Rigidity and Holomorphic Segre Transversality for Holomorphic Segre Maps

Yuan Zhang

Abstract

Let $\mathcal{H}^n$ and $\mathcal{H}^N$ denote the complexifications of Heisenberg hypersurfaces in $\mathbf{C}^n$ and $\mathbf{C}^N$, respectively. We show that non-degenerate holomorphic Segre mappings from $\mathcal{H}^n$ into $\mathcal{H}^N$ with $N \leq 2n-2$ possess a partial rigidity property. As an application, we prove that the holomorphic Segre non-transversality for a holomorphic Segre map from $\mathcal{H}^n$ into $\mathcal{H}^N$ with $N \leq 2n-2$ propagates along Segre varieties. We also give an example showing that this propagation property of holomorphic Segre transversality fails when $N > 2n-2$.

1 Introduction and Main Theorem

Let D be an open subset in $\mathbf{C}^n$ and M be a real analytic hypersurface in D with a real analytic defining function r. Namely, $M = \{z \in D: \ r(z,\overline{z}) = 0\}$ and $\mathrm{d}r \neq 0$. The complexification of M is defined as: $\mathcal{M} = \{(z,\xi) \in D \times \mathrm{Conj}(D): \ r(z,\xi) = 0\}$. Here for a set $E \subset \mathbf{C}^n$, $\mathrm{Conj}(E) := \{\overline{z}: \ z \in E\}$. Assume that $\mathrm{d}r \neq 0$ over $\mathcal{M}$. Then $\mathcal{M}$ is a complex submanifold of complex codimension one in $D \times \mathrm{Conj}(D)$ which contains $\{(z,\bar{z}): \ z \in M\}$ as a maximally totally real submanifold. Define complex analytic varieties $Q_\xi := \{z \in D: \ r(z,\xi) = 0\}$ for $\xi \in \mathrm{Conj}(D)$ and $\hat{Q}_z := \{\xi \in \mathrm{Conj}(D): \ r(z,\xi) = 0\}$ for $z \in D$. We call Q_ξ and $\hat{Q}_z$ the *Segre variety* of M with respect to ξ and z, respectively. (See [Web]). Notice that $\mathcal{M}$ is then holomorphically foliated by $\{Q_\xi \times \{\xi\}\}$ and also by $\{\{z\} \times \hat{Q}_z\}$ for $\xi \in \mathrm{Conj}(D)$ and $z \in D$. As in the literature (see, for instance, [Ch] [Fa2] [HJ2]), we call $\mathcal{M}$ the Segre family associated with M.

A fundamental fact for the Segre family is its invariant property by holomorphic maps. (See the famous paper of S. Webster [Web]). More precisely, let $\widetilde{M}$ be another real analytic hypersurface in $\widetilde{D} \subset \mathbf{C}^N$ and let $\widetilde{\mathcal{M}}$ be its Segre family. f is a holomorphic map from D into $\widetilde{D}$ sending M into $\widetilde{M}$. Then $f(Q_\xi) \subset \widetilde{Q}_{\overline{f}(\xi)}$ and $\overline{f}(\hat{Q}_z) \subset \hat{\widetilde{Q}}_{f(z)}$ for $\xi \in \mathrm{Conj}(D)$ and $z \in D$. Here, for instance, we write $\widetilde{Q}_{\overline{f}(\xi)}$ for the Segre variety of $\widetilde{M}$ with respect to $\overline{f}(\xi)$.

