Weighted Sobolev estimates of the truncated Beurling operator

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Abstract

Given a bounded planar domain D with $W^{k+1,\infty}$ boundary, $k \in \mathbb{Z}^+ \cup \{0\}$, and a weight $\mu \in A_p, 1 < p < \infty$, we show that the corresponding truncated Beurling operator is a bounded operator sending $W^{k,p}(D,\mu)$ into itself. Weighted Sobolev estimates for other Cauchy-type integrals are also obtained.

1 Introduction

Let us first recall the well-known Beurling operator B on $f \in L^p(\mathbb{C}), 1 < p < \infty$:

$$Bf := p.v. \frac{-1}{2\pi i} \int_{\mathbb{C}} \frac{f(\zeta)}{(\zeta - \cdot)^2} d\bar{\zeta} \wedge d\zeta \quad \text{on} \quad \mathbb{C},$$

where $p.v.$ represents the principal value. By the classical Calderón-Zygumund theory [4], B is a bounded operator sending $L^p(\mathbb{C})$ (and $W^{k,p}(\mathbb{C})$) into $L^p(\mathbb{C})$ (and $W^{k,p}(\mathbb{C})$, respectively). In this paper, we study the operator theory of the associated truncated Beurling operator $H := \chi_D B(\chi_D \cdot)$ with respect to a bounded domain $D \subset \mathbb{C}$. Here χ_D is the characteristic function of D. Namely, for $f \in L^p(D), 1 < p < \infty$,

$$Hf := p.v. \frac{-1}{2\pi i} \int_{D} \frac{f(\zeta)}{(\zeta - \cdot)^2} d\bar{\zeta} \wedge d\zeta \quad \text{on} \quad D. \tag{1.1}$$

Unlike the boundedness of B in $W^{k,p}(\mathbb{C})$, the Sobolev regularity of H requires some necessary boundary regularity of the domain D. For instance, consider D to be a polygon in $\mathbb{C}$ and $f \equiv 1$ on D. Then D has $W^{1,\infty}$ boundary and $f \in C^\infty(\bar{D})$. However, $\partial(Hf)(z)$ behaves like $\frac{1}{|z-w_0|}$ near any of its vertices w_0 on the boundary bD (see [AIM, pp. 146]), which does not lie in $W^{1,p}$ any more if $p \geq 2$.

Besides the Beurling operator, there are two other Cauchy-type integral operators that play a crucial role in complex analysis and singular integral theory. For $1 < p < \infty$,

$$\begin{aligned} Tf &:= \frac{-1}{2\pi i} \int_D \frac{f(\zeta)}{\zeta - \cdot} d\bar{\zeta} \wedge d\zeta \quad \text{on} \quad D, \quad \text{for} \quad f \in L^p(D); \\ Sf &:= \frac{1}{2\pi i} \int_{bD} \frac{f(\zeta)}{\zeta - \cdot} d\zeta \quad \text{on} \quad D, \quad \text{for} \quad f \in L^p(bD). \end{aligned} \tag{1.2}$$

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The solid Cauchy operator T is known to provide a weak solution to the Cauchy-Riemann equation $\bar\partial u = f$ on D and to improve the Hölder regularity by one order (see [1,17]); the boundary Cauchy operator S is a singular integral operator when restricting on bD, which sends $L^p(bD)$ continuously into itself according to a result of Riesz. The truncated Beurling operator H is related to T in terms of a distributional derivative of T from the point of view of complex analysis.

Based upon harmonic analysis methods, the sequential works of [10, 12, 16] assert that the truncated Beurling operator H maintains the (unweighted) $W^{k,p}$ regularity provided that $p > 2$ and the unit normal of bD falls into some Besov space. Ever since, such Sobolev regularity of H has found substantial applications to quasi-conformal mapping theory through Beltrami-type equations [2, 11], and the Cauchy-Riemann equations (or the so-called $\bar\partial$ problem). In particular, in the study of the $\bar\partial$ problem on some types of quotient domains that are realized as proper holomorphic images of product domains, a machinery introduced in [3, 8] (see also [18, 19]) has shown that the corresponding Sobolev regularity can eventually be reduced to a weighted Sobolev regularity for those Cauchy-type integrals. See also our subsequent paper [13] for an immediate application to an optimal Sobolev regularity of $\bar\partial$ on product domains and the Hartogs triangle.

