FORMAL COMPLEX CURVES IN REAL SMOOTH HYPERSURFACES

JOHN ERIK FORNÆSS, LINA LEE, AND YUAN ZHANG

Abstract. In this paper, we investigate germs of smooth real hypersurfaces in $\mathbb{C}^n$. We show that if the hypersurface is of infinite D'Angelo type at a point, then there exists a formal complex curve in the hypersurface through that point.

1. Introduction

Let M be a germ at $p \in \mathbb{C}^n$ of a smooth real hypersurface in $\mathbb{C}^n$ and r be a smooth local defining function for M in some open neighborhood $U \subset \mathbb{C}^n$ of p. Hence $M = \{z \in U : r(z) = 0\}$ with $dr(p) \neq 0$. An interesting question is whether there exists a germ of a complex curve in M passing through p. This question was first studied by D'Angelo in [4] and was later on shown to be closely related to the regularity of $\overline{\partial}$-Neumann problems over pseudoconvex domains. See for instance a series of papers of Catlin [1] [2] [3] for references. D'Angelo's approach to this geometric question is by measuring the maximum order of contact to M of all complex curves passing through p, called the D'Angelo type.

Precisely speaking, given a germ of a nonconstant holomorphic curve $\zeta : (\mathbb{C}, 0) \to (\mathbb{C}^n, p)$, denote by $\nu(\zeta)$ the lowest order of vanishing at 0 of all the components of $\zeta(t) - \zeta(0)$, and by $\nu(r \circ \zeta)$ the order of vanishing of the function $r \circ \zeta$ at 0. The *normalized order of contact* of the curve ζ with M at p is

$$\Delta(M, p, \zeta) := \frac{\nu(r \circ \zeta)}{\nu(\zeta)}.$$

The *D'Angelo type* of M at p is defined as follows.

$$\Delta(M, p) := \sup_{\zeta} \Delta(M, p, \zeta) = \sup_{\zeta} \frac{\nu(r \circ \zeta)}{\nu(\zeta)},$$

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where the supremum is taken over all germs ζ of nonconstant holomorphic curves passing through p. We say that p is a *point of finite type* if $\Delta(M,p) < \infty$ and of infinite type otherwise.

We note that the D'Angelo type of M at p defined as above is independent of the choice of its defining function. Moreover, in spite of the algebraic definition of the D'Angelo type, D'Angelo was able to relate the infinite type to the geometric property about the existence of complex curves in M if M is in addition real analytic:

Theorem 1. [5] *If M is a germ at p of a real analytic real hypersurface in $\mathbb{C}^n$, then $\Delta(M,p) = \infty$ if and only if there exists a germ at p of a convergent complex curve $\zeta(t) : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^n, p)$ lying in M.*

In this paper, we prove the formal analogue of this result. Consider a germ M of infinite D'Angelo type at p of a smooth real hypersurface in $\mathbb{C}^n$. Instead of expecting the existence of a convergent complex curve in M through p, we consider formal complex curves through p. Namely, we focus on expressions of the form $\zeta = (\zeta_1, \ldots, \zeta_n) : (\mathbb{C}, 0) \to (\mathbb{C}^n, p)$, where each component ζ_j is a formal power series in the complex variable $t \in \mathbb{C}$ and $\zeta(0) = p$. Here for a formal power series h, the equality $h(0) = c$ means that the constant term of h is equal to c. Write $h \sim 0$ if all the coefficients of the power series of h are 0. Moreover, for any two formal power series h_1 and h_2, we say $h_1 \sim h_2$ if $h_1 - h_2 \sim 0$. As a generalization of D'Angelo's theorem, we show the following formal version for infinite D'Angelo type points:

Theorem 2. *Let M be a germ at p of a smooth real hypersurface in $\mathbb{C}^n$ with a smooth local defining function r. Then $\Delta(M,p) = \infty$ if and only if there exists a nonconstant formal complex curve $\zeta : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^n, p)$ such that $r \circ \zeta \sim 0$.*

