CONTINUOUS SOLUTIONS OF NONLINEAR CAUCHY-RIEMANN EQUATIONS AND PSEUDOHOLOMORPHIC CURVES IN NORMAL COORDINATES

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Abstract. We establish elliptic regularity for nonlinear, inhomogeneous Cauchy-Riemann equations under weak assumptions, and give a counterexample in a borderline case. In some cases where the inhomogeneous term has a separable factorization, the solution set can be explicitly calculated. The methods also give local parametric formulas for pseudoholomorphic curves with respect to some continuous almost complex structures.

1. Introduction

We consider the nonlinear, inhomogeneous Cauchy-Riemann equation: for open sets $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ and a function $u : \Omega_1 \to \Omega_2$, the equation is

$$\frac{\partial u}{\partial \bar{z}} = E(z, u). \tag{1.1}$$

Section 2 starts with the linear case, Theorem 2.6, establishing some regularity of solutions u under minimal assumptions: u is continuous, the partial derivatives u_x and u_y (and the LHS $\frac{1}{2}(u_x + iu_y)$) exist except possibly on some small set, and the linear equation $\frac{\partial u}{\partial \bar{z}} = P(z)$ holds almost everywhere for $P \in L^p_{loc}$, $p \geq 2$. An analogue in the homogeneous case is the Looman-Menchoff Theorem, that a continuous, but not necessarily $\mathcal{C}^1$, function with zero $\bar{z}$-derivative must be analytic. Regularity of u satisfying the nonlinear equation (1.1) then follows in some corollaries of Theorem 2.6. In Section 3 we give a new example of a differentiable function u satisfying $\partial u/\partial \bar{z} = P(z)$, where P is continuous on $\mathbb{C}$ but $\partial u/\partial z$ is not.

In Section 4, we consider the "separable" case of the nonlinear Cauchy-Riemann equation where the RHS of (1.1) factors in the form $E(z, u) =$

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$f(u)g(z)$ with f holomorphic. We state a local existence result in a special case (Theorem 4.7), but our main goal in Section 4 is to explicitly compute local formulas for solutions u without strong *a priori* assumptions on the regularity of u.

In Section 5, we apply the results of Sections 2 and 4 to find formulas for all the J-holomorphic curves in certain coordinate charts in some almost complex 4-manifolds. Example 5.3 uses the counterexample from Section 3 to show that a continuous almost complex structure can admit a J-holomorphic curve which is differentiable but not $\mathcal{C}^1$.

2. Nonlinear Cauchy-Riemann equations

Notation 2.1. For $z = x + iy \in \Omega \subseteq \mathbb{C} = \mathbb{R}^2$, and a function $u : \Omega \to \mathbb{C}$, $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$ are the complex valued pointwise partial derivatives with respect to the real coordinates. If both u_x and u_y exist at a point, then $u_z = \frac{\partial u}{\partial z} = \frac{1}{2}(u_x - iu_y)$ and $u_{\bar{z}} = \frac{\partial u}{\partial \bar{z}} = \frac{1}{2}(u_x + iu_y)$ are the pointwise z- and $\bar{z}$-derivatives. The distributional $\bar{z}$-derivative of u on Ω (and similarly for z) is the operator, denoted by $\partial_{\bar{z}} u$, which maps compactly supported smooth test functions $\varphi \in \mathcal{C}_0^\infty(\Omega)$ to $-\int_\Omega u \frac{\partial \varphi}{\partial \bar{z}}$. We say that $\partial_{\bar{z}} u$ is represented on Ω by a function r to mean that $-\int_\Omega u \frac{\partial \varphi}{\partial \bar{z}} = \int_\Omega r\varphi$.

A distributional derivative represented by r on a domain behaves as expected under restriction: if Ω_2 is an open subset of Ω_1, and $\partial_{\bar{z}} u$ is represented on Ω_1 by r, then $\partial_{\bar{z}}(u|_{\Omega_2})$ is represented on Ω_2 by $r|_{\Omega_2}$.

Notation 2.2. Let $R \Subset \Omega$ denote that R is a bounded, open rectangle of the form $(a_1, b_1) \times (a_2, b_2)$, with closure $\overline{R}$ contained in the open set $\Omega \subseteq \mathbb{C}$. Let ∂R denote the boundary of R.

Usually, Green's Theorem is stated with a $\mathcal{C}^1$ or $W^{1,1}$ hypothesis ([AIM] Theorem 2.9.1). However, in a situation where the partial derivatives exist but may not all be integrable, the following version of Green's Theorem due to Cohen ([C$_1$], [C$_2$], [CV], [GM] Theorem 8) applies.

Proposition 2.3. *Suppose $v : \Omega \to \mathbb{C}$ is continuous and satisfies the following condition:*

(∗) *The partial derivatives v_x, v_y exist at every point in Ω except for countably many.*

Then, for any $R \Subset \Omega$, if $\frac{\partial v}{\partial \bar{z}} \in L^1(R)$, then

$$\int_{\partial R} v(z)dz = 2i \int_R \frac{\partial v}{\partial \bar{z}} dxdy.$$

∎

Remark 2.4. The statement of Proposition 2.3 can be generalized to shapes other than rectangles, and the condition $(*)$ can be weakened to allow a larger exceptional set: see [CV]. The property $(*)$ can also be assumed to hold only on one particular rectangle R, but the above formulation is more convenient for us, and as a practical matter, the condition $(*)$ on the classical derivatives is something more easily checked than properties of distributional derivatives. The main significance of the Proposition is that its hypothesis omits any assumption about the integrability or continuity of the individual partial derivatives v_x, v_y, or v_z. We also remark that the integrand on the RHS is the pointwise derivative (where it exists), not the distributional derivative.

Cohen's proof was motivated by the earlier Looman-Menchoff Theorem, which we recall here from ([N], [GM] Theorem 11) as a Proposition, to be used in Section 4.

Proposition 2.5. *Suppose $v : \Omega \to \mathbb{C}$ is continuous and satisfies condition $(*)$. If*

$$\frac{\partial v}{\partial \bar{z}} = 0 \tag{2.1}$$

almost everywhere in Ω, then v is holomorphic on Ω. ∎

The following Theorem considers an inhomogeneous, linear version of (2.1). In the following Proof, some steps are similar to steps in [BBC] §2 and [CV], and the last two paragraphs recall well-known regularity methods, but we give enough details to show exactly where Proposition 2.3 is used to establish the necessary integration by parts.

Theorem 2.6. *Suppose $u : \Omega_1 \to \Omega_2$ is continuous, satisfies $(*)$, and there is a function $P : \Omega_1 \to \mathbb{C}$ so that $P \in L^p_{loc}(\Omega_1)$ for some p, $2 \le p < \infty$, and*

$$\frac{\partial u}{\partial \bar{z}} = P(z)$$

almost everywhere. Then, for any $R \Subset \Omega_1$, $u|_R \in W^{1,2}(R)$. If, further, $p > 2$, then for $\alpha = 1 - \frac{2}{p}$, $u|_R \in \mathcal{C}^{0,\alpha}(R)$.

Proof. The restriction $u|_R$ is continuous and bounded on R, and an element of $L^2(R)$. The following argument uses the assumption on the classical pointwise derivatives to draw this conclusion about the distributional derivatives: $u|_R \in W^{1,2}(R)$, meaning that its distributional derivatives on R, $\partial_{\bar{z}}(u|_R)$ and $\partial_z(u|_R)$, are represented by functions in $L^2(R)$.

By compactness, there is a larger rectangle with $R \Subset R_1 \Subset \Omega_1$. Let u_1 and P_1 be the restrictions of u and P to R_1, so $P_1 : R_1 \to \mathbb{C}$ satisfies

$$\left.\frac{\partial u}{\partial \bar{z}}\right|_{R_1} =_{a.e.} P_1 \in L^p(R_1) \subseteq L^2(R_1). \tag{2.2}$$

For a test function $\varphi \in \mathcal{C}_0^\infty(R_1)$, the product $u_1\varphi$ satisfies, for all z except in some set of measure 0 (which includes the exceptional set from $(*)$),

$$\begin{aligned} \frac{\partial}{\partial \bar{z}}(u_1(z)\varphi(z)) &= \left(\frac{\partial}{\partial \bar{z}}u_1\right)\varphi(z) + u_1(z)\left(\frac{\partial}{\partial \bar{z}}\varphi\right) \\ &=_{a.e.} P_1(z)\varphi(z) + u_1(z)\left(\frac{\partial}{\partial \bar{z}}\varphi\right). \end{aligned} \tag{2.3}$$

We emphasize that Equations (2.2) and (2.3) are a.e. equalities of functions, not equalities of distributions. P_1 and the RHS of (2.3) are defined for all $z \in R_1$, while $u_{\bar{z}}$ and the LHS of (2.3) may be undefined for some z in a set of measure 0. Because the two functions differ only on a set of measure 0 and $P_1\varphi + u_1\frac{\partial \varphi}{\partial \bar{z}} \in L^p(R_1) \subseteq L^1(R_1)$, $\frac{\partial}{\partial \bar{z}}(u_1\varphi)$ is also in $L^1(R_1)$.

For $p \geq 2$, define this function $P_2 : \mathbb{C} \to \mathbb{C}$:

$$P_2(z) = \begin{cases} P_1(z) & z \in R_1 \\ 0 & z \notin R_1 \end{cases},$$

so $P_2 \in L^p(\mathbb{C}) \cap L^2(\mathbb{C})$. The Cauchy transform $\mathcal{C}(P_2)$ is an element of $L^1_{loc}(\mathbb{C})$ ([AIM] Theorem 4.3.9, Theorem 4.3.13), and its distributional derivative on $\mathbb{C}$, $\partial_{\bar{z}}\mathcal{C}(P_2)$, is represented by $P_2 \in L^2(\mathbb{C})$ ([AIM] Theorem 4.3.10). The restriction $\mathcal{C}(P_2)|_{R_1}$ has distributional derivative $\partial_{\bar{z}}((\mathcal{C}(P_2)|_{R_1}))$ on R_1 represented by $P_2|_{R_1} = P_1$. The restriction $u_1 - (\mathcal{C}(P_2)|_{R_1})$ is integrable on R_1, and the distributional derivative on R_1 satisfies, for $\varphi \in \mathcal{C}_0^\infty(R_1)$,

$$\begin{aligned} \partial_{\bar{z}}(u_1 - (\mathcal{C}(P_2)|_{R_1})) : \varphi &\mapsto -\int_{R_1} (u_1 - (\mathcal{C}(P_2)|_{R_1}))\frac{\partial \varphi}{\partial \bar{z}} \\ &= -\int_{R_1} u_1\frac{\partial \varphi}{\partial \bar{z}} - \int_{R_1} P_1\varphi \qquad (2.4) \\ &= -\int_{R_1} \left(P_1\varphi + u_1\frac{\partial \varphi}{\partial \bar{z}}\right) \\ &= -\int_{R_2} \frac{\partial}{\partial \bar{z}}(u_1\varphi) \qquad (2.5) \\ &= 0. \qquad (2.6) \end{aligned}$$

Line (2.5) follows from Equation (2.3), and $R_2 \Subset R_1$ is a smaller rectangle with interior containing the support of φ. Line (2.6) uses Proposition 2.3, and this is the key technical step using the assumptions on the $\bar{z}$-derivative without any integrability of the z-derivative. It follows from (2.4) and (2.6) that the distributional derivative on R_1, $\partial_{\bar{z}} u_1 = \partial_{\bar{z}} \left(\mathcal{C}(P_2)|_{R_1} \right)$, is represented by P_1, which is a.e. equal to $\frac{\partial u_1}{\partial \bar{z}}$ as in (2.2), so the distributional and a.e. pointwise $\bar{z}$-derivatives coincide. It follows by restriction that $u|_R = u_1|_R$ has distributional derivative on R, $\partial_{\bar{z}}(u|_R)$, represented by $P_1|_R = P|_R \in L^2(R)$.