The goal of this paper, motivated by the results and the call for weighted Sobolev theory to the Cauchy-type integrals, is to study the weighted Sobolev regularity of these operators in (1.1)-(1.2). Here our weight space is taken to be the standard Muckenhoupt's A_p class (see Section 2). Denote by $\mathbb{Z}^+$ the set of positive integers and by $W^{k,p}(D,\mu)$ the weighted Sobolev space on D with respect to a weight μ, $k \in \mathbb{Z}^+ \cup \{0\}, 1 < p < \infty$.

Theorem 1.1. *Let $D \subset \mathbb{C}$ be a bounded domain with $W^{k+1,\infty}$ boundary, $k \in \mathbb{Z}^+ \cup \{0\}$. Assume $\mu \in A_p, 1 < p < \infty$. There exists a constant C dependent only on D, k, p and μ, such that for all $f \in W^{k,p}(D,\mu)$,*

$$\|Hf\|_{W^{k,p}(D,\mu)} \le C\|f\|_{W^{k,p}(D,\mu)} \tag{1.3}$$

and

$$\|Tf\|_{W^{k+1,p}(D,\mu)} \le C\|f\|_{W^{k,p}(D,\mu)}. \tag{1.4}$$

The $k = 0$ case in the theorem above is the classical weighted Calderón-Zygmund theory, which is included for the sake of completeness. We stress particularly that, in the case when $\mu \equiv 1$ (the unweighted case), $p > 2$, and the unit normal $\mathcal{N}$ of bD is in the Besov space $B^{k-\frac{1}{p}}_{p,p}$ (equivalently, $bD \in B^{k+1-\frac{1}{p}}_{p,p} = W^{k+1-\frac{1}{p},p}$), (1.3) was firstly proved by Prats in [10] using technical harmonic analysis approach. In comparison to his result, our Theorem 1.1 recovers the Sobolev regularity with respect to any arbitrary A_p weight over the full range of p, $1 < p < \infty$, by assuming a slightly stronger boundary regularity $bD \in W^{k+1,\infty}$. Our proof to Theorem 1.1, short and analytic, is mostly done through the line of complex analysis and partial differential equations. One of the main ingredients is an induction formula on S, together with a higher order Cauchy-Green formula. Making use of it, we reduce the weighted Sobolev regularity of H and T to the corresponding estimates in the Lebesgue spaces, where the Calderón-Zygmund theory applies.

It is worth pointing out that, under the same assumption $\mu \equiv 1, p > 2$ and $bD \in W^{k+1-\frac{1}{p},p}$ as in [10], our proof can actually recover the above-mentioned result of Prats. For convenience of the reader, we have included the corresponding argument in our proof. The key observation is that when $p > 2$, $W^{k,p}(D)$ forms a multiplication algebra. After our paper was submitted, Di Plinio, Green and Wick recently obtained in [5] a similar weighted estimate (1.3) when $k = 1$ with $\mu \in A_p, p > 2$ and $bD \in W^{2-\frac{1}{p},p}$.

We also obtain the following weighted $W^{k,p}$ regularity for S on domains with $W^{k,\infty}$ boundaries (which is of one order lower than that in Theorem 1.1). Note that the $k=0$ case is excluded since S is not well-defined then.

Theorem 1.2. *Let $D \subset \mathbb{C}$ be a bounded domain with $W^{k,\infty}$ boundary, $k \in \mathbb{Z}^+$. Assume $\mu \in A_p, 1 < p < \infty$. There exists a constant C dependent only on D, k, p and μ, such that for all $f \in W^{k,p}(D, \mu)$,*

$$\|Sf\|_{W^{k,p}(D,\mu)} \le C\|f\|_{W^{k,p}(D,\mu)}. \tag{1.5}$$

If in particular $\mu \equiv 1$ and $p > 2$, then the above holds as long as D has $W^{1,\infty} \cap W^{k-\frac{1}{p},p}$ boundary.

