D_{j-1}} \frac{f_j(\zeta_1, \cdots, \zeta_j, z_{j+1}, \cdots, z_n)\overline{\bar{\partial}^*\phi(z)}}{(\zeta_1 - z_1)\cdots(\zeta_j - z_j)} \\
=& -(-2i)^n(2\pi i)^j(T_jS_1\cdots S_{j-1}\pi_j\mathbf{f}, \bar{\partial}^*\phi)_\Omega.
\end{aligned}$$

(11) is thus proved for T in view of its definition (8).

Finally, combining (10) with (11), we deduce that

$$(\bar{\partial}T\mathbf{f}, \phi)_\Omega = (T\mathbf{f}, \bar{\partial}^*\phi)_\Omega = \lim_{l\to\infty}(T^{(l)}\mathbf{f}, \bar{\partial}^*\phi)_{\Omega^{(l_0)}} = (\mathbf{f}, \phi)_\Omega.$$

The proof of Proposition 5.4 is complete.

∎

Proof of Theorem 1.2. Theorem 1.2 follows directly from Theorem 5.1, Remark 3.3 a) and Proposition 5.4.

∎

Finally, making use of the idea of Kerzman [16], we argue by the following examples the regularity of the $\bar\partial$ solution can not be improved in Hölder spaces over product domains.

Example 5.5. a). For each $k \in \mathbb{Z}^+ \cup \{0\}$ and $0 < \alpha < 1$, consider $\bar\partial u = \mathbf{f} := \bar\partial((z_1-1)^{k+\alpha}\bar z_2)$ on $\mathbb{D}^2$, $\frac{1}{2}\pi < \arg(z_1 - 1) < \frac{3}{2}\pi$. Then $\mathbf{f} = (z_1-1)^{k+\alpha} d\bar z_2 \in C^{k,\alpha}(\mathbb{D}^2)$ is a $\bar\partial$-closed $(0,1)$ form. However, there does not exist a solution $u \in C^{k,\alpha'}(\mathbb{D}^2)$ to $\bar\partial u = \mathbf{f}$ on $\mathbb{D}^2$ for any $\alpha' > \alpha$.

b). For each $k \in \mathbb{Z}^+ \cup \{0\}$, consider $\bar\partial u = \mathbf{f} := \bar\partial(\frac{(z_1-1)^{k+1}}{\log(z_1-1)}\bar z_2)$ on $\mathbb{D}^2$, $\frac{1}{2}\pi < \arg(z_1-1) < \frac{3}{2}\pi$. Then $\mathbf{f} = \frac{(z_1-1)^{k+1}}{\log(z_1-1)} d\bar z_2 \in C^{k,1}(\mathbb{D}^2)$ is a $\bar\partial$-closed $(0,1)$ form. However, there does not exist a solution $u \in C^{k+1,\alpha}(\mathbb{D}^2)$ to $\bar\partial u = \mathbf{f}$ on $\mathbb{D}^2$ for any $\alpha > 0$.

Proof. a). $\mathbf{f}$ is well defined in $\mathbb{D}^2$ and $\mathbf{f} = (z_1-1)^{k+\alpha} d\bar z_2 \in C^{k,\alpha}(\mathbb{D}^2)$. Assume by contradiction that there exists a solution $u \in C^{k,\alpha'}(\mathbb{D}^2)$ to $\bar\partial u = \mathbf{f}$ in $\mathbb{D}^2$ for some α' with $\alpha < \alpha' < 1$. Then $u = h + (z_1-1)^{k+\alpha}\bar z_2$ for some holomorphic function h in $\mathbb{D}^2$.