When $r \circ \zeta \sim 0$, we also call ζ a formal curve in M. The sufficient direction of the theorem is trivial. For the necessary direction, the proof of the formal analogue is slightly different from that of the convergent case. Indeed, in the original proof of D'Angelo's convergent case, the crucial tool is the Nullstellensatz Theorem for holomorphic functions. However to the authors' knowledge, Nullstellensatz Theorem is not available for the formal case in the literature. Moreover, D'Angelo's decomposition of the defining function does not fit well in the formal case, either. In this paper, we will use a different decomposition that works for the formal case and prove a formal Nullstellensatz stated as follows. In detail, denote by ${}_n\mathcal{O}_0$ the *ring of germs of formal complex power series at 0* in $\mathbb{C}^n$. The key ingredient of our result is the following formal Nullstellensatz:

Theorem 3. *Let I be an ideal in ${}_n\mathcal{O}_0$. If $dim({}_n\mathcal{O}_0/I) = \infty$, then there is a formal complex curve $\zeta : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^n, 0)$ such that $e \circ \zeta \sim 0$ for all $e \in I$.*

The outline of the paper is as follows. In Section 2, we introduce some notations and background. In Section 3, we go over the theory of the formal complex power series ring. Section 4 is devoted to the proof of Theorem 3. In the last section, we complete the proof of our main Theorem 2. The authors are grateful to Steven Krantz for suggesting this problem.

2. Notations and background

From now on and throughout the rest of the paper, let $M \subset \mathbb{C}^n$ be a smooth real hypersurface near $p = 0$ defined by a smooth function r in an open subset $U \subset \mathbb{C}^n$, $dr(0) \neq 0$.

2.1. **Decomposing the defining function of M.** Given a multi-index $J = (J_1, \ldots, J_n)$, let $|J| := \sum_{j=1}^{n} J_j$. For two multi-indices $J = (J_1, \ldots, J_n)$, $K = (K_1, \ldots, K_n)$, we say $J < K$ in terms of lexicographic order if either $|J| < |K|$, or $|J| = |K|$ but the first nonequal component in J is less than that in K.

Since $r(0) = 0$, there exists a sequence $(a_{JK})_{|J|+|K|\geq 1}$ such that r can be formally written as follows.

$$\begin{aligned}
r(z) &\sim \Re \sum_{|J|\geq 1} a_{J0} z^J + 4\Re \sum_{|J|\geq 1} \sum_{K\geq J} a_{JK} z^J \overline{z}^K \\
&\sim \Re \sum_{|J|\geq 1} a_{J0} z^J + 4\Re \sum_{|J|\geq 1} z^J \sum_{K\geq J} a_{JK} \overline{z}^K \\
&\sim \Re \sum_{|J|\geq 1} a_{J0} z^J + \sum_{|J|\geq 1} \left| z^J + \sum_{K\geq J} \overline{a}_{JK} z^K \right|^2 - \sum_{|J|\geq 1} \left| z^J - \sum_{K\geq J} \overline{a}_{JK} z^K \right|^2 .
\end{aligned}$$

Write $h := \sum_{|J|\geq 1} a_{J0} z^J$, $f_J := z^J + \sum_{K\geq J} \overline{a}_{JK} z^K$ and $g_J := z^J - \sum_{K\geq J} \overline{a}_{JK} z^K$ for $|J| \geq 1$. Hence we have the following decomposing of r:

$$r \sim \Re h + \sum_{|J|\geq 1} |f_J|^2 - \sum_{|J|\geq 1} |g_J|^2 \tag{1}$$

with $\nu(f_J) \geq |J|$ and $\nu(g_J) \geq |J|$.

We remark that for the above decomposition, if we truncate r at any order k, only finitely many f_J's and g_J's remain in the expression (1). For simplification of notation, we write $f := (f_J)_{|J|\geq 1}$ to denote the formal infinite sequence with f_J the J'th term in the sequence. We similarly write $g := (g_J)_{|J|\geq 1}$.