2 Notations and preliminaries

Denote by dV the Lebesgue integral element on $\mathbb{C}$, and by $|S|$ the Lebesgue measure of a subset S in $\mathbb{C}$. Our weights under consideration are in the standard Muckenhoupt space A_p as follows.

Definition 2.1. *Given $1 < p < \infty$, a weight $\mu : \mathbb{C} \to [0, \infty)$ is said to be in A_p if its A_p constant*

$$A_p(\mu) := \sup \left(\frac{1}{|B|} \int_B \mu(z) dV_z \right) \left(\frac{1}{|B|} \int_B \mu(z)^{\frac{1}{1-p}} dV_z \right)^{p-1} < \infty,$$

where the supremum is taken over all discs $B \subset \mathbb{C}$.

See [15, Chapter V] for an introduction of the A_p class. Clearly, $A_q \subset A_p$ if $1 < q < p < \infty$. A_p spaces also satisfy an open-end property: if $\mu \in A_p$ for some $1 < p < \infty$, then $\mu \in A_{\tilde{p}}$ for some $\tilde{p} < p$. As a direct consequence of this property and Hölder inequality, if $\mu \in A_p, 1 < p < \infty$, there exists some $q > 1$ such that $L^p(D, \mu) \subset L^q(D)$ for a bounded domain D.

Given a non-negative weight μ and $1 < p < \infty$, the weighted function space $W^{k,p}(D, \mu)$ is the set of functions f on D whose weak derivatives up to order k exist, and the $W^{k,p}$ norm

$$\|f\|_{W^{k,p}(D,\mu)} := \left(\sum_{j=0}^{k} \int_D |\nabla^j f(z)|^p \mu(z) dV \right)^{\frac{1}{p}} < \infty.$$

Here $\nabla^j f$ represents all j-th order weak derivatives of f. When $\mu \equiv 1$, $W^{k,p}(D, \mu)$ is reduced to the (unweighted) $W^{k,p}(D)$ space. From now on, we shall say $a \lesssim b$ if $a \le Cb$ for a constant $C > 0$ dependent only possibly on D, k, p and the A_p constant of μ.

3 Weighted Sobolev estimates

Let D be a bounded domain in $\mathbb{C}$ with $W^{1,\infty}$ (equivalently, Lipschitz) boundary. For $f \in L^p(D)$, it is known that the solid Cauchy integral T is a solution operator to the $\bar{\partial}$ equation on D:

$$\bar{\partial} T f = f \quad \text{on} \quad D$$

in the sense of distributions. Moreover, for a.e. $z \in D$,

$$\partial T f(z) = H f(z) \left(= \lim_{\epsilon \to 0} \frac{-1}{2\pi i} \int_{D \setminus B_\epsilon(z)} \frac{f(\zeta)}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta \right), \tag{3.1}$$

where $B_\epsilon(z)$ is the disc centered at $z \in D$ with radius ϵ. See for instance [1, 17]. According to the weighted Calderón-Zygmund theory, if $\mu \in A_p, 1 < p < \infty$, then

$$\|Hf\|_{L^p(D,\mu)} \lesssim \|f\|_{L^p(D,\mu)}. \tag{3.2}$$

The following proposition follows immediately from the above (in)equalities and [18, Proposition 3.1] about a weighted estimate on T.

Proposition 3.1. *[15, 17] Let D be a bounded domain in $\mathbb{C}$ with $W^{1,\infty}$ boundary and $\mu \in A_p, 1 < p < \infty$. If $f \in L^p(D,\mu)$, then $Tf \in W^{1,p}(D,\mu)$ with*

$$\|Tf\|_{W^{1,p}(D,\mu)} \lesssim \|f\|_{L^p(D,\mu)}.$$

We first estimate the boundary Cauchy integral S. Recall that S is related to T in terms of the following Cauchy-Green formula: for all $f \in W^{1,q}(D), q > 1$.

$$Sf = f - T\bar{\partial} f \quad \text{on} \quad D \tag{3.3}$$

in the sense of distributions. Since $W^{1,p}(D,\mu) \subset W^{1,q}(D)$ for some $q > 1$, by the trace theorem $W^{1,p}(D,\mu)$ is continuously embedded in $L^q(bD)$. So Sf is well defined on $W^{1,p}(D,\mu)$. Moreover, we have the following estimate for ∂S.