Also, Weyl's Lemma ([AIM] Lemma A.6.10, [GM] Theorem 9) applies, so there exists a holomorphic function $\Phi : R_1 \to \mathbb{C}$ equal to $u_1 - (\mathcal{C}(P_2)|_{R_1})$ as an element of $L^1(R_1)$. The Beurling transform, $\mathcal{S}(P_2) \in L^2(\mathbb{C})$, is a function defined almost everywhere in $\mathbb{C}$ ([AIM] Theorem 4.0.10) that represents the distributional derivative of $\mathcal{C}(P_2)$ on $\mathbb{C}$, $\partial_z(\mathcal{C}(P_2))$ ([AIM] Theorem 4.3.10). So, the distributional derivative of u_1 on R_1, $\partial_z(u_1) = \partial_z(\Phi + (\mathcal{C}(P_2)|_{R_1}))$, is represented by $\frac{\partial \Phi}{\partial z} + (\mathcal{S}(P_2)|_{R_1})$. The restrictions $\Phi|_R$ and $\mathcal{S}(P_2)|_R$ are both in $L^2(R)$, so the distributional derivative of u on R, $\partial_z(u|_R)$, is represented by $\Phi|_R + (\mathcal{S}(P_2)|_R) \in L^2(R)$.

For $p > 2$, $\mathcal{C}(P_2) \in \mathcal{C}^{0,\alpha}(\mathbb{C})$ ([AIM] Theorem 4.3.13), and the restriction $\Phi|_R$ is in $\mathcal{C}^{0,\alpha}(R)$, so by continuity, $u|_R = \Phi|_R + (\mathcal{C}(P_2)|_R)$ pointwise everywhere in R and $u|_R \in \mathcal{C}^{0,\alpha}(R)$. ∎

Corollary 2.7. *Let $E : \Omega_1 \times \Omega_2 \to \mathbb{C}$, let $u : \Omega_1 \to \Omega_2$ be continuous, and suppose that u satisfies $(*)$, and*

$$\frac{\partial u}{\partial \bar{z}} = E(z, u(z)) \tag{2.7}$$

almost everywhere.

- *If E is continuous, then for any $R \Subset \Omega_1$, $u|_R \in \mathcal{C}^{0,\alpha}(R)$ for all $0 < \alpha < 1$.*
- *If $0 < \beta < 1$ and $E \in \mathcal{C}^{0,\beta}_{loc}(\Omega_1 \times \Omega_2)$, then for any $R \Subset \Omega_1$, $u|_R \in \mathcal{C}^{1,\beta}(R)$.*
- *For $r \in \mathbb{N}$, $r = \infty$, or $r = \omega$, if $E \in \mathcal{C}^r(\Omega_1 \times \Omega_2)$, then $u \in \mathcal{C}^r(\Omega_1)$.*

Proof. First, if E is continuous on $\Omega_1 \times \Omega_2$, then for any $p \geq 2$, u satisfies the hypotheses of Theorem 2.6, with $\frac{\partial u}{\partial \bar{z}}$ equal almost everywhere to the continuous function $P(z) = E(z, u(z)) \in L^p_{loc}(\Omega_1)$. The conclusion from the Theorem is that for any $R \Subset \Omega_1$ and any $0 < \alpha < 1$,

$$u|_R \in W^{1,2}(R) \cap \mathcal{C}^{0,\alpha}(R). \tag{2.8}$$

For the second claim of the Corollary, consider larger rectangles $R \Subset R_2 \Subset R_1 \Subset \Omega_1$. $E(z, w)$ is $\mathcal{C}^{0,\beta}$ on the compact product $\overline{R_1} \times u(\overline{R_1})$, and the composite $E(z, u(z))$ is continuous, with Hölder exponent $\alpha\beta$. Because the RHS of (2.7), restricted to $z \in R_1$, is in $\mathcal{C}^{0,\alpha\beta}(R_1)$, it follows from (2.8) and [AIM] Theorem 15.0.7 that $u|_{R_1} \in \mathcal{C}^{1,\alpha\beta}_{loc}(R_1)$. The composite $E(z, u(z))$ is now in $\mathcal{C}^{0,\beta}(R_2)$, and [AIM] Theorem 15.0.7 applies again to establish the claim.

For the third claim with $r = 1$, because the conclusion is a local property of u, it is enough to work with the same rectangle R as the previous case and $u|_R$ as in (2.8). If $E \in \mathcal{C}^1(\Omega_1 \times \Omega_2)$, then the composite $E(z, u(z))$ is $\mathcal{C}^{0,\alpha}$ on R, and again by [AIM] Theorem 15.0.7, $u|_R \in \mathcal{C}^{1,\alpha}_{loc}(R)$. So, $u \in \mathcal{C}^1(\Omega_1)$. If $E \in \mathcal{C}^2(\Omega_1 \times \Omega_2)$, then the composite $E(z, u(z)) \in \mathcal{C}^{1,\alpha}_{loc}(R)$, so $u|_R \in \mathcal{C}^{2,\alpha}_{loc}(R)$. For $r > 1$, the bootstrap technique applies, iterating r times when E is $\mathcal{C}^r$, and if E is smooth, then u is smooth.

When $E(z, w)$ is real analytic, u is smooth, and using the chain rule ([AIM] §2.9.1) gives:

$$\begin{aligned} \Delta(u) &= 4\frac{\partial}{\partial z}\frac{\partial u}{\partial \bar{z}} = 4\frac{\partial}{\partial z}(E(z, u(z)) \\ &= 4(E_z(z, u(z)) + E_w(z, u(z))\frac{\partial u}{\partial z} + E_{\bar{w}}(z, u(z))\overline{\frac{\partial u}{\partial \bar{z}}}). \end{aligned}$$

This complex equation (or the system of two real equations $\Delta(\mathrm{Re}(u)) = \mathrm{Re}(\Delta(u))$ and $\Delta(\mathrm{Im}(u)) = \mathrm{Im}(\Delta(u))$) is a second order, nonlinear, elliptic system where the RHS is a real analytic expression in z, u, (or their real and imaginary parts) and the first derivatives of u. For such a system, $\mathcal{C}^3$ solutions u must be real analytic ([M]). ∎

Corollary 2.8. *For a connected open set $\Omega_1 \subseteq \mathbb{C}$, suppose $u : \Omega_1 \to \Omega_2$ is continuous and satisfies $(*)$. Given $w_0 \in \Omega_2$, let $Z_0 = \{z \in \Omega_1 : u(z) = w_0\}$. If there is a function $A : \Omega_1 \setminus Z_0 \to \mathbb{C}$ so that $A \in L^p(\Omega_1 \setminus Z_0)$ for some $p > 2$ and*

$$\frac{\partial u}{\partial \bar{z}} = (u(z) - w_0)A(z) \tag{2.9}$$

almost everywhere in $\Omega_1 \setminus Z_0$, then either Z_0 is a set of isolated points in Ω_1 or $Z_0 = \Omega_1$.

Remark 2.9. Proposition C of [GR] is similar to the above statement, but its hypothesis includes the distributional derivative version of (2.9) (see also [IS_1], [IS_2]). In view of the Proof of Theorem 2.6, the distributional derivative equation is equivalent to the a.e. pointwise property under these conditions, so Corollary 2.8 is also a corollary of [GR]

Proposition C. Because we need formula (2.10) in the Proof of Theorem 4.9, here we sketch a Proof of Corollary 2.8 using the same methods as the Proof of Theorem 2.6.

Proof of Corollary 2.8. Let z_0 be an arbitrary point of Z_0, and let $R_1 \Subset \Omega_1$ be a neighborhood of z_0. Define this function $A_1 : \mathbb{C} \to \mathbb{C}$:

$$A_1(z) = \begin{cases} -A(z) & z \in R_1 \setminus Z_0 \\ 0 & z \notin R_1 \setminus Z_0 \end{cases},$$

so $A_1 \in L^p(\mathbb{C}) \cap L^2(\mathbb{C})$. The Cauchy transform $\mathcal{C}(A_1)$ is in $\mathcal{C}^{0,\alpha}(\mathbb{C})$, and its distributional derivative on $\mathbb{C}$, $\partial_{\bar{z}}\mathcal{C}(A_1)$, is represented by $A_1 \in L^2(\mathbb{C})$. The function $\mathcal{S}(A_1) \in L^2(\mathbb{C})$ represents the distributional derivative of $\mathcal{C}(A_1)$ on $\mathbb{C}$, $\partial_z(\mathcal{C}(A_1))$. So, the restriction $\mathcal{C}(A_1)|_{R_1}$ is bounded and in $W^{1,2}(R_1)$, with distributional $\bar{z}$-derivative on R_1 represented by $A_1|_{R_1}$. This is enough ([GT], [BBC] §8) for the weak chain rule to apply: the composite $\exp(\mathcal{C}(A_1)|_{R_1})$ is in $W^{1,1}(R_1)$ and its distributional $\bar{z}$-derivative on R_1 is represented by $\exp(\mathcal{C}(A_1)|_{R_1})(A_1|_{R_1})$.