2.2. **Infinite matrices.** Denote by $\mathcal{M}$ the collection of all infinite square matrices with complex numbers as entries. For any matrix $A = \{a_{ij}\}_{i,j=1}^{\infty} \in \mathcal{M}$, let $A^* = \{\overline{a_{ji}}\}_{i,j=1}^{\infty}$ denote the adjoint matrix of A. Let $Id \in \mathcal{M}$ be the infinite diagonal matrix with 1 for all diagonal entries and 0 otherwise. Consider the two subsets of $\mathcal{M}$:

$$\mathcal{M}_0 := \left\{ A \in \mathcal{M} : \sum_{j\geq 1} |a_{ij}|^2 \leq 1 \text{ for all } i, \sum_{i\geq 1} |a_{ij}|^2 \leq 1 \text{ for all } j \right\};$$

$$\mathcal{M}_1 := \left\{ A \in \mathcal{M}_0 : AA^* = A^*A = Id \right\}.$$

Here given two matrices $A = \{a_{ij}\}_{i,j=1}^{\infty}, B = \{b_{k\ell}\}_{k,\ell=1}^{\infty} \in \mathcal{M}_0$, $AB = \{c_{i\ell}\}_{i,\ell=1}^{\infty}$ is defined in terms of formal matrix product. Namely, $c_{i\ell} = \sum\limits_{k\geq 1} a_{ik}b_{k\ell}$. Let ℓ^2 be the infinite sequences of complex numbers $c = (c_j)_{j\geq 1}$ with norm $\|c\|^2 = \sum\limits_{j\geq 1} |c_j|^2$ finite. Hence $\mathcal{M}_1 \subset \mathcal{M}_0$ is the set of unitary linear operators between ℓ^2.

In the upcoming Lemma 5 of a later section, we will construct elements in $\mathcal{M}_1$ of the following form:

$$A = \begin{bmatrix} A_k & 0 \\ 0 & Id \end{bmatrix},$$

where the first block A_k is some k by k unitary matrix.

Here are some facts of $\mathcal{M}_1$ from basic functional analysis:

- If $A \in \mathcal{M}_1$, then $\|Ac\| = \|c\|$ for any $c \in \ell^2$.
- For any sequence $\{U_k = \left(u_{ij}^k\right)_{i,j=1}^{\infty}\}_{k=1}^{\infty} \subset \mathcal{M}_1$, after passing to a subsequence if necessary, one can always assume that U_k converges weakly to some $U = (u_{ij})_{i,j=1}^{\infty} \in \mathcal{M}$. Equivalently, for each fixed $i, j \in \mathbb{Z}^+$, $u_{ij}^k \to u_{ij}$ as $k \to \infty$. Moreover,
- If $U \in \mathcal{M}$ is a weak limit of some sequence in $\mathcal{M}_1$, then $U \in \mathcal{M}_0$ and $\|Uc\| \leq \|c\|$ for any $c \in \ell^2$.

3. Ideals in the formal complex power series ring

Let $e(z) = \sum b_J z^J$ be a germ of a formal power series at 0. Then e is said to be a germ of a convergent series if e is convergent as a power series in some neighborhood of 0. The ring of germs of all convergent power series at 0 is called the ring of germs of holomorphic functions at 0, denoted by ${}_nO_0$. Standard holomorphic function theory states that ${}_nO_0$ is a local ring, hence Noetherian. Equivalently, every ideal in ${}_nO_0$ has a finite basis. In ${}_nO_0$, there are two useful tools: the Weierstrass preparation Theorem and the Weierstrass division Theorem. Associated with each ideal $I \subset {}_nO_0$, there is a complex analytic variety consisting of the common zeroes of all elements in

I. Those varieties have a nice local parametrization by further constructing a regular system of coordinates for I.

Even though one does not expect the formal version of the geometric parametrization to hold for the formal complex series ring, the formal complex power series ring do inherit many parallel algebraic properties from its convergent counterpart. For convenience of the reader, we outline in the following some properties of the formal complex power series ring ${}_n\mathcal{O}_0$ that will be used in our paper.