In particular, f induces a holomorphic map $\mathcal{F} := (f(z), \overline{f}(\xi))$ from $\mathcal{M}$ into $\widetilde{\mathcal{M}}$. Here, as usual, we write $\overline{f}(z)$ for $\overline{f(\overline{z})}$. More generally, we introduce the following notion as in [HJ2]:

Definition 1.1: Φ is called a holomorphic Segre map from $\mathcal{M}$ into $\widetilde{\mathcal{M}}$ if Φ is a holomorphic map from $\mathcal{M}$ to $\widetilde{\mathcal{M}}$ such that Φ sends each $Q_\xi \times \{\xi\}$ of $\mathcal{M}$ into a certain $Q_{\widetilde{\xi}} \times \{\widetilde{\xi}\}$ of $\widetilde{\mathcal{M}}$ and sends each $\{z\} \times \hat{Q}_z$ into a certain $\{\widetilde{z}\} \times \hat{\widetilde{Q}}_{\widetilde{z}}$ of $\widetilde{\mathcal{M}}$ for $\xi \in \mathrm{Conj}(D)$ and $z \in D$.

We remark that there is another important but very different class of real-analytic maps closely related to the Segre families introduced by Baouendi-Ebenfelt-Rothschild (See [BER1]), which are called the Segre maps in many references. A holomorphic Segre map Φ is called a *holomorphic Segre embedding* if it is also a holomorphic embedding. Holomorphic Segre maps from $\mathcal{M}$ into $\widetilde{\mathcal{M}}$ are the natural generalizations of holomorphic mappings from M into $\widetilde{M}$. As already demonstrated by E. Cartan, holomorphic Segre maps play a very role in the study of holomorphic equivalence problems and many other related fields. (See [Car] [Ch] [BER2] [Hua3] and the references therein, for instance).

In this paper, we focus our attention on holomorphic Segre maps between the Segre family of the model manifolds: Heisenberg hypersurfaces. Holomorphic Segre maps are much less restricted than holomorphic maps induced from CR maps between Heisenberg hypersurfaces. It is thus not surprising that many properties for the latter are no longer true for holomorphic Segre maps. For instance, it is an easy consequence of the classical Hopf lemma that any non-constant holomorphic map between Heisenberg hypersurfaces must have non-vanishing normal derivative in its normal component. (This property is called the Hopf Lemma property by Baouendi-Rothschild [BR] or the CR transversality by Ebenfelt-Rothschild [ER] and Ebenfelt-Huang-Zaitsev [EHZ]). However, such a property does not hold anymore for general holomorphic Segre maps. Also, easy examples show that there are many non-rational holomorphic Segre embeddings from $\mathcal{H}^n$ into $\mathcal{H}^N$ for $N \geq n+1$, which can not occur for holomorphic maps between the Heisenberg hypersurfaces as is well known from the work of Forstneric [Fr]. (See below for the precise definition of $\mathcal{H}^n$ and Remark 5.3 for related examples.)

Making use of the recent work of Baouendi-Huang [BH] and Ebenfelt-Huang-Zaitsev [EHZ], we will provide, in this paper, two theorems for holomorphic Segre maps from $\mathcal{H}^n$ into $\mathcal{H}^N$ with $N \leq 2n-2$. As was done in those papers, we first normalize the holomorphic Segre maps by composing the maps with automorphisms in the source and in the target. For those normalized holomorphic Segre maps, we then prove that although full linearity fails, linearity for certain components still holds. Afterwards, we use this semi-linearity property to prove a propagation theorem concerning the failure of a holomorphic notion of the Hopf lemma. Namely, we will show that if the transversality breaks down at one point, then it breaks down along one of the two Segre varieties through this point. One may compare this with the work of Baouendi-Huang [BH], where the transversality breaks down at one point if and only if it does for all points.

We next give more notation to state our main theorems.

Denote the Heisenberg hypersurface in $\mathbf{C}^n$ by

$$\mathbf{H}^n := \{(z_1, \ldots, z_{n-1}, w) \in \mathbf{C}^n : \Im w = \sum_{j=1}^{n-1} z_j \bar{z}_j\}.$$

Then its complexification is

$$\mathcal{H}^n := \{(z_1, \ldots, z_{n-1}, w, \xi_1, \ldots, \xi_{n-1}, \eta) \in \mathbf{C}^{2n} : w - \eta = 2i \sum_{j=1}^{n-1} z_j \xi_j\}.$$