Now Theorem 2.6 applies to u on the open set $R_1 \setminus Z_0$, with $P = ((u(z) - w_0)A(z))\,|_{R_1 \setminus Z_0} \in L^p(R_1 \setminus Z_0)$. Let z_1 be any point of $R_1 \setminus Z_0$, and let $R_2 \Subset R_1 \setminus Z_0$ be a neighborhood of z_1. The restriction $u|_{R_2}$ is in $W^{1,2}(R_2) \cap \mathcal{C}^{0,\alpha}(R_2)$, and its distributional $\bar{z}$-derivative on R_2 is represented by $P|_{R_2}$, from (2.4). This is enough for the weak product rule to apply: the product $(\exp(\mathcal{C}(A_1)|_{R_1}))\,|_{R_2}(u(z)-w_0)|_{R_2}$ is in $W^{1,1}(R_2)$, with distributional $\bar{z}$-derivative represented on R_2 by

$$\begin{aligned} & (\exp(\mathcal{C}(A_1)|_{R_1}))\,|_{R_2}P|_{R_2} + (\exp(\mathcal{C}(A_1)|_{R_1})(A_1|_{R_1}))\,|_{R_2}(u(z) - w_0)|_{R_2} \\ = \; & (\exp(\mathcal{C}(A_1)|_{R_1}))\,|_{R_2}\left(((u(z) - w_0)A(z))\,|_{R_2} + (A_1)|_{R_2}(u(z) - w_0)|_{R_2}\right) \\ = \; & (\exp(\mathcal{C}(A_1)|_{R_1}))\,|_{R_2}(u(z) - w_0)|_{R_2}\left((A(z))|_{R_2} + (-A(z))|_{R_2}\right) \\ = \; & 0. \end{aligned}$$

By Weyl's Lemma and continuity, $(\exp(\mathcal{C}(A_1)|_{R_1}))\,|_{R_2}\,(u(z) - w_0))\,|_{R_2}$ is holomorphic on R_2. Since z_1 was arbitrary, every point in $R_1 \setminus Z_0$ is contained in some neighborhood where $\sigma : R_1 \to \mathbb{C}$,

$$\sigma(z) = \exp(\mathcal{C}(A_1)|_{R_1})((u(z) - w_0)|_{R_1}), \tag{2.10}$$

restricts to a holomorphic function, so σ is holomorphic on $R_1 \setminus Z_0$. Because σ is continuous on R_1 and equal to 0 exactly on $R_1 \cap Z_0$, Radó's Theorem ([N]) implies σ is holomorphic on R_1, so z_0 is either an isolated zero of $u(z) - w_0$ or $u(z) \equiv w_0$ on R_1. It follows that the set of non-isolated points in Z_0 is both open and closed in Ω_1, so it is either empty or all of Ω_1. ∎

3. Examples in a borderline case

The following two Examples give solutions of $\frac{\partial v}{\partial \bar{z}} = P(z)$ where v and P are continuous but v is not $\mathcal{C}^1$. This can be considered a borderline case, as $\alpha \to 1^-$ in Theorem 2.6, or $\beta \to 0^+$ in Corollary 2.7. The function v in Example 3.2 is well-known and elementary, but $\frac{\partial v}{\partial \bar{z}}$ fails to exist at one point. The goal of Example 3.3 is to improve Example 3.2 by finding a continuous function V where the partial derivatives exist at every point, and $\partial V/\partial \bar{z}$ is continuous, while $\partial V/\partial z$ is not locally bounded. These examples are of interest from the point of view of the foundations of classical complex analysis, not motivated by any particular application.

Notation 3.1. The notation $D_{a,r}$ refers to an open disk in $\mathbb{C}$ with center a and radius $r > 0$.

Example 3.2. The following function (adapted from [AIM] §15.1) is continuous but not $\mathcal{C}^1$; it satisfies the assumptions of Theorem 2.6 for all $p \geq 2$, and the first part of Corollary 2.7, but not the second or third. Using the real-valued natural logarithm ln and positive square root, define this function for $z \in D_{0,1}$:

$$v(z) = \begin{cases} 0 & z = 0 \\ z\sqrt{-\ln(|z|^2)} & 0 < |z| < 1 \end{cases} . \tag{3.1}$$

v is real analytic except at the origin, where the partial derivatives $v_x(0)$ and $v_y(0)$ do not exist. For $z \neq 0$, the derivatives are:

$$\begin{aligned} v_{\bar{z}} &= \frac{-z^2}{2\sqrt{-\ln(|z|^2)}|z|^2}, \\ v_z &= \sqrt{-\ln(|z|^2)} + \frac{-1}{2\sqrt{-\ln(|z|^2)}}. \end{aligned} \tag{3.2}$$

So, $v_{\bar{z}}$ has a removable discontinuity: there is a continuous function P equal almost everywhere to $v_{\bar{z}}$, but there is no continuous function equal almost everywhere to the unbounded function v_z.

Example 3.3. We start with a smooth cutoff function: let $\kappa : (0, \infty) \to [0, 1]$ be a fixed, weakly decreasing, $\mathcal{C}^\infty$ function satisfying $\kappa(x) \equiv 1$ for $0 < x \leq \frac{1}{2}$, and $\kappa(x) \equiv 0$ for $x \geq e^{-1/2} \approx 0.6$.

Next, define the following family of functions $V_t(z) : \mathbb{C} \to \mathbb{C}$, depending on a parameter $0 < t \leq \frac{1}{2}$:

$$V_t(z) = \begin{cases} 0 & z = 0 \\ \kappa(|z|)z|z|^{2t}\sqrt{-\ln(|z|^2)} & 0 < |z| < 1 \\ 0 & |z| \geq 1 \end{cases} .$$

Each $V_t(z)$ is a smoothed modification of $v(z)$ from (3.1): the cutoff κ makes V_t smooth on $\mathbb{C}\setminus\{0\}$, and a calculation using the positivity of the exponent $2t$ shows that the x, y partial derivatives exist at the origin, where $\frac{\partial V_t}{\partial x}(0)=\frac{\partial V_t}{\partial y}(0)=0$. A little calculus shows that $V_t(z)$ is bounded by a constant not depending on t: $|V_t(z)|\le e^{-1/2}$.

For $0<|z|<1$, the expression:

$$\begin{aligned}\frac{\partial V_t}{\partial\bar z} &= \frac{\partial}{\partial\bar z}(\kappa(|z|))z|z|^{2t}\sqrt{-\ln(|z|^2)}\\ &\quad+\kappa(|z|)t\frac{z}{\bar z}|z|^{2t}\sqrt{-\ln(|z|^2)}\\ &\quad-\kappa(|z|)\frac{1}{2}\frac{z}{\bar z}|z|^{2t}/\sqrt{-\ln(|z|^2)}\end{aligned}\tag{3.3}$$

shows that $\frac{\partial V_t}{\partial\bar z}$ is continuous on $\mathbb{C}$. The following calculation shows that $\frac{\partial V_t}{\partial\bar z}$ is bounded by a constant not depending on t. Recalling that κ does not depend on t, $\frac{\partial}{\partial\bar z}(\kappa(|z|))$ is bounded by some $B_1>0$.

$$\begin{aligned}\left|\frac{\partial V_t}{\partial\bar z}\right| &\le B_1\max_{\frac12\le|z|\le e^{-1/2}}\left\{|z|^{1+2t}\sqrt{-\ln(|z|^2)}\right\}\\ &\quad+\max_{0<|z|\le e^{-1/2}}\left\{t|z|^{2t}\sqrt{-\ln(|z|^2)}\right\}\\ &\quad+\max_{0<|z|\le e^{-1/2}}\left\{\frac12|z|^{2t}/\sqrt{-\ln(|z|^2)}\right\}\\ &\le B_1e^{-1/2}+\frac{\sqrt{t}}{\sqrt{2e}}+\frac12e^{-t}\le B_2.\end{aligned}$$

Next, choose any real sequence R_k decreasing to limit 0, and another positive sequence r_k with $R_k-r_k>R_{k+1}+r_{k+1}$. (For example, $R_k=10^{-k}$ and $r_k=10^{-(k+1)}$.)

Finally, define

$$V(z)=\sum_{k=1}^{\infty}2^{-k}r_kV_{2^{-4k}}\left(\frac{z-R_ke^{\pi i/4}}{r_k}\right).$$

The expression $V_{2^{-4k}}\left(\frac{z-R_ke^{\pi i/4}}{r_k}\right)$ is the result of re-scaling $V_{2^{-4k}}(z)$, supported in $D_{0,1}$, to $V_{2^{-4k}}(z/r_k)$, supported in D_{0,r_k}, and then translating along the diagonal, so that $V_{2^{-4k}}\left(\frac{z-R_ke^{\pi i/4}}{r_k}\right)$ is supported in $D_{R_ke^{\pi i/4},r_k}$. These support disks approach, but do not contain, the origin as $k\to\infty$, and are disjoint from each other, so the above infinite sum trivially converges for each $z \in \mathbb{C}$. Every point in $\mathbb{C}$ except the origin has a neighborhood intersecting at most one of these disks, so V is continuous and its partial derivatives exist on $\mathbb{C} \setminus \{0\}$.

Some of the disks $D_{R_k e^{\pi i/4}, r_k}$ may intersect the x and y axes, but by the choice of the cutoff function κ and the numerical inequality $e^{-1/2} < 1/\sqrt{2}$, the support of $V_{2^{-4k}}\left(\frac{z - R_k e^{\pi i/4}}{r_k}\right)$ is actually contained in $D_{R_k e^{\pi i/4}, r_k/\sqrt{2}}$, which is contained in the open first quadrant and disjoint from the x-axis and the y-axis. The partial derivatives of V exist at the origin, where $\frac{\partial V}{\partial x}(0) = \frac{\partial V}{\partial y}(0) = 0$, because $V \equiv 0$ along both the axes.

V is continuous at the origin, $\lim_{z\to 0} V(z) = V(0) = 0$, and in fact satisfies the stronger condition of complex differentiability at that point: $\lim_{z\to 0} \dfrac{V(z)}{z} = 0$. For z in the k^{th} disk, $V(z) = 0$ for $\left|z - R_k e^{\pi i/4}\right| > e^{-1/2} r_k$, and otherwise,

$$
\begin{aligned}
(3.4) \qquad \frac{|V(z)|}{|z|} &= \frac{\left|2^{-k} r_k V_{2^{-4k}}\left(\frac{z - R_k e^{\pi i/4}}{r_k}\right)\right|}{|z|} \\
&\leq \frac{2^{-k} r_k \max_{z\in\mathbb{C}} |V_{2^{-4k}}(z)|}{R_k - e^{-1/2} r_k} \\
&\leq \frac{2^{-k} e^{-1/2}}{\frac{R_k}{r_k} - e^{-1/2}} \leq \frac{2^{-k} e^{-1/2}}{1 - e^{-1/2}}.
\end{aligned}
$$

The derivative $\frac{\partial V}{\partial \bar{z}}$ is continuous at every point of $\mathbb{C}$, including the origin. To show

$$
\lim_{z\to 0} \frac{\partial V}{\partial \bar{z}}(z) = \frac{\partial V}{\partial \bar{z}}(0) = 0,
$$

apply the chain rule at an arbitrary point z_0 in any particular disk:

$$
\begin{aligned}
\frac{\partial V}{\partial \bar{z}}(z_0) &= \frac{\partial}{\partial \bar{z}}\left(2^{-k} r_k V_{2^{-4k}}\left(\frac{z - R_k e^{\pi i/4}}{r_k}\right)\right)\Bigg|_{z=z_0} \\
&= 2^{-k} r_k \frac{\partial V_{2^{-4k}}}{\partial \bar{z}}\left(\frac{z_0 - R_k e^{\pi i/4}}{r_k}\right)\frac{1}{r_k} \\
\implies \left|\frac{\partial V}{\partial \bar{z}}(z_0)\right| &\leq 2^{-k} B_2.
\end{aligned}
$$