Denote by ${}_{n-1}\mathcal{O}_0[z_n]$ (as a subring of ${}_n\mathcal{O}_0$), the polynomial ring over ${}_{n-1}\mathcal{O}_0$ in the variable z_n. An element $e \in {}_{n-1}\mathcal{O}_0[z_n]$ is called a Weierstrass polynomial of degree ℓ in z_n if it is of the form $e(z) = z_n^\ell + \sum_{j=0}^{\ell-1} b_j z_n^j$ where the coefficients $b_j \in {}_{n-1}\mathcal{O}_0$ are nonunits (i.e., $b_j(0) = 0$) for $0 \le j \le \ell - 1$. Then we have

- ${}_n\mathcal{O}_0$ is a local ring. In particular, ${}_n\mathcal{O}_0$ is Noetherian.
- Formal version of the Weierstrass preparation Theorem: Let $e \in {}_n\mathcal{O}_0$ be regular of order ℓ in z_n. Then there is a unique Weierstrass polynomial $\tilde{e} \in {}_{n-1}\mathcal{O}_0[z_n]$ of degree ℓ such that $e = u\tilde{e}$ for some unit $u \in {}_n\mathcal{O}_0$.
- Formal version of the Weierstrass division Theorem: Let $\tilde{e} \in {}_{n-1}\mathcal{O}_0[z_n]$ be a Weierstrass polynomial in z_n of degree ℓ. Then any $e \in {}_n\mathcal{O}_0$ can be written uniquely as $e = p\tilde{e} + q$, where $p \in {}_n\mathcal{O}_0$ and $q \in {}_{n-1}\mathcal{O}_0[z_n]$ is a polynomial of degree less than ℓ in z_n.

The interested reader may refer to [7], [8] for more details about the properties of ${}_n\mathcal{O}_0$.

Given $f_1, \ldots, f_\ell \in {}_n\mathcal{O}_0$, denote by $I(f_1, \ldots, f_\ell)$ the ideal in ${}_n\mathcal{O}_0$ generated by $f_1, \ldots, f_\ell$. Let H_0 be the maximal ideal in ${}_n\mathcal{O}_0$ consisting of those formal complex power series whose constant term vanishes. For a proper ideal $I \subset {}_n\mathcal{O}_0$, we define the *dimension* of I by

$$D(I) := \dim_{\mathbb{C}} {}_n\mathcal{O}_0/I.$$

Since $D(I) < \infty$ if and only if $H_0^k \subset I$ for some $k \in \mathbb{Z}^+$, we have the following lemma:

Lemma 1. *Let I_1, I_2 be two proper ideals in ${}_n\mathcal{O}_0$. Then $D(I_1 \cap I_2) = \infty$ implies either $D(I_1) = \infty$ or $D(I_2) = \infty$.*

Proof of Lemma 1: Suppose that both $D(I_j) < \infty$. Then for some positive integers k_1, k_2 we have that $H_0^{k_j} \subset I_j, j = 1, 2$. Hence $H_0^{\max\{k_1,k_2\}} \subset I_1 \cap I_2$ and $D(I_1 \cap I_2) < \infty$. This is a contradiction. $\square$

We also similarly define the formal radical of an ideal I as follows.

$$\sqrt{I} := \{e : \text{there exists } \ k \in \mathbb{Z}^+ \text{ such that } e^k \in I\}.$$

Lemma 2. *Let I be a proper ideal in ${}_n\mathcal{O}_0$. $D(I) = \infty$ if and only if $D(\sqrt{I}) = \infty$.*

Proof of Lemma 2: Suppose that $D(\sqrt{I}) < \infty$. Then there exists some positive integer k such that $H_0^k \subset \sqrt{I}$. Let $e_1, ..., e_s$ be generators of H_0^k. Then any $e \in H_0^k$ can be written as $e = \sum g_j e_j$ for some $g_j \in {}_n\mathcal{O}_0$, and $e_i^{r_i} \in I$ for some positive integer r_i. Therefore, for any $e \in H_0^k$, $e^{r_1+\cdots+r_s} \in I$, or equivalently, $H_0^K \subset I$ with $K = r_1 + \cdots + r_s$. This would imply $D(I) < \infty$, This is a contradiction. The other direction is trivial. □

Given a proper ideal I of ${}_n\mathcal{O}_0$, following the idea in Gunning [6], we construct a *regular system of coordinates* $z_1, \ldots, z_n$ such that there exists an integer k satisfying

a) ${}_k\mathcal{O}_0 \cap I = \{0\}$;

b) ${}_{j-1}\mathcal{O}_0[z_j] \cap I$ contains a defining Weierstrass polynomial p_j in z_j for each $j = k+1, \ldots, n$.