Proposition 3.2. *Let D be a bounded domain with $W^{1,\infty}$ boundary and $\mu \in A_p, 1 < p < \infty$. Let $f \in W^{1,p}(D,\mu)$, $1 < p < \infty$. Then*

$$\partial S f = H\bar{\partial} f + \partial f. \tag{3.4}$$

in the sense of distributions. Consequently,

$$\|Sf\|_{W^{1,p}(D,\mu)} \lesssim \|f\|_{W^{1,p}(D,\mu)}. \tag{3.5}$$

Proof. Formula (3.4) follows from (3.3) by taking ∂ directly in the sense of distributions. By (3.2), $\partial S f \in L^p(D,\mu)$. To prove the estimate (3.5), we first use the Cauchy-Green formula (3.3) to obtain

$$\|Sf\|_{L^p(D,\mu)} \lesssim \|f\|_{L^p(D,\mu)} + \|T\bar{\partial} f\|_{L^p(D,\mu)} \lesssim \|f\|_{W^{1,p}(D,\mu)}. \tag{3.6}$$

The rest of the proof of (3.5) is a direct consequence of (3.6), (3.4), (3.2) and the holomorphy of S on D.

□

On the higher order Sobolev spaces, S satisfies an induction formula below, similar to [18, Lemma 3.5]. We include the proof here for the reader's convenience, and meanwhile in order to chase down precisely how the regularity of bD affects the regularity of S. Let

$$\partial S f = \frac{1}{2\pi i} \int_{bD} \frac{f(\zeta)}{(\zeta - \cdot)^2} d\zeta =: \tilde{S} f \ \text{ on } \ D.$$

In particular, Proposition 3.2 states that if $D \in W^{1,\infty}, \mu \in A_p, 1 < p < \infty$, then

$$\|\tilde{S} f\|_{L^p(D,\mu)} \lesssim \|f\|_{W^{1,p}(D,\mu)}. \tag{3.7}$$

Lemma 3.3. *Let D be a bounded domain in $\mathbb{C}$ with $W^{k,\infty}$ boundary, $k \in \mathbb{Z}^+$, and $\mu \in A_p, 1 < p < \infty$. For any $f \in W^{k,p}(D,\mu)$, we have*

$$\partial^k Sf = \tilde{S}\tilde{f} \quad on \quad D$$

for some $\tilde{f} \in W^{1,p}(D,\mu)$ such that

$$\|\tilde{f}\|_{W^{1,p}(D,\mu)} \lesssim \|f\|_{W^{k,p}(D,\mu)}. \tag{3.8}$$

Furthermore, if $\mu \equiv 1$ and $p > 2$, then the above still holds as long as D is of $W^{1,\infty} \cap W^{k-\frac{1}{p},p}$ boundary.

Proof. The $k = 1$ case is by the definition of $\tilde{S}$, with $\tilde{f} = f$. When $k = 2$, let ζ be a parameterization of bD in terms of the arclength $s \in (0, s_0)$. For each fixed $z \in D$, since $\frac{1}{\cdot - z} \in C^\infty(bD)$ and $f \in W^{2,p}(D,\mu) \subset W^{2,1}(D) \subset W^{1,1}(bD)$ by the trace theorem, we can use the Divergence theorem on bD to compute $\partial^2 Sf$ directly as follows.

$$\begin{aligned}
\partial^2 Sf(z) = \partial\tilde{S}f(z) &= \frac{1}{2\pi i}\int_0^{s_0} \partial_z\left(\frac{1}{(\zeta(s)-z)^2}\right) f(\zeta(s))\zeta'(s)ds \\
&= -\frac{1}{2\pi i}\int_0^{s_0} \partial_s\left(\frac{1}{(\zeta(s)-z)^2}\right) f(\zeta(s))ds \\
&= \frac{1}{2\pi i}\int_0^{s_0} \frac{\partial_s\left(f(\zeta(s))\right)}{(\zeta(s)-z)^2}ds \\
&= \frac{1}{2\pi i}\int_0^{s_0} \frac{\partial_\zeta f(\zeta(s))\zeta'(s) + \partial_{\bar{\zeta}} f(\zeta(s))\bar{\zeta}'(s)}{(\zeta(s)-z)^2}ds \\
&= \frac{1}{2\pi i}\int_{bD} \frac{\partial f(\zeta) + \bar{\zeta}'^2(s)\bar{\partial} f(\zeta)}{(\zeta - z)^2}d\zeta = \tilde{S}\tilde{f}(z),
\end{aligned}$$

where

$$\tilde{f} = \partial f + \bar{\zeta}'^2\bar{\partial}f \quad \text{on} \quad bD. \tag{3.9}$$