Now, consider the z-derivative. For $0 < |z| < 1$, the expression:

$$\begin{aligned}\frac{\partial V_t}{\partial z} &= \frac{\partial}{\partial \bar{z}}(\kappa(|z|))z|z|^{2t}\sqrt{-\ln(|z|^2)}\\ &\quad +\kappa(|z|)(1+t)|z|^{2t}\sqrt{-\ln(|z|^2)}\\ &\quad -\kappa(|z|)\frac{1}{2}|z|^{2t}/\sqrt{-\ln(|z|^2)}\end{aligned} \tag{3.5}$$

shows that $\frac{\partial V_t}{\partial z}$ is continuous on $\mathbb{C}$. However, the coefficient $(1+t)$ in term (3.5) is significantly larger than the corresponding coefficient t in (3.3) as $t \to 0^+$; this gain is the key step in this example. The continuity of both $\frac{\partial V_t}{\partial z}$ and $\frac{\partial V_t}{\partial \bar{z}}$ imply that V_t is $\mathcal{C}^1$ on $\mathbb{C}$, and V is $\mathcal{C}^1$ on $\mathbb{C} \setminus \{0\}$. This, together with (3.4), shows that V is differentiable (real differentiable, in the sense of multivariable calculus) on $\mathbb{C}$.

To show $\frac{\partial V}{\partial z}$ is not locally bounded, define a sequence

$$z_k = r_k e^{\left(-2^{4k-2}\right)} + R_k e^{\pi i/4},$$

so $\lim\limits_{k\to\infty} z_k = 0$.

$$\begin{aligned}&\frac{\partial V}{\partial z}(z_k)\\ =\ & \frac{\partial}{\partial z}\left(2^{-k} r_k V_{2^{-4k}}\left(\frac{z - R_k e^{\pi i/4}}{r_k}\right)\right)\bigg|_{z=z_k}\\ =\ & 2^{-k} r_k \frac{\partial V_{2^{-4k}}}{\partial z}\left(e^{\left(-2^{4k-2}\right)}\right)\frac{1}{r_k}\\ =\ & 2^{-k}\left(0 + \left(1+2^{-4k}\right)\left|e^{\left(-2^{4k-2}\right)}\right|^{\left(2^{-4k+1}\right)}\sqrt{-\ln\left(\left|e^{\left(-2^{4k-2}\right)}\right|^2\right)}\right.\\ & \left. -\frac{1}{2}\left|e^{\left(-2^{4k-2}\right)}\right|^{\left(2^{-4k+1}\right)}/\sqrt{-\ln\left(\left|e^{\left(-2^{4k-2}\right)}\right|^2\right)}\right)\\ =\ & 2^{-k}\left(\left(1+2^{-4k}\right)e^{-1/2}\sqrt{2^{4k-1}} - \frac{1}{2}e^{-1/2}/\sqrt{2^{4k-1}}\right)\\ =\ & 2^k/\sqrt{2e}.\end{aligned}$$

It follows that the derivatives $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ are also not locally bounded.

Remark 3.4. The above piecewise construction, with smooth cutoffs and a sequence of exponents $t = 2^{-4k}$, is similar to examples constructed in [R] and [CP], of $\mathcal{C}^\infty$ vector valued functions where the $\bar{z}$-derivative is small compared to the z-derivative.

Remark 3.5. It is well-known that the Beurling transform $\mathcal{S} : v_{\bar{z}} \mapsto v_z$ need not preserve the $\mathcal{C}^0$ or L^∞ properties. Example 3.3 shows that this still holds even when $v_{\bar{z}}$ is the continuous derivative of a differentiable function.

Remark 3.6. In the theory of one real variable, the function $x^2 \sin(1/x^2)$ extends to a differentiable function with an unbounded derivative. We do not know of an analogous elementary expression in x and y with the same properties as $V(z)$. Any function where $v_{\bar{z}} = \frac{1}{2}(v_x + iv_y)$ is locally bounded, and $v_z = \frac{1}{2}(v_x - iv_y)$ is not, cannot be real valued; v must be complex valued.

4. The separable Cauchy-Riemann equation

Let Ω_1 and Ω_2 be open subsets of $\mathbb{C}$. Here we consider the "separable" case of the nonlinear Cauchy-Riemann equation where the RHS of (1.1) factors in the form:

$$\frac{\partial u}{\partial \bar{z}} = f(u)g(z), \tag{4.1}$$

for $u : \Omega_1 \to \Omega_2$, $g : \Omega_1 \to \mathbb{C}$, and where $f : \Omega_2 \to \mathbb{C}$ is holomorphic. We have already considered one separable equation in Corollary 2.8. The goal is to compute explicit (as in (4.4), (4.9)), or implicit (as in (4.10)) local formulas for all solutions u of (4.1) satisfying minimal regularity properties.

We consider three cases: first, where f is nonvanishing, Subsection 4.1 uses only results of single-variable complex analysis (as in [Conway] Ch. IV and [N]) without appealing to integral transforms as in Section 2. Second, where f has a simple zero, Subsection 4.2 solves an auxiliary ODE (4.7) to find a substitution that establishes existence and uniqueness for (4.1). It is not until the third case, where f has a zero of multiplicity greater than one, that we need to use the results of Section 2, in Subsection 4.3.

4.1. Nonvanishing f.

The following Lemma is an existence result; it is essentially the first-year calculus method for solving a separable first-order ODE.

Lemma 4.1. *For functions $f : \Omega_2 \to \mathbb{C}$ and $g : \Omega_1 \to \mathbb{C}$, suppose there exist a holomorphic function $F : \Omega_2 \to \mathbb{C}$ such that $\frac{\partial F}{\partial w} = \frac{1}{f(w)}$, and a continuous function $G : \Omega_1 \to \mathbb{C}$ such that the partial derivatives G_x, G_y exist and satisfy $\frac{\partial G}{\partial \bar{z}} = g(z)$. For any points $z_0 \in \Omega_1$, $w_0 \in \Omega_2$, there exists a non-constant function $u : \Omega_1^0 \to \Omega_2$ on some neighborhood of z_0, $\Omega_1^0 \subseteq \Omega_1$, such that $\frac{\partial u}{\partial \bar{z}} = f(u)g(z)$ and $u(z_0) = w_0$.*

Proof. For the existence of a primitive F, it is necessary that f is holomorphic and nonvanishing on Ω_2, and it would further be sufficient for Ω_2 to be simply connected.

Because $F'(w_0) = \frac{1}{f(w_0)} \neq 0$, there is some neighborhood Ω_2^0 of w_0 such that F is one-to-one on Ω_2^0, the image $F(\Omega_2^0)$ is an open subset of $\mathbb{C}$, and there is a holomorphic local inverse $H : F(\Omega_2^0) \to \Omega_2^0$. The derivative of H is $H'(\zeta) = \frac{1}{F'(H(\zeta))} = f(H(\zeta))$.

Let Ω_1^1 be any neighborhood of z_0 in Ω_1, and let $\theta : \Omega_1^1 \to \mathbb{C}$ be any holomorphic function. The following function,

$$G_1(z) = G(z) - G(z_0) + \theta(z) - \theta(z_0) + F(w_0), \tag{4.2}$$

is continuous on Ω_1^1 and satisfies $G_1(z_0) = F(w_0)$. The set

$$\Omega_1^0 = G_1^{-1}(F(\Omega_2^0)) = \{z \in \Omega_1^1 : G_1(x) \in F(\Omega_2^0)\}$$

is an open neighborhood of z_0, and is the domain of the composite function

$$u = H \circ (G_1|_{\Omega_1^0}). \tag{4.3}$$

By construction, $u(z_0) = w_0$, and

$$\frac{\partial u}{\partial \bar{z}} = H'(G_1(z)) \frac{\partial G_1}{\partial \bar{z}} = f(H(G_1(z)))g(z) = f(u)g(z).$$

Similarly,

$$\frac{\partial u}{\partial z} = H'(G_1(z)) \frac{\partial G_1}{\partial z} = f(u) \left(\frac{\partial G}{\partial z} + \frac{\partial \theta}{\partial z} \right),$$

and θ can be chosen so that the derivative is non-zero at z_0. ∎

The above method constructs a local solution, of the form $u = H \circ G_1$, which has the same $\mathcal{C}^r$ regularity as G. Theorem 4.3 shows all continuous solutions are locally of the same form, using the following Lemma and the $(*)$ property.

Lemma 4.2. *For f and F as in Lemma 4.1 and any $g : \Omega_1 \to \mathbb{C}$, suppose $u : \Omega_1 \to \Omega_2$ and $v : \Omega_1 \to \Omega_2$ are continuous functions both satisfying property $(*)$, and*

$$\frac{\partial u}{\partial \bar{z}} = f(u)g(z), \quad \frac{\partial v}{\partial \bar{z}} = f(v)g(z)$$

almost everywhere in Ω_1. Then there exists $C : \Omega_1 \to \mathbb{C}$ which is holomorphic and satisfies $F(v(z)) = F(u(z)) + C(z)$.

Proof. Applying the chain rule off the union (still countable) of the exceptional sets for u and v from $(*)$, and Proposition 2.5,

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}}(F(v)-F(u)) &= F'(v(z))\frac{\partial v}{\partial \bar{z}} - F'(u(z))\frac{\partial u}{\partial \bar{z}} \\
&=_{a.e.} \frac{1}{f(v(z))}f(v(z))g(z) - \frac{1}{f(u(z))}f(u(z))g(z) \\
&\equiv 0 \\
\implies F(v)-F(u) &= C(z).
\end{aligned}$$

∎

The following Theorem is stated as a regularity result, but our main interest is in uniqueness — showing that, for nonvanishing f, all continuous solutions of (4.1) that satisfy $(*)$ must be locally of the form (4.4).