If in addition I is prime, the theorem of the primitive element guarantees that, by making a linear change of coordinates in the $z_{k+1}, \ldots, z_n$ plane if necessary, the quotient field ${}_n\tilde{H}_0$ of ${}_n\mathcal{O}_0/I$ is an algebraic extension of ${}_k\tilde{\mathcal{O}}_0$ so that ${}_n\tilde{H}_0 = {}_k\tilde{\mathcal{O}}_0[\tilde{z}_{k+1}]$. Here ${}_k\tilde{\mathcal{O}}_0$ is the image of ${}_k\mathcal{O}_0$ in ${}_n\mathcal{O}_0/I$ and $\tilde{z}_{k+1}$ is the image of z_{k+1} in ${}_n\mathcal{O}_0/I$. We call the above regular system of coordinates a *strictly regular system of coordinates.*

Assuming such a strictly regular system of coordinates as above, denote by $E \in {}_k\mathcal{O}_0$ *the discriminant* of the unique irreducible defining polynomial p_{k+1} of z_{k+1}. Then for each coordinate function $z_j, n \geq j \geq k+2$, we can construct $q_j(z) := E \cdot z_j - Q_j(z_{k+1}) \in I \cap {}_k\mathcal{O}_0[z_{k+1}, z_j]$ for some $Q_j(\cdot) \in {}_k\mathcal{O}_0[\cdot]$. The ideal $I(p_{k+1}, q_{k+2}, \ldots, q_n)$ generated by the elements $p_{k+1}, q_{k+2}, \ldots, q_n$ is called *the associated ideal* for I.

Using exactly the same argument as in [6], the relationship between the above constructed associated ideal $I(p_{k+1}, q_{k+2}, \ldots, q_n)$ and the original ideal I in the formal case can be formulated as follows.

Lemma 3. [6] *There exists an integer ν such that*

$$E^\nu I \subseteq I(p_{k+1}, q_{k+2} \ldots, q_n) \subseteq I.$$

Here E is the discriminant of p_{k+1} in z_{k+1} and $I(p_{k+1}, q_{k+2} \ldots, q_n)$ is the associated ideal of I.

4. Formal Nullstellensatz

In the convergent case, the zero set of a Weierstrass polynomial contains at least a complex curve. The next lemma generalizes this fact to the formal case:

Lemma 4. *Let e be a formal Weierstrass polynomial in w. Denote by E^e the discriminant of e. Suppose that $E^e(z_1, \ldots, z_k) \not\equiv 0$, then there exists a formal curve ζ such that $e \circ \zeta \sim 0$.*

Proof of Lemma 4: Write $e(z_1, \ldots, z_k, w) = w^\ell + \sum_{j<\ell} b_j(z_1, \ldots, z_k)w^j$, $b_j(0) = 0$. After a linear change of coordinates in $(z_1, \ldots, z_k)$, we can assume that $E^e(z_1, 0, \ldots, 0) \not\equiv 0$. Restricting to the subspace defined by $\{z_2 = \cdots = z_k = 0\}$, it suffices to find a formal curve passing through 0 in $\mathbb{C}^2(\ni (z_1, w))$ on which e vanishes. Hence we assume we are in $\mathbb{C}^2$. Write $E^e(z_1) = \sum_{j=s}^{\infty} a_j z_1^j$ where $a_s \neq 0$. For a fixed $r >> s$, we consider a truncation e_r of e up to order r and the corresponding discriminant E^e_r. Then E^e_r is a symmetric function in the roots of e_r, and can still be written as $E^e_r(z_1) = a_s z_1^s + o(z_1^s)$.