Consider $w(\xi) := \int_{|z_2|=\frac{1}{2}} u(\xi, z_2) dz_2$ for $\xi \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Since $u \in C^{k,\alpha'}(\mathbb{D}^2)$, we have $w \in C^{k,\alpha'}(\mathbb{D})$ as well. On the other hand, by Cauchy's Theorem,

$$w(\xi) = \int_{|z_2|=\frac{1}{2}} (\xi-1)^{k+\alpha}\bar z_2 dz_2 = (\xi-1)^{k+\alpha}\int_{|z_2|=\frac{1}{2}} \frac{1}{4z_2} dz_2 = \frac{\pi i}{2}(\xi-1)^{k+\alpha}.$$

This is a contradiction since $(\xi-1)^{k+\alpha} \notin C^{k,\alpha'}(\mathbb{D})$ for any $\alpha' > \alpha$.

b). Argue in a similar way as in a) by noticing that $\mathbf{f} = \frac{(z_1-1)^{k+1}}{\log(z_1-1)} d\bar z_2 \in C^{k,1}(\mathbb{D}^2)$. If $u \in C^{k+1,\alpha}(\mathbb{D}^2)$ solves $\bar\partial u = \mathbf{f}$ in $\mathbb{D}^2$ for some $\alpha > 0$, then $u = h + \frac{(z_1-1)^{k+1}}{\log(z_1-1)}\bar z_2$ for some holomorphic function h in $\mathbb{D}^2$ and $w(\xi) := \int_{|z_2|=\frac{1}{2}} u(\xi, z_2) dz_2 \in C^{k+1,\alpha}(\mathbb{D}^2)$. However by Cauchy's Theorem,

$$w(\xi) = \int_{|z_2|=\frac{1}{2}} \frac{(\xi-1)^{k+1}}{\log(\xi-1)}\bar z_2 dz_2 = \frac{\pi i}{2}\frac{(\xi-1)^{k+1}}{\log(\xi-1)} \notin C^{k+1,\alpha}(\mathbb{D})$$

for any $\alpha > 0$.

∎

A Appendix

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\Omega_j := \{x \in \Omega : dist(x, \partial\Omega) > \frac{1}{j}\}$ when j is large, and ρ be a smooth function in $\mathbb{R}^n$ by

$$\rho(x) := \begin{cases} C\exp(\frac{1}{|x|^2-1}), & |x| < 1; \\ 0, & |x| \ge 1, \end{cases}$$

where C is selected such that $\int_{\mathbb{R}^n} \rho(y)dy = 1$. ρ is called the standard mollifier. Let $f \in L^1_{loc}(\Omega)$ and define for $x \in \Omega_j$,

$$f_j(x) := \int_{|y|\le 1} \rho(y) f(x - \frac{y}{j}) dy. \tag{12}$$

Then $f_j \in C^\infty(\Omega_j)$. The mollifier argument is a standard method dealing with weak derivatives in Sobolev spaces (See, for instance, [7] p. 717). The following theorem ought to be well-known for Hölder spaces, however we could not locate a reference. For convenience of the reader, we include the proof below.

Theorem A.1. *Let $\tilde{\Omega} \subset\subset \Omega$ and $0 < \alpha' < \alpha$. If $f \in C^\alpha(\Omega)$, then $f_j \to f$ in $C^{\alpha'}(\tilde{\Omega})$. I.e., $\|f_j - f\|_{C^{\alpha'}(\tilde{\Omega})} \to 0$ as $j \to \infty$.*

Proof. Let j_0 be such that $\tilde{\Omega} \subset \Omega_{j_0}$ and assume $j \ge j_0$. $\|f_j - f\|_{C(\tilde{\Omega})} \to 0$ due to the uniform continuity of f on Ω ([7] p.718). Write $\phi_j(x) := f_j(x) - f(x) = \int_{|y|\le 1} \rho(y)(f(x - \frac{y}{j}) - f(x))dy$. We next show for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that when $j \ge N$,

$$\frac{|\phi_j(x) - \phi_j(x')|}{|x - x'|^{\alpha'}} \le \epsilon,$$

for all $x, x' \in \tilde{\Omega}$. Indeed, choose $\delta_0 > 0$ satisfing $\|f\|_{C^\alpha(\Omega)} \delta_0^{\alpha - \alpha'} \le \frac{\epsilon}{2}$.