Theorem 4.3. *Let $f : \Omega_2 \to \mathbb{C}$ be holomorphic, and suppose $g : \Omega_1 \to \mathbb{C}$ is equal to $\frac{\partial G}{\partial \bar{z}}$ for some $G \in \mathcal{C}^r(\Omega_1)$, $r = 0, 1, 2, \ldots, \infty, \omega$. For a continuous function $v : \Omega_1 \to \Omega_2$, define the open set $\Omega_0 = \{z \in \Omega_1 : f(v(z)) \neq 0\}$. If v satisfies $(*)$ on Ω_0 and $\frac{\partial v}{\partial \bar{z}} = f(v)g(z)$ almost everywhere in Ω_0, then $v \in \mathcal{C}^r(\Omega_0)$.*

Proof. Let z_0 be an arbitrary point in Ω_0, so $f(v(z_0)) \neq 0$ and there is some simply connected neighborhood of $v(z_0)$, $\Omega_2^1 \subseteq \Omega_2$, so that f is nonvanishing on Ω_2^1. There exists a holomorphic $F : \Omega_2^1 \to \mathbb{C}$ such that $\frac{\partial F}{\partial w} = \frac{1}{f(w)}$. Let $\Omega_0^1 = v^{-1}(\Omega_2^1)$, so Ω_0^1 is an open neighborhood of z_0 in Ω_0. Lemma 4.1 applies to the restrictions $g : \Omega_0^1 \to \mathbb{C}$ and $f : \Omega_2^1 \to \mathbb{C}$. There exists a solution $u : \Omega_0^2 \to \Omega_2^1$ on some neighborhood of z_0, $\Omega_0^2 \subseteq \Omega_0^1$, such that $\frac{\partial u}{\partial \bar{z}} = f(u)g(z)$, $u(z_0) = v(z_0)$, and $F \circ u = G_1|_{\Omega_0^2}$, where $G_1(z) = G(z) - G(z_0) + F(v(z_0))$ (from (4.2) with $\theta \equiv 0$; for this Theorem, u is not necessarily non-constant.) From (4.3), where u is defined as a composite of a holomorphic function with G_1, $u \in \mathcal{C}^r(\Omega_0^2)$. By Lemma 4.2, there exists a holomorphic function $C : \Omega_0^2 \to \mathbb{C}$ such that $F(v(z)) = F(u(z)) + C(z)$ and $C(z_0) = 0$. As in the Proof of Lemma 4.1, there is some neighborhood of $v(z_0)$, $\Omega_2^2 \subseteq \Omega_2^1$, where F is one-to-one, so $F(\Omega_2^2)$ is open in $\mathbb{C}$ and $H : F(\Omega_2^2) \to \Omega_2^2$ is a holomorphic local inverse of F. Define this open neighborhood of z_0,

$$\Omega_0^3 = (F \circ u + C)^{-1}(F(\Omega_2^2)) = \{z \in \Omega_0^2 : F(u(z)) + C(z) \in F(\Omega_2^2)\}.$$

Then, for all $z \in \Omega_0^3$, $F(v(z)) = F(u(z)) + C(z) \in F(\Omega_2^2)$, and plugging into H gives

$$\begin{aligned}
v(z) &= H(F(u(z)) + C(z)) \\
&= H(G_1(z) + C(z)).
\end{aligned} \tag{4.4}$$

It follows from (4.4) that $v \in \mathcal{C}^r(\Omega_0^3)$, which since z_0 was arbitrary, is enough to show $v \in \mathcal{C}^r(\Omega_0)$. ∎

Example 4.4. Let $f(w) = e^w$, $g(z) \equiv 1$, and choose $F(w) = -e^{-w}$, $G(z) = \bar{z}$. Let Ω_2^0 be a neighborhood of $w_0 \in \mathbb{C}$ where F is one-to-one, so there is a branch of the complex logarithm which is a holomorphic local inverse of F, $H(\zeta) = -\mathrm{Log}(-\zeta)$. Then, for any $z_0 \in \mathbb{C}$, if Ω_1 is a neighborhood of z_0, and $v : \Omega_1 \to \mathbb{C}$ is continuous, satisfies $(*)$, is a solution of

$$\frac{\partial v}{\partial \bar{z}} = e^v$$

almost everywhere in Ω_1, with initial condition $v(z_0) = w_0$, then by Theorem 4.3, v is real analytic on Ω_1, and locally near z_0,

$$v(z) = -\mathrm{Log}\left(-(\overline{(z - z_0)} + C(z) - e^{-w_0})\right),$$

for some holomorphic function $C(z)$ with $C(z_0) = 0$. Conversely, choosing any such C gives an example of a local solution. One such solution, with $z_0 = w_0 = 0$ and $C(z) = z$, is real valued on the domain $\{\mathrm{Re}(z) < \frac{1}{2}\}$,

$$v(x + iy) = -\ln(-2x + 1).$$

The level sets are lines, unlike the isolated points as in Corollary 2.8.

4.2. **Simple zeros of f.**

Informally considering the equation $\frac{\partial u}{\partial \bar{z}} = ug(z)$, the obvious solutions are of the form $u(z) = B(z)\exp(G(z))$, where B is holomorphic and G is a $\bar{z}$-antiderivative of g. To apply this idea to the more general separable equation $\frac{\partial u}{\partial \bar{z}} = f(u)g(z)$, where f has a simple zero, the following Lemma leads to a useful substitution.

Lemma 4.5. *Given $f : \Omega_2 \to \mathbb{C}$ holomorphic, with a simple zero at w_0, there exist a disk D_{0,r_0} and a holomorphic function $h : D_{0,r_0} \to \Omega_2$ such that $h(0) = w_0$, $h : D_{0,r_0} \to h(D_{0,r_0})$ is invertible, and for $\zeta \in D_{0,r_0}$,*

$$f'(w_0)\zeta h'(\zeta) = f(h(\zeta)). \tag{4.5}$$

Proof. On some disk $D_{w_0,r_1} \subseteq \Omega_2$, $f(w) = \sum_{j=1}^{\infty} f_j (w - w_0)^j$, where by hypothesis, $f_1 = f'(w_0) \neq 0$. On D_{w_0,r_1}, define

$$\tilde{f}(w) = \sum_{j=2}^{\infty} \left(\frac{f_j}{f_1}\right)(w - w_0)^j,$$

so $\tilde{f}$ is holomorphic and $f(w) = f_1(w - w_0 + \tilde{f}(w))$. For some $r_2 > 0$, $r_3 > 0$ with $r_2(1 + r_3) < r_1$, the following two-variable function is holomorphic and bounded on the bidisk $D_{0,r_2} \times D_{0,r_3} \subseteq \mathbb{C}^2$, defined by an absolutely convergent power series:

$$\begin{aligned} \mathbf{F}(W_1, W_2) &= \begin{cases} \frac{1}{W_1^2}\tilde{f}(w_0 + W_1 + W_1W_2) & W_1 \neq 0 \\ \frac{f_2}{f_1}(1 + W_2)^2 & W_1 = 0 \end{cases} \\ &= \sum_{j=2}^{\infty} \left(\frac{f_j}{f_1}\right) W_1^{j-2}(1 + W_2)^j = \sum_{j,\ell} F_{j\ell} W_1^j W_2^\ell. \end{aligned} \tag{4.6}$$

The differential equation

$$H'(\zeta) = \mathbf{F}(\zeta, H(\zeta)), \tag{4.7}$$

with initial condition $H(0) = 0$, has a formal solution $H(\zeta) = \sum_{j=1}^{\infty} H_j\zeta^j$, where the coefficient sequence H_j is defined uniquely by the coefficients $F_{j\ell}$, which uniquely depend on $f_1, f_2, \dots$ by re-centering the power series in step (4.6). The series is convergent ([H] Theorem 2.5.1, proved by a majorization method), so H is holomorphic on some disk D_{0,r_4}. Define

$$h(\zeta) = w_0 + \zeta + \zeta H(\zeta),$$

so h is holomorphic and invertible on some disk D_{0,r_0}, and by construction, satisfies:

$$\begin{aligned} f'(w_0)\zeta h'(\zeta) &= f_1\zeta(1 + H(\zeta) + \zeta H'(\zeta)) \\ &= f_1(\zeta + \zeta H(\zeta) + \zeta^2\mathbf{F}(\zeta, H(\zeta))) \\ &= f_1(h(\zeta) - w_0 + \tilde{f}(w_0 + \zeta + \zeta H(\zeta))) \\ &= f(h(\zeta)). \end{aligned}$$

∎

Remark 4.6. Equation (4.5) is a special case of an equation considered by [H] §11.1; the above proof shows how h can be computed in terms of f.

Theorem 4.7. *Given $f : \Omega_2 \to \mathbb{C}$ holomorphic with only simple zeros, $g : \Omega_1 \to \mathbb{C}$, and any points $z_0 \in \Omega_1$, $w_0 \in \Omega_2$, if there is a continuous function $G : \Omega_1 \to \mathbb{C}$ such that $\frac{\partial G}{\partial \bar{z}} = g(z)$, then there exists a non-constant, continuous function $u : \Omega_1^0 \to \Omega_2$ on some neighborhood of z_0, $\Omega_1^0 \subseteq \Omega_1$, such that $\frac{\partial u}{\partial \bar{z}} = f(u)g(z)$ and $u(z_0) = w_0$.*

Proof. For the case where f is non-vanishing at w_0, Lemma 4.1 applies locally near w_0, so we assume that f has a zero of order 1 at w_0.

Let $f_1 = f'(w_0)$ and $h : D_{0,r_0} \to \Omega_2$ be as in Lemma 4.5. The function $U(z) = (z - z_0)\exp(f_1 G(z))$ is continuous on Ω_1, with $U(z_0) = 0$. Let $\Omega_1^0 = U^{-1}(D_{0,r_0})$, and define $u = h \circ (U|_{\Omega_1^0})$, so that

$$u(z) = h(U(z)) = h((z - z_0)\exp(f_1 G(z))) \tag{4.8}$$

is continuous on Ω_1^0, and its partial derivatives satisfy, using (4.5):

$$\begin{aligned}
\frac{\partial u}{\partial \bar z} &= h'((z - z_0)\exp(f_1 G(z)))(z - z_0)\exp(f_1 G(z)) f_1 \frac{\partial G}{\partial \bar z} \\
&= f(h((z - z_0)\exp(f_1 G(z))))g(z) = f(u)g(z), \\
\left.\frac{\partial u}{\partial z}\right|_{z=z_0} &= h'(0)\exp(f_1 G(z_0)) \neq 0.
\end{aligned}$$

∎

As remarked after Lemma 4.1, for any f, the solution u constructed in (4.8) has the same $\mathcal{C}^r$ regularity as the antiderivative G.

The following Theorem is a generalization of Theorem 4.3; again our interest is in computing a local formula (4.9) for any continuous solution.