On the other hand, if we write $e_r(z_1, w) = \Pi_{j\le\ell}(w - \alpha^r_j(z_1))$, then the expression for E^e_r leads to the estimate $|\alpha^r_j(z_1)-\alpha^r_i(z_1)| \ge c|z_1|^s$ for some small c independent of $z_1, |z_1|$ small, when $i \neq j$. For a fixed z_1, let $\Delta(\alpha^r_j(z_1), \epsilon|z_1|^s)$ be the disc in w plane centered at $\alpha^r_j(z_1)$ with radius $\epsilon|z_1|^s$ for each j with ϵ small enough. Then those discs do not intersect with each other and outside the union of these discs, $|e_r| \ge (\epsilon|z_1|^s)^\ell$. In the same manner for all higher truncations with $\rho > r >> s\ell$, we have as well that $|e_\rho| \ge (\epsilon|z_1|^s)^\ell$ and that the zeroes of e_ρ are contained in the union of these discs by shrinking z_1 if necessary. This means that for all $\rho >> s\ell$, the zero curves for e_ρ are all trapped in the union of those discs with radius $\epsilon|z_1|^s$, which can be again made arbitrarily small by shrinking z_1. Thus by letting ρ go to infinity and passing to a subsequence of those zero curves for e_ρ if necessary, some formal limit, say ζ, exists. By construction, e vanishes to infinite order on ζ. □

We generalize the above lemma and show that formal ideals of infinite dimensions must vanish on some formal curve.

Proposition 1. *Let $P \subset {}_n\mathcal{O}_0$ be a prime ideal with $D(P) = \infty$. Then there exists a formal curve $\zeta : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ such that $e \circ \zeta \sim 0$ for all $e \in P$.*

Proof of Proposition 1: We first choose coordinates $z_1, \ldots, z_n$ such that P is strictly regular under this system of coordinates in the spirit of Section 3. Note that since $D(P) = \infty$, $k \ge 1$.

Consider the associated ideal $I(p_{k+1}, q_{k+2} \ldots, q_n)$ of P. Since the discriminant E of p_{k+1} is not identically zero, apply Lemma 4 to p_{k+1} and we get a formal power curve $\zeta' = (\zeta_1, \zeta_2, \ldots, \zeta_{k+1}) : (\mathbb{C}, 0) \to (\mathbb{C}^{k+1}, 0)$ such that $p_{k+1} \circ \zeta' \sim 0$. We note that the curve can always be chosen so that $E \circ (\zeta_1, \ldots, \zeta_k) \not\equiv 0$.

Next, we add the remaining components $\zeta_{k+2}, ..., \zeta_n$ so that the ideal P formally vanish on the formal curve given by $\zeta := (\zeta_1, \ldots, \zeta_n)$. Recall that for each $j > k + 1$, $q_j = E \cdot z_j - Q_j(z_{k+1}) \in P$ for some $Q_j \in {}_kO_0[\cdot]$. We define

$\zeta_j(t) := (Q_j/E)|_{(z_1,\ldots,z_{k+1})=\zeta'(t)}$ for $j = k+2,\ldots,n$. Hence the associated ideal $I(p_{k+1}, q_{k+2}\ldots, q_n)$ formally vanishes on $\zeta(t) = (\zeta_1(t),\ldots,\zeta_n(t))$.