$\mathcal{H}^n$ is the Segre family associated with $\mathbf{H}^n$. $\mathcal{H}^n$ is holomorphically foliated by $\{Q_{(\xi,\eta)} \times \{(\xi, \eta)\}\}$ and also by $\{\{(z, w)\} \times \hat{Q}_{(z,w)}\}$ where $Q_{(\xi,\eta)} = \{(z, w) \in \mathbf{C}^n : w - \eta = 2i \sum_{j=1}^{n-1} z_j \xi_j\}$ for any $(\xi, \eta) \in \mathbf{C}^n$ and $\hat{Q}_{(z,w)} = \{(\xi, \eta) \in \mathbf{C}^n : w - \eta = 2i \sum_{j=1}^{n-1} z_j \xi_j\}$ for any $(z, w) \in \mathbf{C}^n$.

Next let Φ be a holomorphic Segre map from an open piece $\mathcal{M}$ of $\mathcal{H}^n$ into $\mathcal{H}^N$. It is known that Φ takes the following form: $\Phi(z, w, \xi, \eta) = (\Phi_1(z, w), \Phi_2(\xi, \eta))$ for certain holomorphic maps Φ_1, Φ_2 defined in a neighborhood of $\mathcal{M}$ in $\mathbf{C}^{2n}$ into $\mathbf{C}^N$. (See [Fa2] [Hua3] [HJ2]). In the following, we always assume that $N \geq n$.

We say $\sigma \in Aut(\mathcal{H}^n)$ if σ is a bimeromorphic self-map of $\mathcal{H}^n$ which is also a holomorphic Segre map away from its pole. Further, we write $\sigma \in Aut_0(\mathcal{H}^n)$ if $\sigma \in Aut(\mathcal{H}^n)$, σ is holomorphic near 0 and $\sigma(0) = 0$. For any $p_0 = (z_0, w_0, \xi_0, \eta_0) \in \mathcal{H}^n$, define $\sigma^0_{p_0}$ by sending $(z, w, \xi, \eta) \in \mathbf{C}^{n-1} \times \mathbf{C}^1 \times \mathbf{C}^{n-1} \times \mathbf{C}^1$ to $(z + z_0, w + w_0 + 2iz \cdot \xi_0, \xi + \xi_0, \eta + \eta_0 - 2i\xi \cdot z_0)$. Then $\sigma^0_{p_0} \in Aut(\mathcal{H}^n)$ with $\sigma^0_{p_0}(0) = p_0$. (See Theorem 6.3 of [HJ2] for the explicit formula for elements in $Aut(\mathcal{H}^n)$)

Theorem 1.2: Let $\mathcal{M}$ be a connected neighborhood of p_0 in $\mathcal{H}^n$. Let Φ be a holomorphic Segre map from $\mathcal{M}$ into $\mathcal{H}^N$ and write $\Phi(z, w, \xi, \eta) = (\Phi_1(z, w), \Phi_2(\xi, \eta)) = (\tilde{f}_1(z, w), \ldots, \tilde{f}_{N-1}(z, w), g(z, w), \tilde{h}_1(\xi, \eta), \ldots, \tilde{h}_{N-1}(\xi, \eta), e(\xi, \eta))$ for $(z, w, \xi, \eta) \in \mathcal{M}$. Assume that Φ is holomorphic in a neighborhood $\mathcal{U}$ of $\mathcal{M}$ in $\mathbf{C}^{2n}$, $p_0 \in \mathcal{M}$ with $\Phi(p_0) = \widetilde{p_0}$ and $N \leq 2n - 2$. For $p \in \mathcal{M}$, write $\Phi_p = ((\widetilde{f}_1)_p, \ldots, (\widetilde{f}_{N-1})_p, g_p, (\widetilde{h}_1)_p, \ldots, (\widetilde{h}_{N-1})_p, e_p) := (\widetilde{\sigma}^0_{\Phi(p)})^{-1} \circ \Phi \circ \sigma^0_p$. Then the following holds:

(1). If $\frac{\partial g_{p_0}}{\partial w}(0) \neq 0$, then there exists a $\tau \in Aut_0(\mathcal{H}^N)$ such that

$$\tau \circ \Phi_{p_0}(z, w, \xi, \eta) = (z, \phi(z, w), w, \xi, \psi(\xi, \eta), \eta)$$

with $\phi_j(z, w)\psi_j(\xi, \eta) \equiv 0$ for $(z, w, \xi, \eta) \in \mathcal{U}$ for each $j = 1, \cdots, N - n$.