Theorem 4.8. *Let $f : \Omega_2 \to \mathbb{C}$ be holomorphic, with only simple zeros, and suppose $g : \Omega_1 \to \mathbb{C}$ is equal to $\frac{\partial G}{\partial \bar z}$ for some $G \in \mathcal{C}^r(\Omega_1)$, $r = 0, 1, 2, \ldots, \infty, \omega$. If $v : \Omega_1 \to \Omega_2$ is continuous, satisfies $(*)$, and $\frac{\partial v}{\partial \bar z} = f(v)g(z)$ almost everywhere in Ω_1, then $v \in \mathcal{C}^r(\Omega_1)$.*

Proof. Let z_0 be an arbitrary point in Ω_0. If $f(v(z_0)) \neq 0$, then Theorem 4.3 applies, to show that there is some neighborhood of z_0 where v is $\mathcal{C}^r$. Otherwise, $w_0 = v(z_0)$ is a simple zero of f, so Lemma 4.5 applies to give h on D_{0,r_0}. Let $\Omega_1^1 = v^{-1}(h(D_{0,r_0}))$ be a neighborhood of z_0 in Ω_1. The function $B : \Omega_1^1 \to \mathbb{C}$ defined by

$$B(z) = h^{-1}(v(z))\exp(-f_1 G(z))$$

is continuous, satisfies $(*)$ and $B(z_0) = 0$, and almost everywhere in Ω_1^1,

$$\begin{aligned}
&\frac{\partial B}{\partial \bar z} \\
=_{a.e.}\ & \frac{\partial v/\partial \bar z \exp(-f_1 G(z))}{h'(h^{-1}(v(z)))} + h^{-1}(v(z))\exp(-f_1 G(z))(-f_1)\frac{\partial G}{\partial \bar z} \\
=_{a.e.}\ & \frac{f(v)g(z)\exp(-f_1 G(z))}{h'(h^{-1}(v(z)))} - B(z) f_1 g(z) \\
=\ & f_1 h^{-1}(v(z))g(z)\exp(-f_1 G(z)) - B(z) f_1 g(z) \equiv 0.
\end{aligned}$$

The conclusion is that, on Ω_1^1, B is holomorphic by Proposition 2.5, and

$$v(z) = h(B(z)\exp(f_1 G(z))) \tag{4.9}$$

is $\mathcal{C}^r$, so v is $\mathcal{C}^r$ on a neighborhood of every point in Ω_1. ∎

Any holomorphic B with $B(z_0) = 0$ in (4.9) gives a local solution v, by the same calculation as the example in Theorem 4.7.

4.3. **Zeros of f with higher multiplicity.**

The result of the following Theorem is a local implicit formula (4.10) for continuous solutions u of the separable equation $u_{\bar{z}} = f(u)g(z)$, when f has a zero of order > 1. Unlike the expression from Equation (4.9), involving a substitution function h depending on f, the expression in Equation (4.10) uses only antiderivatives F and G of the given factors f and g.

Theorem 4.9. *Given an open set $\Omega_1 \subseteq \mathbb{C}$, $p > 2$, and $g \in L^p_{loc}(\Omega_1)$, suppose there is some $G : \Omega_1 \to \mathbb{C}$ so that G is continuous, satisfies $(*)$, and $\frac{\partial G}{\partial \bar{z}} = g(z)$ almost everywhere. Let $f : \Omega_2 \to \mathbb{C}$ be continuous with $f(w_0) = 0$, and suppose for some disk $D_{w_0,r} \subseteq \Omega_2$, there is $F : D_{w_0,r} \setminus \{w_0\} \to \mathbb{C}$ so that F is holomorphic and satisfies $F'(w) = \frac{1}{f(w)}$. For any $z_0 \in \Omega_1$, if there exist a neighborhood of z_0, $\Omega_1^0 \subseteq \Omega_1$, and a non-constant, continuous function $u : \Omega_1^0 \to \Omega_2$ satisfying $(*)$, $u(z_0) = w_0$, and $\frac{\partial u}{\partial \bar{z}} = f(u)g(z)$ almost everywhere on Ω_1^0, then there exist an integer $M \geq 1$ and a nonvanishing holomorphic function $\phi(z)$ on some neighborhood of z_0, with*

$$(z - z_0)^M F(u(z)) = \phi(z) + (z - z_0)^M G(z). \tag{4.10}$$

Proof. From $F' = \frac{1}{f}$, f is holomorphic and nonvanishing on $D_{w_0,r} \setminus \{w_0\}$, and because f is continuous, f is holomorphic on $D_{w_0,r}$. Let $k \geq 1$ be the order of vanishing of $f(w)$ at w_0, so there is a series expression converging on $D_{w_0,r}$,

$$f(w) = (w - w_0)^k (f_k + \sum_{j=k+1}^{\infty} f_j (w - w_0)^{j-k}) \tag{4.11}$$

with $f_k \neq 0$. The reciprocal has a Laurent expansion

$$\frac{1}{f(w)} = (w - w_0)^{-k} (\frac{1}{f_k} + \sum_{\ell=1}^{\infty} q_\ell (w - w_0)^\ell). \tag{4.12}$$

The existence of the primitive F is equivalent to $k > 1$ and $q_{k-1} = 0$ (this is the Residue of $\frac{1}{f}$ at w_0). By integrating the above Laurent series, any holomorphic primitive F has a pole of order exactly $k-1$ at w_0. So,

$$(w-w_0)^{k-1}F(w) \tag{4.13}$$

extends to a holomorphic function on $D_{w_0,r}$, which is nonvanishing on some possibly smaller disk D_{w_0,r_0}; denote the extension $\tilde{F}: D_{w_0,r_0} \to \mathbb{C}\setminus\{0\}$.

Let $R_0 \Subset u^{-1}(D_{w_0,r_0}) \subseteq \Omega_1^0$ be a neighborhood of z_0, so that, using (4.11), u satisfies the hypotheses of Corollary 2.8 on R_0. As in (2.10), on a neighborhood of z_0, $R_1 \Subset R_0$, $u(z)-w_0 = e^U\sigma$, where σ is holomorphic on R_1 and $U \in \mathcal{C}^{0,\alpha}(R_1)$. On R_1, the composite $\tilde{F}(u(z))$ is continuous and nonvanishing. There is a neighborhood of z_0, $\Omega_1^1 \subseteq R_1$, where z_0 is the only point where $u(z)=w_0$, the holomorphic factor σ has a series expansion at z_0 with order of vanishing $m \geq 1$, and

$$u(z)-w_0 = (z-z_0)^m p(z), \tag{4.14}$$

for some nonvanishing continuous function $p(z)$. An expression for $\tilde{F}(u(z))$ can be computed on $\Omega_1^1\setminus\{z_0\}$, using (4.13) and (4.14):

$$\tilde{F}(u(z)) = (u(z)-w_0)^{k-1}F(u(z)) = ((z-z_0)^m p(z))^{k-1}F(u(z)).$$

It follows that the product

$$(z-z_0)^{m(k-1)}F(u(z))$$

extends from $\Omega_1^1\setminus\{z_0\}$ to a nonvanishing, continuous function on Ω_1^1, and by the continuity of G, the expression

$$\phi = (z-z_0)^{m(k-1)}F(u(z)) - (z-z_0)^{m(k-1)}G(z)$$

is also nonvanishing and continuous on some neighborhood of z_0, $\Omega_1^2 \subseteq \Omega_1^1$. For all z except z_0 and possibly countably many more from the exceptional sets from $(*)$ for u and G,

$$\begin{aligned}
\frac{\partial}{\partial\bar{z}}\phi(z) &= \frac{\partial}{\partial\bar{z}}\left((z-z_0)^{m(k-1)}(F(u(z))-G(z))\right)\\
&= (z-z_0)^{m(k-1)}\left(F'(u(z))\frac{\partial u}{\partial\bar{z}} - \frac{\partial G}{\partial\bar{z}}\right)\\
&=_{a.e.} (z-z_0)^{m(k-1)}\left(\frac{1}{f(u(z))}f(u(z))g(z) - g(z)\right) \equiv 0,
\end{aligned}$$

so by Proposition 2.5, ϕ is holomorphic on Ω_1^2. ∎

Note that the exponent $M = m(k-1)$ depends on the order of vanishing of u and f, but not on the choices of primitives F and G.

Example 4.10. Let $0 < \alpha < 1$, $g = |z|^{-1+\alpha} \in L^p_{loc}(\mathbb{C})$, $f(w) = w^2$, and $w_0 = 0$, so $k = 2$ and $p > 2$. Choose antiderivatives $F(w) = -w^{-1}$, and $G(z) = \frac{2}{1+\alpha}\bar{z}|z|^{-1+\alpha}$ extended to $G(0) = 0$. Then for any z_0, if u is a non-constant, continuous solution of

$$\frac{\partial u}{\partial \bar{z}} = u^2|z|^{-1+\alpha} \tag{4.15}$$

almost everywhere, satisfying $(*)$ and $u(z_0) = 0$, then there exist a positive integer m and a holomorphic function ϕ with $\phi(z_0) \neq 0$, so that for non-zero z near z_0,

$$(z - z_0)^m(-u(z))^{-1} = \phi(z) + (z - z_0)^m G(z) \tag{4.16}$$

$$\implies u(z) = \frac{-(z - z_0)^m}{\phi(z) + (z - z_0)^m \frac{2}{1+\alpha}\bar{z}|z|^{-1+\alpha}}. \tag{4.17}$$

In this case, choosing any m and ϕ gives an example of a local solution u with order of vanishing m as in (4.14). When extended by continuity to $u(0) = \frac{-(-z_0)^m}{\phi(0)}$, u is Hölder continuous on rectangles, as in Theorem 2.6, and if g and u are restricted to a domain not containing $z = 0$, then g and u are real analytic, as in Corollary 2.7.

Example 4.11. If, in Example 4.10, $\alpha = 1$, then (4.15) becomes the autonomous equation $\frac{\partial u}{\partial \bar{z}} = u^2$. All solutions with initial condition $u(z_0) = 0$ are real analytic, but the form of the solution set does not change: non-constant solutions still satisfy (4.17), with $\alpha = 1$. Equations with higher powers, $\frac{\partial u}{\partial \bar{z}} = u^k$, have implicit solutions (4.10) similar to (4.16), but require selecting a local root to get an explicit solution for u as in (4.17).

5. An application to almost complex geometry

5.1. Normal coordinates in $\mathbb{R}^4$.

Let $J(\vec{x})$ be a smooth almost complex structure on a neighborhood of the origin in $\mathbb{R}^4$. For example, if J_{std} is the 2×2 constant matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then the constant matrix $J_0 = \begin{bmatrix} J_{std} & 0 \\ 0 & J_{std} \end{bmatrix}_{4\times 4}$ is the standard complex structure operator for $\mathbb{C}^2 = (\mathbb{R}^4, J_0)$.

For an open set $\Omega \subseteq \mathbb{C} = (\mathbb{R}^2, J_{std})$, a J-holomorphic curve is a differentiable map $\mathbf{u} : \Omega \to \mathbb{R}^4$, so that the differential $d\mathbf{u}$ satisfies $d\mathbf{u}(x, y) \circ J_{std} = J(\mathbf{u}(x, y)) \circ d\mathbf{u}(x, y)$.

We very briefly recall the geometric construction of "normal coordinates" from [S], [ST], [T], but then, starting with Equation (5.2), go into some detail regarding computations in this coordinate system.