We will show that $(Q_j/E)|_{(z_1,\ldots,z_{k+1})=0} = 0$ and hence $\zeta(0) = 0$. It will suffice to prove that $z_j = \zeta_j(t)$ is one of the formal roots for the defining Weierstrass polynomials $p_j \in {}_k\mathcal{O}_0[z_j], j = k+2,\ldots,n$. Indeed, for each $j > k+1$, denote by n_j the degree of $p_j \in {}_k\mathcal{O}_0[z_j]$ with respect to z_j and consider $R_j := E^{n_j} p_j$. Then $R_j \in {}_k\mathcal{O}_0[E\cdot z_j]$. Substitute $E\cdot z_j$ in the expression of R_j by $q_j + Q_j$. We get $R_j = H(q_j) + G(Q_j)$ for some $H(q_j) = \sum_{\ell=1}^{n_j} b_j\cdot(q_j)^\ell$ with $b_j \in {}_k\mathcal{O}_0$ and $G(\cdot) \in {}_k\mathcal{O}_0[\cdot]$. Since $R_j, H(q_j) \in P$, so $G(Q_j) \in P$ and hence $G(Q_j) \in {}_k\mathcal{O}_0[z_{k+1}] \cap P$. Since moreover $p_{k+1} \in {}_k\mathcal{O}_0[\cdot]$ is the defining polynomial in P for z_{k+1}, $G(Q_j)$ is divisible by p_{k+1}. By the construction of ζ', we have thus $G(Q_j(\zeta')) \sim 0$. On the other hand, $H(q_j(\zeta)) \sim 0$. Hence $R_j \circ \zeta = H(q_j(\zeta)) + G(Q_j(\zeta')) \sim 0$. This further implies $p_j \circ \zeta \sim 0$ since $E \circ (\zeta_1,\ldots,\zeta_k) \not\equiv 0$. Therefore, $z_j = \zeta_j(t)$ is one of the formal roots for the defining Weierstrass polynomials p_j.

Finally we show that for any $e \in P$, e vanishes on the formal curve ζ. Indeed, by Lemma 3, there exists some large positive number ν, such that $E^\nu e \in I(p_{k+1}, q_{k+2}\ldots, q_n)$. Therefore $(E^\nu e)\circ\zeta \sim 0$. Since E vanishes only to finite order on the curve, we get $e\circ\zeta \sim 0$. □

Recall that an ideal I is primary if, whenever $xy \in I$, either $x \in I$ or $y^m \in I$ for some $m \in \mathbb{Z}^+$. We are now in a position to prove Theorem 3.

Proof of Theorem 3: By the Lasker-Noether decomposition theorem, we can write $I = P_1 \cap \cdots \cap P_s$, where P_j's are primary ideals. Since $D(I) = \infty$, applying an induction process by Lemma 1, we have $D(P_j) = \infty$ for some $j \in \{1,\ldots,s\}$. On the other hand, Lemma 2 implies $D(\sqrt{P_j}) = \infty$. Notice that $\sqrt{P_j}$ is also prime. Proposition 1 thus implies the existence of a formal curve ζ such that $\zeta(0) = 0$ and $e\circ\zeta \sim 0$ for all $e \in \sqrt{P_j}$. Since $I \subset P_j \subset \sqrt{P_j}$, $e\circ\zeta \sim 0$ for all $e \in I$. The proof of Theorem 3 is thus complete. □

5. Existence of formal complex curves

Given a formal complex power series h and a positive integer k, we call $j_k(h)$, the k-jet of h, the truncation in h up to order k. A slight change of D'Angelo's theorem gives the following lemma.

Lemma 5. *Let h, f and g be defined by (1) as in Section 2.1 for the given defining function r of M. Let $\zeta : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ be a formal complex curve. If $j_{2k\nu(\zeta)}(r\circ\zeta) = 0$, then there is an infinite unitary matrix $U_k \in \mathcal{M}_1$ such that*

$$j_{2k\nu(\zeta)}(h\circ\zeta) = j_{k\nu(\zeta)}((f - U_k g)\circ\zeta) = j_{k\nu(\zeta)}((U_k^* f - g)\circ\zeta) = 0. \tag{2}$$

Proof of Lemma 5: If $j_{2k\nu(\zeta)}(r\circ\zeta)=0$, then $j_{2k\nu(\zeta)}(h\circ\zeta)=0$ and $j_{2k\nu(\zeta)}(\|f\circ\zeta\|^2)=j_{2k\nu(\zeta)}(\|g\circ\zeta\|^2)$. Thus $\|j_{k\nu(\zeta)}(f\circ\zeta)\|^2=\|j_{k\nu(\zeta)}(g\circ\zeta)\|^2$. Note that for each k, the above norms involve only summations of finite many terms since there are only finitely many components in f and g with non-vanishing k-jets by our construction. For these finitely many f_J's and g_J's, applying Theorem 3.5 [4], one can find an element $\tilde{U}$ of the group of unitary matrices with a finite size such that

$$j_{k\nu(\zeta)}((f-\tilde{U}g)\circ\zeta)=0.$$

Extending $\tilde{U}$ to an infinite unitary matrix $U\in\mathcal{M}_1$ by letting the rest of the diagonal entries be 1 and the other terms 0, we obtain (2). This choice of U also makes $j_{k\nu(\zeta)}((U^*f-g)\circ\zeta)=0$. □