(2). If $\frac{\partial g_p}{\partial w}(0) = 0$ for all $p \in \mathcal{M}$ near p_0, then $g_{p_0} \equiv e_{p_0} \equiv 0$ and $\sum_{j=1}^{N-1} (\widetilde{f}_j)_{p_0}(z, w)(\widetilde{h}_j)_{p_0}(\xi, \eta) \equiv 0$ for $(z, w, \xi, \eta) \in \mathcal{U}$.

Motivated by the concept of CR transversality, we introduce the following notion:

Definition 1.3: A holomorphic Segre map $\mathcal{F} : (\mathcal{M}, p)(\subset (\mathbf{C}^{2n}, p)) \to (\tilde{\mathcal{M}}, \tilde{p})(\subset (\mathbf{C}^{2N}, \widetilde{p}))$ with $\mathcal{F}(p) = \widetilde{p} = (q_1, q_2) \in \mathbf{C}^N \times \mathbf{C}^N$ is called to be holomorphic Segre transversal to $\mathcal{M}$ at p if:

$$d\mathcal{F}(T_p^{(1,0)}\mathcal{M}) + T_{\widetilde{p}}^{(1,0)}\widetilde{Q}_{q_2} + T_{\widetilde{p}}^{(1,0)}\hat{\widetilde{Q}}_{q_1} = T_{\widetilde{p}}^{(1,0)}\mathcal{H}^N,$$

where $T_p^{(1,0)}\mathcal{M}$ is the holomorphic tangent space of $\mathcal{M}$ at p.

In §4, we shall show that the assumption in Theorem 1.2 (1) is equivalent to the statement that Φ is holomorphic Segre transveral to $\mathcal{H}^N$ at p_0.

Definition 1.4: Let $\mathcal{M}$ be an open subset of $\mathcal{H}^n$. A connected non-empty complex analytic variety $\mathcal{E} \subset \mathcal{M}$ is called a holomorphic Segre-related set of $\mathcal{M}$ if either for any $(z_0, w_0, \xi_0, \eta_0) \in \mathcal{E}$, the connected component of $\{(z_0, w_0, \xi, \eta) \in \mathcal{M} : (\xi, \eta) \in \hat{Q}_{(z_0,w_0)}\}$ containing $(z_0, w_0, \xi_0, \eta_0)$ is a subset of $\mathcal{E}$, or for any $(z_0, w_0, \xi_0, \eta_0) \in \mathcal{E}$, the connected component of $\{(z, w, \xi_0, \eta_0) \in \mathcal{M} : (z, w) \in Q_{(\xi_0,\eta_0)}\}$ containing $(z_0, w_0, \xi_0, \eta_0)$ is a subset of $\mathcal{E}$.

Write $\mathrm{Hol}(\mathcal{M}, \mathbf{C})$ for the collection of holomorphic functions from $\mathcal{M}$ into $\mathbf{C}$. Then as an immediate application of Theorem 1.2, we have the following characterization of holomorphic non-transversal points of a holomorphic Segre map:

Theorem 1.5: Let $\mathcal{M}$ be a connected open piece in $\mathcal{H}^n$. Let Φ be a holomorphic Segre map from $\mathcal{M}$ into $\mathcal{H}^N$ and write $\Phi(z, w, \xi, \eta) = (\Phi_1(z, w), \Phi_2(\xi, \eta)) = (\tilde{f}_1(z, w), \ldots, \tilde{f}_{N-1}(z, w), g(z, w), \tilde{h}_1(\xi, \eta), \ldots, \tilde{h}_{N-1}(\xi, \eta), e(\xi, \eta))$ for $(z, w, \xi, \eta) \in \mathcal{M}$. Assume that Φ is holomorphic in a neighborhood $\mathcal{U}$ of $\mathcal{M}$ in $\mathbf{C}^{2n}$, $p_0 \in \mathcal{M}$ with $\Phi(p_0) = \widetilde{p_0}$ and $N \leq 2n-2$. Let $\mathcal{E}_\Phi$ be the collection of points, where Φ fails to be holomorphically Segre transversal. Then the following holds:

(1) $\mathcal{E}_\Phi$, if not empty nor the whole space $\mathcal{M}$, must be a complex analytic variety of codimension one, whose irreducible components are holomorphic Segre-related sets of codimension one in $\mathcal{M}$. Moreover for each irreducible component $\mathcal{E}_j$ of $\mathcal{E}_\Phi$, there is a point $(z_0, w_0, \xi_0, \eta_0) \in \mathcal{M}$ such that $\Phi(\mathcal{E}_j) \subset \widetilde{Q}_{\Phi_2(\xi_0,\eta_0)} \times \{\Phi_2(\xi_0, \eta_0)\}$ or $\Phi(\mathcal{E}_j) \subset \{\Phi_1(z_0, w_0)\} \times \hat{\widetilde{Q}}_{\Phi_1(z_0,w_0)}$.

(2)When $N = n+1$, $n \geq 3$, either there is a $\chi_1 \in \mathrm{Hol}(\mathcal{M}, \mathbf{C})$ depending only on (z, w)-variables such that $\mathcal{E}_\Phi$ is precisely the zero set of χ_1, or there is a $\chi_2 \in \mathrm{Hol}(\mathcal{M}, \mathbf{C})$ depending only on (ξ, η)-variables such that $\mathcal{E}_\Phi$ is precisely the zero set of χ_2.

(3)When $2n-2 \geq N > n+1$, there are a $\chi_1 \in \mathrm{Hol}(\mathcal{M}, \mathbf{C})$ depending only on (z, w)-variables and a $\chi_2 \in \mathrm{Hol}(\mathcal{M}, \mathbf{C})$ depending only on (ξ, η)-variables such that $\mathcal{E}_\Phi$ is precisely the union of the zero sets of χ_1 and χ_2.

Proposition 1.6: Let $\mathcal{E} \subset \mathcal{M}$ be a holomorphic Segre-related set of $\mathcal{M}$ of codimension one, where $\mathcal{M}$ is a connected open subset of $\mathcal{H}^n$ $(n \geq 2)$. Suppose that either $\mathcal{E} = \{(z, w, \xi, \eta) \in \mathcal{M} : \ \chi_1(z, w) = 0, \ \chi_1 \in \mathrm{Hol}(\mathcal{M}, \mathbf{C})\}$ or $\mathcal{E} = \{(z, w, \xi, \eta) \in \mathcal{M} : \ \chi_2(\xi, \eta) = 0, \ \chi_2 \in \mathrm{Hol}(\mathcal{M}, \mathbf{C})\}$. Assume that $\chi_1(\mathcal{M}) \neq \mathbf{C}$ in the first case and $\chi_2(\mathcal{M}) \neq \mathbf{C}$ in the latter. Then there is a holomorphic Segre map Φ from $\mathcal{M}$ into $\mathcal{H}^N$ with $N = n + 1$ such that $\mathcal{E}$ is precisely the collection of points, where Φ fails to be holomorphic Segre transversal.