In the above we have used the fact that $\bar{\zeta}' = \frac{1}{\zeta'}$ on bD.

If bD has $W^{2,\infty}$ boundary, then $\bar{\zeta}' \in W^{1,\infty}(bD)$. Namely, $\bar{\zeta}'$ is Lipschitz continuous on bD. Making use of Kirszbraun theorem [6], we can extend $\bar{\zeta}'$ to D, still denoted as $\bar{\zeta}'$, such that $\bar{\zeta}' \in W^{1,\infty}(D)$. Thus by (3.9), $\tilde{f}$ extends as an element in $W^{1,p}(D,\mu)$, with

$$\|\tilde{f}\|_{W^{1,p}(D,\mu)} \lesssim \|\nabla f\|_{W^{1,p}(D,\mu)} \le \|f\|_{W^{2,p}(D,\mu)}.$$

If $\mu \equiv 1$, $p > 2$ and $bD \in W^{2-\frac{1}{p},p}$ instead, then $\zeta' \in W^{1-\frac{1}{p},p}(bD)$ (note that the unit normal $\mathcal{N}$ is $\pm i\zeta'$). By the trace theorem for Sobolev–Slobodeckij spaces [7,9], ζ' can be extended as an element, still denoted by ζ', lying in $W^{1,p}(D)$. On the other hand, by the Sobolev embedding theorem, $\bar{\zeta}', \nabla f \in W^{1,p}(D) \subset L^\infty(D)$ with $\|\bar{\zeta}'\|_{L^\infty(D)} \lesssim \|\bar{\zeta}'\|_{W^{1,p}(D)} \lesssim 1$ and $\|\nabla f\|_{L^\infty(D)} \lesssim \|f\|_{W^{2,p}(D)}$. Then

$$\|\bar{\zeta}'^2\|_{W^{1,p}(D)} \lesssim \|\bar{\zeta}'\|_{W^{1,p}(D)}\|\bar{\zeta}'\|_{L^\infty(D)} \lesssim 1.$$

Consequently,

$$\|\bar{\zeta}'^2\bar{\partial}f\|_{W^{1,p}(D)} \lesssim \|\bar{\zeta}'^2\bar{\partial}f\|_{L^\infty(D)} + \|\bar{\zeta}'^2\|_{W^{1,p}(D)}\|\bar{\partial}f\|_{L^\infty(D)} + \|\bar{\partial}f\|_{W^{1,p}(D)}\|\bar{\zeta}'^2\|_{L^\infty(D)} \lesssim \|f\|_{W^{2,p}(D)},$$

from which and (3.9) we have $\tilde{f} \in W^{1,p}(D)$ and (3.8) follows. This completes the proof of the lemma when $k = 2$. The remaining part of the lemma follows by a standard induction on k. □

Proof of Theorem 1.2: Let $f \in W^{k,p}(D,\mu)$. The $k=1$ case is due to Proposition 3.2. When $k \geq 2$, since Sf is holomorphic on D, we only need to estimate $\|\partial^k Sf\|_{L^p(D,\mu)}$. According to Lemma 3.3, $\partial^k Sf = \tilde{S}\tilde{f}$ for some $\tilde{f} \in W^{1,p}(D,\mu)$ such that $\|\tilde{f}\|_{W^{1,p}(D,\mu)} \lesssim \|f\|_{W^{k,p}(D,\mu)}$. Then (3.7) gives

$$\|\partial^k Sf\|_{L^p(D,\mu)} = \|\tilde{S}\tilde{f}\|_{L^p(D,\mu)} \lesssim \|\tilde{f}\|_{W^{1,p}(D,\mu)} \lesssim \|f\|_{W^{k,p}(D,\mu)}.$$

(1.5) is thus proved.