From now on, we consider the ideal generated by all the components in $(h,f-Ug,U^*f-g)$ for some $U\in\mathcal{M}_1$. The following proposition reveals the connection between the null set of $I(h,f-Ug,U^*f-g)$ and that of $I(r)$.

Proposition 2. *Let h,f and g be defined by (1) as in Section 2.1 for the given defining function r of M. Let $\zeta:(\mathbb{C},0)\to(\mathbb{C}^n,0)$ be a formal complex curve. If there exists an operator $U\in\mathcal{M}_0$ as a weak limit of a sequence in $\mathcal{M}_1$ such that $e\circ\zeta\sim 0$ for any $e\in I(h,f-Ug,U^*f-g)$, then $r\circ\zeta\sim 0$.*

Proof of Proposition 2: If a formal curve ζ satisfies $h\circ\zeta\sim(f-Ug)\circ\zeta\sim(U^*f-g)\circ\zeta\sim 0$, then for any positive integer k, $j_k(h\circ\zeta)=0$ and

$$\begin{aligned}\|j_k(f\circ\zeta)\|&=\|j_k(Ug\circ\zeta)\|=\|U(j_k(g\circ\zeta))\|\\&\le\|j_k(g\circ\zeta)\|\\&=\|j_k(U^*f\circ\zeta)\|=\|U^*(j_k(f\circ\zeta))\|\\&\le\|j_k(f\circ\zeta)\|.\end{aligned}$$

Hence, $\|j_k(f\circ\zeta)\|=\|j_k(g\circ\zeta)\|$ for any k. Letting k go to infinity, we see that $h\circ\zeta\sim\|f\circ\zeta\|^2-\|g\circ\zeta\|^2\sim 0$ and therefore we get $r\circ\zeta\sim 0$. □

Theorem 2 follows directly by combining Proposition 2 and Proposition 3, together with Theorem 3 in the previous section.

Proposition 3. *Let h,f and g be defined by (1) as in Section 2.1 for the given defining function r of M. Assume M is of infinite D'Angelo type at 0, then there exists $U\in\mathcal{M}_0$ which is a weak limit of a sequence in $\mathcal{M}_1$, such that $D(I(h,f-Ug,U^*f-g))=\infty$.*

Proof of Proposition 3: Since the hypersurface is of infinite type at 0, for any order $k\in\mathbb{Z}^+$, there is a unitary matrix $U_k\in\mathcal{M}_1$ as in Lemma 5 such that (2) holds. Let $U\in\mathcal{M}_0$ be a weak limit of $\{U_k\}$ by passing to a subsequence if necessary. Suppose that $I(h, f - Ug, U^*f - g)$ has finite dimension. Then there exists an integer ℓ such that $H_0^\ell \subset I(h, f - Ug, U^*f - g)$. By the upper semi-continuity of $D(I)$, this would then imply $H_0^\ell \subset I(h, f - U_k g, U_k^* f - g)$ that for all k larger than some $k_0 \in \mathbb{Z}^+$. This is a contradiction to Lemma 5. $\square$

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Mathematics Department, NTNU, Alfred Getz vei 1, Gløshaugen, 7491 Trondheim, Norway

E-mail address: `johnefo@math.ntnu.no`

Mathematics Department, University of California Riverside, Riverside CA 92521, USA

E-mail address: `linalee@math.ucr.edu`

Department of Mathematics, Indiana University - Purdue University Fort Wayne, Fort Wayne, IN 46805, USA

E-mail address: `zhangyu@ipfw.edu`