Proposition 1.7: Let $\mathcal{M}$ is a connected open subset of $\mathcal{H}^n$ $(n \geq 2)$. Suppose that $\mathcal{E} = \{(z, w, \xi, \eta) \in \mathcal{M} : \ \chi_1(z, w) = 0, \ \chi_1 \in \mathrm{Hol}(\mathcal{M}, \mathbf{C})\} \cup \{(z, w, \xi, \eta) \in \mathcal{M} : \ \chi_2(\xi, \eta) = 0, \ \chi_2 \in \mathrm{Hol}(\mathcal{M}, \mathbf{C})\}$, where $\chi_1(\mathcal{M}) \neq \mathbf{C}$ and $\chi_2(\mathcal{M}) \neq \mathbf{C}$. Then there is a holomorphic Segre map Φ from $\mathcal{M}$ into $\mathcal{H}^N$ with $N = n + 2$ such that $\mathcal{E}$ is precisely the collection of points, where Φ fails to be holomorphic Segre transversal.

The new phenomenon in Theorem 1.5 is the propagation property for the holomorphic Segre non-transversality along the Segre varieties for the case of $N \leq 2n-2$. Interestingly, the following example shows that this property does not hold for $N > 2n - 2$:

Example 1.8: Let Φ be the holomorphic Segre embedding $\Phi : \mathcal{H}^n \to \mathcal{H}^N$ with $N = 2n - 1$, where

$$\Phi(z_1, \cdots, z_{n-1}, w, \xi_1, \cdots, \xi_{n-1}, \eta) =$$

$$(z_1 w, \cdots, z_{n-1} w, z_1, \cdots, z_{n-1}, w^2, \xi_1, \cdots, \xi_{n-1}, \xi_1 \eta, \cdots, \xi_{n-1} \eta, \eta^2).$$

We will see at the end of §4 that Φ fails to be holomorphic Segre transversal at $(z, w, \xi, \eta) \in \mathcal{H}^n$ if and only if $w + \eta = 0$. We also shows there, however, the connected complex submanifold of codimension one defined by $w + \eta = 0$ is not a holomorphic Segre-related set of $\mathcal{H}^n$.

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2 Normalization and a curvature equation

In this section, we shall use the strategy of Huang([Hua1]) and Huang-Ji([HJ1]). Let $n \geq 2$ and $\mathcal{M}$ be a connected open piece of $\mathcal{H}^n$ containing the origin. Let

$$\Phi = (\Phi_1(z, w), \Phi_2(\xi, \eta)) : \mathcal{M}(\subset \mathcal{H}^n) \to \mathcal{H}^N$$

be a holomorphic Segre map, where

$$\Phi := (\widetilde{f}, g, \widetilde{h}, e) = (\tilde{f}_1, \ldots, \tilde{f}_{N-1}, g, \tilde{h}_1, \ldots, \tilde{h}_{N-1}, e)$$

which is also written as $\Phi = (f, \phi, g, h, \psi, e) = (f_1, \ldots, f_{n-1}, \phi_1, \ldots, \phi_{N-n}, h_1, \ldots, h_{n-1}, \psi_1, \ldots, \psi_{N-n}, e)$. Write

$$\mathcal{L}_j = 2i\xi_j \frac{\partial}{\partial w} + \frac{\partial}{\partial z_j},\ \mathcal{K}_j = -2iz_j \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi_j},\ \mathcal{T} = \frac{\partial}{\partial w} + \frac{\partial}{\partial \eta}.$$

Then $\{\mathcal{L}_j, \mathcal{K}_j, \mathcal{T}\}_{j=1}^n$ forms a global basis for the space of sections of the complex tangent bundle $T^{(1,0)}\mathcal{H}^n$ of $\mathcal{H}^n$.

Notice that $\Phi(\mathcal{M}) \subset \mathcal{H}^N$ gives the following equation:

$$g(z, w) - e(\xi, \eta) = 2i\tilde{f}(z, w) \cdot \tilde{h}(\xi, \eta) \ \text{ over } \ w - \eta = 2iz \cdot \xi \tag{2.1}$$

where $a \cdot b := \sum_{j=1}^m a_j b_j$ for $a, b \in \mathbf{C}^m$.
Let $\mathbf{Z}_+$ be the set of non-negative integers. Then applying $\mathcal{L}^\alpha$, $\mathcal{K}^\alpha$, $\mathcal{T}$ and $\mathcal{K}_j\mathcal{L}_k$ to (2.1), respectively, where $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbf{Z}_+^{n-1}$ and $\mathcal{L}^\alpha = \mathcal{L}_1^{\alpha_1} \cdots \mathcal{L}_{n-1}^{\alpha_{n-1}}$, we have