Near a given point Z_0 on an embedded J-holomorphic curve $\mathbf{u}$, there exists a family of local perturbations of the curve, parametrized by a complex variable w, which together with a complex coordinate ζ for the original curve, defines a smooth local coordinate system (ζ, w), with Z_0 at the origin. The matrix representation of J in this coordinate system is:

$$J(\zeta, w) = \begin{bmatrix} J_{std} & B_1 \\ 0 & J_{std} + B_2 \end{bmatrix}, \tag{5.1}$$

where the blocks B_1, B_2 are smooth 2×2 matrix functions of the coordinates (ζ, w), satisfying $B_1(\zeta, 0) = 0$ and $B_2(0,0) = 0$, so $J(0,0) = J_0$. By construction, the previously given curve $\mathbf{u}$ in these coordinates is the complex ζ-axis, parametrized by $z \mapsto (z, 0)$, and the nearby J-holomorphic curves are parametrized by $z \mapsto (z, c)$, for complex constants c. The mapping $z \mapsto (\zeta, w) = (h(z), c)$ is J-holomorphic for any holomorphic h and constant c.

The real entries of the 4×4 matrix (5.1) (depending on ζ, w) are constrained by the property $J^2 = -Id_{\mathbb{R}^4}$, so they must be of the following form. It can be assumed that $|b_2| < 1$ for (ζ, w) near $\vec{0}$:

$$J(\zeta, w) = \begin{bmatrix} 0 & -1 & a_1 & a_2 \\ 1 & 0 & \frac{a_1 b_1 b_2 - a_2 b_1^2 - a_1 b_1 - a_2}{b_2 - 1} & a_1 b_2 - a_2 b_1 - a_1 \\ 0 & 0 & b_1 & -1 + b_2 \\ 0 & 0 & 1 + \frac{b_1^2 + b_2}{1 - b_2} & -b_1 \end{bmatrix}. \tag{5.2}$$

The $-i$ eigenspace can be calculated (5.9) and then written in complex coordinates with smooth complex coefficients β_1, β_2:

$$\begin{aligned} T^{0,1} &= \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{\zeta}}, \frac{\partial}{\partial \bar{w}} + \beta_1 \frac{\partial}{\partial w} + \beta_2 \frac{\partial}{\partial \zeta} \right\} \\ \beta_1(\zeta, w) &= \frac{b_2 - i b_1}{b_2 - 2 + i b_1} \\ \beta_2(\zeta, w) &= \frac{a_2 + i(a_1 b_2 - a_2 b_1 - a_1)}{b_2 - 2 + i b_1}. \end{aligned} \tag{5.3}$$

Conversely, given complex coefficients β_1, β_2 in an expression of the form (5.3) with $|\beta_1| < 1$, the real entries a_1, a_2, b_1, b_2 in a complex structure operator of the form (5.2) are uniquely determined by:

$$\begin{aligned} a_1 + i a_2 &= \frac{2i(\beta_1 \overline{\beta_2} + \beta_2)}{\beta_1 \overline{\beta_1} - 1} \\ b_1 + i b_2 &= \frac{2i \beta_1 (\overline{\beta_1} + 1)}{\beta_1 \overline{\beta_1} - 1}. \end{aligned}$$

In terms of β_1, β_2, the matrix (5.2) for $J(\zeta, w)$ is:

$$\left[\begin{array}{cccc} 0 & -1 & \frac{2(\mathrm{Im}(\beta_2)\mathrm{Re}(\beta_1)-\mathrm{Im}(\beta_1)\mathrm{Re}(\beta_2)-\mathrm{Im}(\beta_2))}{|\beta_1|^2-1} & \frac{2(\mathrm{Im}(\beta_2)\mathrm{Im}(\beta_1)+\mathrm{Re}(\beta_2)\mathrm{Re}(\beta_1)+\mathrm{Re}(\beta_2))}{|\beta_1|^2-1} \\ 1 & 0 & -\frac{2(\mathrm{Im}(\beta_2)\mathrm{Im}(\beta_1)+\mathrm{Re}(\beta_2)\mathrm{Re}(\beta_1)-\mathrm{Re}(\beta_2))}{|\beta_1|^2-1} & \frac{2(\mathrm{Im}(\beta_2)\mathrm{Re}(\beta_1)-\mathrm{Im}(\beta_1)\mathrm{Re}(\beta_2)+\mathrm{Im}(\beta_2))}{|\beta_1|^2-1} \\ 0 & 0 & -\frac{2\mathrm{Im}(\beta_1)}{|\beta_1|^2-1} & -1+\frac{2(|\beta_1|^2+\mathrm{Re}(\beta_1))}{|\beta_1|^2-1} \\ 0 & 0 & 1-\frac{2(|\beta_1|^2-\mathrm{Re}(\beta_1))}{|\beta_1|^2-1} & \frac{2\mathrm{Im}(\beta_1)}{|\beta_1|^2-1} \end{array}\right].$$

The integrability condition $[T^{0,1}, T^{0,1}] \subseteq T^{0,1}$ is satisfied when $\frac{\partial \beta_1}{\partial \bar{\zeta}}$ and $\frac{\partial \beta_2}{\partial \bar{\zeta}}$ are both 0, so β_1, β_2 are holomorphic in ζ.

If $\vec{f} : \Omega \to \mathbb{R}^4$ is a real variable parametrization, $\vec{f}(x, y) = (f^1, f^2, f^3, f^4)$, of a J-holomorphic curve in a neighborhood of $\vec{0} \in \mathbb{R}^4$, then

$$\begin{align} d\vec{f}(x,y) \circ J_{std} &= J(\vec{f}(x,y)) \circ d\vec{f}(x,y) \tag{5.4} \\ \implies \frac{\partial \vec{f}}{\partial y} &= J(\vec{f}(x,y))\frac{\partial \vec{f}}{\partial x}. \tag{5.5} \end{align}$$

If the parametric equation is written in complex form as

$$\mathbf{u} : z \mapsto (\zeta, w) = (h(z), k(z)) = (f^1 + if^2, f^3 + if^4), \tag{5.6}$$

then the $\bar{z}$-derivatives of the components are related to the z-derivatives using a 2×2 complex matrix $\mathbf{Q}(\zeta, w)$, in the following complex non-linear system of equations:

$$\left[\begin{array}{c} h_{\bar{z}} \\ k_{\bar{z}} \end{array}\right] = \mathbf{Q}(h,k) \left[\begin{array}{c} \overline{h_z} \\ \overline{k_z} \end{array}\right]. \tag{5.7}$$

The calculation deriving $\mathbf{Q}$ in terms of J is well-known ([IS$_2$], [S]). However, in this coordinate system, it is more convenient to express the entries of $\mathbf{Q}$ in terms of the coefficients β_1, β_2 from the complex eigenvectors (5.3).

Lemma 5.1. *In a coordinate system for a neighborhood of $\mathbb{R}^4$ where J is of the form* (5.1) *with complex eigenvectors as in* (5.3)*, the matrix $\mathbf{Q}$ from* (5.7) *is of the form*

$$\mathbf{Q}(\zeta, w) = \left[\begin{array}{cc} 0 & \beta_2(\zeta, w) \\ 0 & \beta_1(\zeta, w) \end{array}\right]. \tag{5.8}$$

Proof. The diagonalizing matrix of eigenvectors, its inverse, and the diagonalization of J are:

$$
\begin{aligned}
P &= \begin{bmatrix} 1 & 1 & \beta_2 & \bar{\beta}_2 \\ i & -i & -i\beta_2 & i\bar{\beta}_2 \\ 0 & 0 & 1+\beta_1 & 1+\bar{\beta}_1 \\ 0 & 0 & i-i\beta_1 & -i+i\bar{\beta}_1 \end{bmatrix}, \qquad (5.9) \\
P^{-1} &= \frac{1}{2}\begin{bmatrix} 1 & -i & \frac{\bar{\beta}_2(1-\beta_1)}{\beta_1\bar{\beta}_1-1} & \frac{i\bar{\beta}_2(1+\beta_1)}{\beta_1\bar{\beta}_1-1} \\ 1 & i & \frac{\beta_2(1-\bar{\beta}_1)}{\beta_1\bar{\beta}_1-1} & \frac{-i\beta_2(1+\bar{\beta}_1)}{\beta_1\bar{\beta}_1-1} \\ 0 & 0 & \frac{\bar{\beta}_1-1}{\beta_1\bar{\beta}_1-1} & \frac{i(1+\bar{\beta}_1)}{\beta_1\bar{\beta}_1-1} \\ 0 & 0 & \frac{\beta_1-1}{\beta_1\bar{\beta}_1-1} & \frac{-i(1+\beta_1)}{\beta_1\bar{\beta}_1-1} \end{bmatrix}, \\
D &= \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix}.
\end{aligned}
$$

Then, from (5.5),

$$
\frac{\partial \vec{f}}{\partial y} = J(\vec{f}(x,y))\frac{\partial \vec{f}}{\partial x} = PDP^{-1}\frac{\partial \vec{f}}{\partial x},
$$

and this equality of vectors follows:

$$
\begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix} P^{-1} \begin{bmatrix} f_x^1 \\ f_x^2 \\ f_x^3 \\ f_x^4 \end{bmatrix} = P^{-1} \begin{bmatrix} f_y^1 \\ f_y^2 \\ f_y^3 \\ f_y^4 \end{bmatrix}
$$

$$
\begin{aligned}
&\Longrightarrow \begin{bmatrix} -if_x^1 - f_x^2 - i\frac{\bar{\beta}_2(1-\beta_1)}{\beta_1\bar{\beta}_1-1}f_x^3 + \frac{\bar{\beta}_2(1+\beta_1)}{\beta_1\bar{\beta}_1-1}f_x^4 \\ if_x^1 - f_x^2 + i\frac{\beta_2(1-\bar{\beta}_1)}{\beta_1\bar{\beta}_1-1}f_x^3 + \frac{\beta_2(1+\bar{\beta}_1)}{\beta_1\bar{\beta}_1-1}f_x^4 \\ -i\frac{\bar{\beta}_1-1}{\beta_1\bar{\beta}_1-1}f_x^3 + \frac{1+\bar{\beta}_1}{\beta_1\bar{\beta}_1-1}f_x^4 \\ i\frac{\beta_1-1}{\beta_1\bar{\beta}_1-1}f_x^3 + \frac{1+\beta_1}{\beta_1\bar{\beta}_1-1}f_x^4 \end{bmatrix} \qquad (5.10) \\
&= \begin{bmatrix} f_y^1 - if_y^2 + \frac{\bar{\beta}_2(1-\beta_1)}{\beta_1\bar{\beta}_1-1}f_y^3 + i\frac{\bar{\beta}_2(1+\beta_1)}{\beta_1\bar{\beta}_1-1}f_y^4 \\ f_y^1 + if_y^2 + \frac{\beta_2(1-\bar{\beta}_1)}{\beta_1\bar{\beta}_1-1}f_y^3 - i\frac{\beta_2(1+\bar{\beta}_1)}{\beta_1\bar{\beta}_1-1}f_y^4 \\ \frac{\bar{\beta}_1-1}{\beta_1\bar{\beta}_1-1}f_y^3 + i\frac{1+\bar{\beta}_1}{\beta_1\bar{\beta}_1-1}f_y^4 \\ \frac{\beta_1-1}{\beta_1\bar{\beta}_1-1}f_y^3 - i\frac{1+\beta_1}{\beta_1\bar{\beta}_1-1}f_y^4 \end{bmatrix}. \qquad (5.11)
\end{aligned}
$$