When $|x - x'| \le \delta_0$,

$$\begin{aligned}\frac{|\phi_j(x) - \phi_j(x')|}{|x - x'|^{\alpha'}} &\le \int_{|y|\le 1} \rho(y) \frac{|f(x - \frac{y}{j}) - f(x' - \frac{y}{j})|}{|x - x'|^{\alpha'}} dy + \int_{|y|\le 1} \rho(y) \frac{|f(x) - f(x')|}{|x - x'|^{\alpha'}} dy \\ &\le 2\|f\|_{C^\alpha(\Omega)} |x - x'|^{\alpha - \alpha'} \le \epsilon.\end{aligned}$$

When $|x - x'| > \delta_0$, choose $N \in \mathbb{N}$ such that $\|f\|_{C^\alpha(\Omega)} \delta_0^{-\alpha'} N^{-\alpha} \le \frac{\epsilon}{2}$. Then for any $j \ge N$, $|x - x'| > \delta_0$, we have

$$\begin{aligned}\frac{|\phi_j(x) - \phi_j(x')|}{|x - x'|^{\alpha'}} &\le \int_{|y|\le 1} \rho(y) \frac{|f(x - \frac{y}{j}) - f(x)|}{|x - x'|^{\alpha'}} dy + \int_{|y|\le 1} \rho(y) \frac{|f(x' - \frac{y}{j}) - f(x')|}{|x - x'|^{\alpha'}} dy \\ &\le 2\|f\|_{C^\alpha(\Omega)} |x - x'|^{-\alpha'} j^{-\alpha} \le \epsilon.\end{aligned}$$

∎

Given $f \in C^\alpha(\Omega)$, although only the $C^{\alpha'}$ convergence of the family $\{f_j\}$ defined by (12) for some $\alpha' > 0$ is needed in Proposition 5.4, we note that the C^α convergence of $\{f_j\}$ can not be achieved in general. The following simple counter-example was provided by Liding Yao.

Example A.2. Let $\Omega = (-1, 1) \in \mathbb{R}$ and

$$f(x) = \begin{cases} 0, & x \leq 0; \\ x^\alpha, & x > 0. \end{cases}$$

Then $f \in C^\alpha(\Omega)$. However, for any $\tilde{\Omega} \subset\subset \Omega$ containing the origin, $\|f_j - f\|_{C^\alpha(\tilde{\Omega})} \geq \int_0^1 \rho(y) y^\alpha dy > 0$ for sufficiently large j.

Proof. Let j_0 be such that $\tilde{\Omega} \subset \Omega_{j_0}$ and assume $j \geq j_0$. Write $\phi_j(x) := f_j(x) - f(x) = \int_{-1}^1 \rho(y)(f(x - \frac{y}{j}) - f(x))dy$. For each fixed j, it can be verified that

$$\phi_j(-\frac{1}{j}) = \int_{-1}^1 \rho(y) f(-\frac{1+y}{j}) dy = 0$$

and

$$\phi_j(0) = \int_{-1}^1 \rho(y) f(-\frac{y}{j}) dy = (\frac{1}{j})^\alpha \int_0^1 \rho(y) y^\alpha dy.$$

However for all j,

$$\|\phi_j\|_{C^\alpha(\tilde{\Omega})} \geq \frac{\phi_j(0) - \phi_j(-\frac{1}{j})}{(\frac{1}{j})^\alpha} = \int_0^1 \rho(y) y^\alpha dy > 0.$$

∎

References

[1] Bertrams, J.: *Randregularität von Lösungen der $\bar{\partial}$-Gleichung auf dem Polyzylinder und zweidimensionalen analytischen Polyedern.* Bonner Math. Schriften, 176(1986), 1–164.

[2] Chakrabarti, D.; Shaw, M.:*The Cauchy-Riemann equations on product domains.* Math. Ann. **349** (2011), no. 4, 977–998.

[3] Chen, l.; McNeal, J.: *A solution operator for $\bar{\partial}$ on the Hartogs triangle and L^p estimates.* Math. Ann. **376** (2020), no. 1-2, 407–430.