□

Proof of Theorem 1.1: Since (1.3) is a consequence of (1.4) and (3.1), we will only estimate T on $W^{k,p}(D,\mu), k \geq 1$. Let

$$Lf := \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - \cdot} d\bar{\zeta} \quad \text{on} \quad D.$$

Similar to S, Lf is well defined for $f \in W^{1,p}(D,\mu)$ and is holomorphic in D. L is associated with T in terms of the following formula (see, [17] or [18, Theorem 3.3]):

$$\partial Tf = T\partial f - Lf \quad \text{on} \quad D$$

in the sense of distributions. Consequently, for $j \in \mathbb{Z}^+, j \leq k$,

$$\partial^{j+1}Tf = \partial^j T\partial f - \partial^j Lf \quad \text{on} \quad D$$

in the sense of distributions. Thus to prove (1.4), we only need to show

$$\|Lf\|_{W^{k,p}(D,\mu)} \lesssim \|f\|_{W^{k,p}(D,\mu)}. \tag{3.10}$$

As in the proof of Lemma 3.3, let $\zeta(s)|_{s\in(0,s_0)}$ be a parametrization of bD in terms of the arclength parameter s. Then for $z \in D$,

$$Lf(z) = \frac{1}{2\pi i}\int_0^{s_0} \frac{f(\zeta(s))}{\zeta(s)-z}\bar{\zeta}'(s)ds = \frac{1}{2\pi i}\int_0^{s_0} \frac{f(\zeta(s))(\bar{\zeta}')^2(s)}{\zeta(s)-z}\zeta'(s)ds = S\left(\bar{\zeta}'^2 f\right)(z). \tag{3.11}$$

If $bD \in W^{k+1,\infty}$, then $\zeta' \in W^{k,\infty}(bD)$ and

$$\|\bar{\zeta}'^2 f\|_{W^{k,p}(D,\mu)} \leq \|f\|_{W^{k,p}(D,\mu)}\|(\bar{\zeta}')^2\|_{W^{k,\infty}(bD)} \leq \|f\|_{W^{k,p}(D,\mu)}\|\bar{\zeta}'\|^2_{W^{k,\infty}(bD)} \lesssim \|f\|_{W^{k,p}(D,\mu)}.$$

In the case when $\mu \equiv 1$, $p > 2$ and $bD \in W^{k+1-\frac{1}{p},p}$ (as assumed in [10]), we argue similarly as in the last paragraph of the proof to Lemma 3.3. Indeed, since $\bar{\zeta}' \in W^{k-\frac{1}{p},p}(bD)$, by the trace theorem $\bar{\zeta}'$ can be extended as an element in $W^{k,p}(D)$. Recall that $W^{k,p}(D)$ forms a multiplication algebra when $p > 2$. See [11, 14] etc. Or, directly by the Sobolev embedding theorem, we have $\bar{\zeta}', f \in W^{k,p}(D) \subset W^{k-1,\infty}(D)$. Then $\bar{\zeta}'^2 f \in W^{k,p}(D)$ with

$$\begin{aligned}\|\bar{\zeta}'^2 f\|_{W^{k,p}(D)} \lesssim& \|\bar{\zeta}'^2\|_{W^{k,p}(D)}\|f\|_{W^{k-1,\infty}(D)} + \|f\|_{W^{k,p}(D)}\|\bar{\zeta}'^2\|_{W^{k-1,\infty}(D)}\\ \lesssim& \|(\bar{\zeta}')^2\|_{W^{k,p}(D)}\|f\|_{W^{k,p}(D)} \lesssim \|\bar{\zeta}'\|_{W^{k,p}(D)}\|\bar{\zeta}'\|_{W^{k-1,\infty}(D)}\|f\|_{W^{k,p}(D)}\\ \lesssim& \|\bar{\zeta}'\|^2_{W^{k,p}(D)}\|f\|_{W^{k,p}(D)} \lesssim \|f\|_{W^{k,p}(D)}.\end{aligned}$$

In both cases, we combine (1.5) with (3.11) to obtain (3.10). The proof is complete.

□

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