The first and second entries in each vector (5.10), (5.11), are complex conjugates, and the third and fourth entries are also conjugates, so for $|\beta_1| \neq 1$, the above vector equality is equivalent to a system of two complex equations (5.12), (5.13). Setting the fourth entries of (5.10), (5.11) equal and multiplying by $|\beta_1|^2 - 1$:

$$\begin{aligned} i(\beta_1 - 1)f^3_x + (1+\beta_1)f^4_x &= (\beta_1 - 1)f^3_y - i(1+\beta_1)f^4_y \\ \implies \frac{\partial}{\partial \bar z}(f^3 + if^4) &= \beta_1(\vec f(x,y)) \cdot \overline{\frac{\partial}{\partial z}(f^3+if^4)}. \end{aligned} \tag{5.12}$$

Setting the second entries of (5.10), (5.11) equal and multiplying by $|\beta_1|^2 - 1$:

$$\begin{aligned} & (\beta_1\bar\beta_1 - 1)(if^1_x - f^2_x) - i\beta_2(\bar\beta_1 - 1)f^3_x + \beta_2(1+\bar\beta_1)f^4_x \\ = \ & (\beta_1\bar\beta_1 - 1)(f^1_y + if^2_y) - \beta_2(\bar\beta_1 - 1)f^3_y - i\beta_2(1+\bar\beta_1)f^4_y \end{aligned} \tag{5.13}$$

$$\begin{aligned} \implies \frac{\partial}{\partial \bar z}(f^1 + if^2) &= \frac{1}{1-\beta_1\bar\beta_1}\left(-\beta_2\bar\beta_1\frac{\partial}{\partial \bar z}(f^3+if^4) + \beta_2\overline{\frac{\partial}{\partial z}(f^3+if^4)}\right) \\ &= \beta_2(\vec f(x,y)) \cdot \overline{\frac{\partial}{\partial z}(f^3+if^4)}. \end{aligned}$$

Equation (5.13) looks more complicated than (5.12), but there is a simplification using (5.12) in the last step. The claim that $\mathbf{Q}$ as in (5.7) is of the form (5.8) follows. ∎

It follows from Lemma 5.1 that for $\mathbf{u} = (h,k)$ as in (5.6), h satisfies a nonlinear, inhomogeneous Cauchy-Riemann equation

$$h_{\bar z} = \beta_2(h,k)\overline{k_z}, \tag{5.14}$$

and k satisfies a Beltrami equation

$$k_{\bar z} = \beta_1(h,k)\overline{k_z}. \tag{5.15}$$

5.2. **The pseudoholomorphically fibered case.**

The results of Sections 2 and 4 apply to (5.14), so at this point we consider the special case where the complex structure in normal coordinates satisfies

$$\beta_1 \equiv 0.$$

We also drop the assumption that $\beta_2(\zeta, w)$ is smooth. The matrix (5.2) for the complex structure operator $J(\zeta, w)$ is:

$$J(\zeta,w) = \begin{bmatrix} 0 & -1 & a_1 & a_2 \\ 1 & 0 & a_2 & -a_1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 2\mathrm{Im}(\beta_2) & -2\mathrm{Re}(\beta_2) \\ 1 & 0 & -2\mathrm{Re}(\beta_2) & -2\mathrm{Im}(\beta_2) \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{5.16}$$

The projection $(\zeta, w) \mapsto w$ is a pseudoholomorphic map $D_{\vec{0},\rho} \times D_{\vec{0},\rho} \to D_{\vec{0},\rho}$; the fibers are the J-holomorphic curves (z, c) — this is called the "pseudoholomorphically fibered" case by [ST] §3.

Equations (5.14) and (5.15), for a parametric map $\mathbf{u}$ as in (5.6), become:

$$\begin{aligned} h_{\bar{z}} &= \beta_2(h(z), k(z))\overline{k_z}, \\ k_{\bar{z}} &\equiv 0. \end{aligned} \tag{5.17}$$

So (5.15) reduces to the homogeneous Cauchy-Riemann equation, and h satisfies a nonlinear, inhomogeneous Cauchy-Riemann equation.

The previously stated differentiability assumption in the definition of J-holomorphic curve has been weakened by some authors (e.g., [IS$_2$]) to $\mathbf{u} \in \mathcal{C}^0 \cap W^{1,2}$, when working with lower regularity $\mathbf{u}$ and J. However, for this special case where $\mathbf{Q}$ is strictly upper-triangular, the z-derivative of h does not appear, and k is already holomorphic by Proposition 2.5, so as in Section 2, one may consider solutions of the system without assuming $W^{1,2}$. More precisely, suppose $\mathbf{u} = (h, k)$ is a parametric map $\Omega \to \mathbb{C}^2$, where h and k are continuous, satisfy $(*)$ on Ω, and satisfy the system (5.17) almost everywhere in Ω. Then k is holomorphic, and if β_2 is continuous, then Theorem 2.6 and Corollary 2.7 apply to h, so the $W^{1,2}$ property follows as a conclusion. Further, it follows immediately from (5.17) and Liouville's Theorem that for any β_2, if $\mathbf{u} : \mathbb{C} \to \mathbb{C}^2$ has bounded image then it is constant.

If $\beta_2(\zeta, w)$ has a factorization of the separable form $f(\zeta)\frac{\partial v(w)}{\partial \bar{w}}$, then using the chain rule,

$$\frac{\partial}{\partial \bar{z}}(v(k(z))) = v_w(k(z))k_{\bar{z}} + v_{\bar{w}}(k(z))\overline{k_z},$$

(5.17) can be re-written as:

$$h_{\bar{z}} = f(h(z))v_{\bar{w}}(k(z))\overline{k_z} = f(h(z))\frac{\partial}{\partial \bar{z}}(v(k(z))). \tag{5.18}$$

Example 5.2. Consider the function $\beta_2(\zeta, w) = \zeta^2\bar{w}$ and the corresponding almost complex structure J (5.16) on $\mathbb{C}^2$. The system (5.17) for $\mathbf{u} : \Omega \to \mathbb{C}^2$ becomes:

$$\begin{aligned} h_{\bar{z}} &= \beta_2(h(z), k(z))\overline{k_z} = h^2\overline{kk_z} \\ k_{\bar{z}} &\equiv 0, \end{aligned} \tag{5.19}$$

so a continuous $\mathbf{u}$ satisfying $(*)$ and (5.19) almost everywhere on Ω must be real analytic by Corollary 2.7. In fact, this J defines an integrable almost complex structure on $\mathbb{C}^2$, so we do not expect the local qualitative behavior of J-holomorphic curves to be different from standard holomorphic curves. However, the results of Section 4 allow us to explicitly compute local parametric formulas for all the J-holomorphic curves in this coordinate system. As in (5.18), Equation (5.19) can be re-written

$$h_{\bar{z}} = h^2\overline{\frac{\partial}{\partial z}(\frac{1}{2}k^2)} = h^2\frac{\partial}{\partial \bar{z}}\left(\overline{\frac{1}{2}k^2}\right),$$

so this is a separable equation, to which Theorem 4.3 and Theorem 4.9 apply, with $G(z) = \overline{\frac{1}{2}(k(z))^2}$, $f(w) = w^2$, and $F(w) = -\frac{1}{w}$. If z_0 is any point in Ω_1 with $h(z_0) = \zeta_0 \neq 0$, then by the constructions in the Proofs of Lemma 4.1 and Theorem 4.3, for z near z_0

$$h(z) = \frac{-1}{\overline{\frac{1}{2}(k(z))^2} - \overline{\frac{1}{2}(k(z_0))^2} + C(z) - \frac{1}{\zeta_0}},$$

for some holomorphic C with $C(z_0) = 0$.

If $h(z_0) = 0$ (so $\mathbf{u} = (h, k)$ meets the w-axis), then h is either $\equiv 0$ ($\mathbf{u} = (0, k(z))$ is J-holomorphic), or has the following form, by Theorem 4.9:

$$h(z) = \frac{-(z - z_0)^m}{\phi(z) + (z - z_0)^m\overline{\frac{1}{2}(k(z))^2}}. \tag{5.20}$$

In this case, choosing any $m \geq 1$, holomorphic ϕ with $\phi(z_0) \neq 0$, and holomorphic $k(z)$ gives an example of a solution h.

Example 5.3. Consider the function

$$\beta_2(\zeta, w) = \frac{\partial V}{\partial \bar{w}}(w),$$

where V is the function constructed in Example 3.3, depending on w. The corresponding almost complex structure J (5.16) is continuous on $\mathbb{C}^2$, and equal to the standard complex structure J_0 outside a neighborhood of the origin. The system (5.17) for $\mathbf{u} : \Omega \to \mathbb{C}^2$ becomes, as in (5.18):

$$
\begin{aligned}
(5.21)\quad h_{\bar{z}} &= \beta_2(h(z), k(z))\overline{k_z} = \frac{\partial V}{\partial \bar{w}}(k(z))\overline{k_z} = \frac{\partial}{\partial \bar{z}}(V(k(z))) \\
k_{\bar{z}} &\equiv 0.
\end{aligned}
$$

If $\mathbf{u} : \Omega \to \mathbb{C}^2$ is continuous, satisfies $(*)$, and satisfies (5.21) almost everywhere on Ω, then k is holomorphic, and by Theorem 2.6, for any $R \Subset \Omega$ and $0 < \alpha < 1$, $h|_R \in W^{1,2}(R) \cap \mathcal{C}^{0,\alpha}(R)$. By Lemma 4.2,

$$h(z) = V(k(z)) + C(z)$$

for some holomorphic function C. One example of such a solution $\mathbf{u} : \mathbb{C} \to \mathbb{C}^2$ is $(h, k) = (V(z), z)$.

This Example shows that there exists a continuous almost complex structure J, admitting a differentiable J-holomorphic curve $\mathbf{u} = (h, k)$ which is a solution of the matrix equation (5.7) such that both LHS and RHS of (5.7) are defined everywhere and continuous (after the matrix multiplication), but $\mathbf{u}$ is not $\mathcal{C}^1$ because the LHS and RHS of (5.5) are not locally bounded.

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