[4] Chen, l.; McNeal, J.: *Product domains, multi-Cauchy transforms, and the $\bar{\partial}$ equation.* Adv. Math. **360** (2020), 106930, 42 pp.

[5] Diederich, K.; Fischer, B.; Fornæss, J. E.: *Hölder estimates on convex domains of finite type.* Math. Z. **232** (1999), no. 1, 43–61.

[6] Dong, X.; Li, S.; Treuer, J.: *Sharp pointwise and uniform estimates for $\bar\partial$.* Preprint.

[7] Evans, L.: *Partial differential equations. Second edition.* Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010. xxii+749 pp.

[8] Ehsani, D.: *The $\bar\partial$-Neumann problem on product domains in $\mathbb{C}^n$.* Math. Ann. **337** (2007), 797–816.

[9] Fassina, M.; Pan, Y.: *Supnorm estimates for $\bar\partial$ on product domains in $\mathbb{C}^n$.* Preprint. https://arxiv.org/pdf/1903.10475.pdf.

[10] Grauert, H.; Lieb, I.: *Das Ramirezsche Integral und die Lösung der Gleichung $\bar\partial f = \alpha$ im Bereich der beschränkten Formen.* (German) Rice Univ. Studies **56** (1970), no. 2, (1971), 29–50.

[11] Gong, X.: *Hölder estimates for homotopy operators on strictly pseudoconvex domains with C^2 boundary.* Math. Ann. **374** (2019), no. 1-2, 841–880.

[12] Gong, X.; Lanzani, L.: *Regularity of a $\bar\partial$-solution operator for strongly $\mathbb{C}$-linearly convex domains with minimal smoothness.* To appear in J. Geom. Anal.

[13] Henkin, G. M.: *Integral representation of functions in strictly pseudoconvex domains and applications to the $\bar\partial$-problem.* Mat. Sbornik. **124** (1970), no. 2, 300–308.

[14] Henkin, G. M.: *A uniform estimate for the solution of the $\bar\partial$-problem in a Weil region.* (Russian) Uspehi Mat. Nauk **26** (1971), no. 3(159), 211–212.

[15] Henkin, G. M.; Romanov, A. V.: *Exact Hölder estimates of the solutions of the $\bar\delta$-equation.* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 1171–1183.

[16] Kerzman, N.: *Hölder and L^p estimates for solutions of $\bar\partial u = f$ in strongly pseudoconvex domains.* Comm. Pure Appl. Math. **24**(1971) 301–379.

[17] Lieb, I.; Range, R. M.: *Lösungsoperatoren für den Cauchy-Riemann-Komplex mit C^k-Abschätzungen.* (German) Math. Ann. **253** (1980), no. 2, 145–164.

[18] Muskhelishvili, N. I.: *Singular integral equations. Boundary problems of function theory and their application to mathematical physics.* Dover Publications, Inc., New York, 1992. 447 pp.

[19] Nijenhuis, A.; Woolf, W.: *Some integration problems in almost-complex and complex manifolds.* Ann. of Math. (2) **77** (1963), 424–489.

[20] Pan, Y.; Zhang, Y.: *Cauchy singular integral operator with parameters in Log-Hölder spaces.* To appear in Journal d'Analyse Mathématique.

[21] Tumanov, A.: *On the propagation of extendibility of CR functions.* Complex analysis and geometry (Trento, 1993), 479–498, Lecture Notes in Pure and Appl. Math., 173, Dekker, New York, 1996.

[22] Siu, S. T.: *The $\bar{\partial}$ problem with uniform bounds on derivatives.* Math. Ann. **207** (1974), 163–176.

[23] Vekua, I. N.: *Generalized analytic functions*, vol. 29, Pergamon Press Oxford, 1962.

Yifei Pan, pan@pfw.edu, Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, IN 46805-1499, USA

Yuan Zhang, zhangyu@pfw.edu, Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, IN 46805-1499, USA