$$\mathcal{L}^\alpha g = 2i\mathcal{L}^\alpha \widetilde{f} \cdot \widetilde{h}, \qquad \mathcal{K}^\alpha e = 2i\widetilde{f} \cdot \mathcal{K}^\alpha \tilde{h}, \tag{2.2}$$

$$\frac{\partial g}{\partial w} - \frac{\partial e}{\partial \eta} = 2i\frac{\partial \tilde{f}}{\partial w} \cdot \tilde{h} + 2i\tilde{f} \cdot \frac{\partial \tilde{h}}{\partial \eta}, \qquad \delta_j^k \frac{\partial g}{\partial w} = 2i\delta_j^k \frac{\partial \tilde{f}}{\partial w} \cdot \tilde{h} + \mathcal{L}_k \tilde{f} \cdot \mathcal{K}_j \tilde{h} \tag{2.3}$$

where δ_j^k is the standard Kronecker function. Assuming that $\Phi(0) = 0$ and letting $(z, w, \xi, \eta) = (0, 0, 0, 0)$ in (2.2),(2.3), we have

$$e(\xi, 0) \equiv 0,\ \frac{\partial g}{\partial w}(0) = \frac{\partial e}{\partial \eta}(0) = \mathcal{L}_j \tilde{f}(0) \cdot \mathcal{K}_j \tilde{h}(0),\ \ \mathcal{L}_j \tilde{f} \cdot \mathcal{K}_k \tilde{h} = 0 \ (j \neq k). \tag{2.4}$$

Let $M_{n \times N}(n \leq N)$ denote the set of n by N matrixes with all entries in $\mathbf{C}$. We then have the following elementary lemma:

Lemma 2.1: Let A, B $\in M_{n \times N}$ and $A \cdot B^t = Id_{n \times n}$. Then there exist $\tilde{A}$, $\tilde{B} \in M_{N \times N}$ whose first n rows are A and B, respectively, such that $\tilde{A} \cdot \tilde{B}^t = Id_{N \times N}$.

Proof of Lemma 2.1 : Consider the linear equation $A \cdot y^t = 0$ with $y \in \mathbf{C}^N$. Then its solution space has dimension $N - n$. Choose a basis $\{y^1, \cdots, y^{N-n}\}$ and define D to be the matrix whose k^{th}-row is precisely y^k. Then $A \cdot D^t = 0$. Considering the new matrix $\begin{pmatrix} B \\ D \end{pmatrix}$, it has full rank. In fact, suppose that $\begin{pmatrix} B \\ D \end{pmatrix}^t \cdot y^t = 0$ with $y = (y_1, \cdots, y_N) \in \mathbf{C}^N$. Then multiplying from the left by A, we obtain $y_j = 0$ for $j \leq n$. Hence, we get

$(y_{n+1}, \cdots, y_N)\cdot D = 0$. Since D has rank $N-n$, we conclude that $y=0$. Similarly, we can construct $C \in M_{(N-n)\times N}$ with rank $N-n$ such that $C\cdot B^t = 0$ and Rank $\begin{pmatrix} A \\ C \end{pmatrix} = N$. Since $\begin{pmatrix} A \\ C \end{pmatrix}\cdot\begin{pmatrix} B \\ D \end{pmatrix}^t = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & C\cdot D^t \end{pmatrix}$, $C\cdot D^t$ is invertible. Let $\tilde{A} = \begin{pmatrix} A \\ (C\cdot D^t)^{-1}\cdot C \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} B \\ D \end{pmatrix}$. Then we see the proof of the Lemma. ∎

Now assume $\lambda := \frac{\partial g}{\partial w}(0) \neq 0$. Write

$$A = \begin{pmatrix} \mathcal{K}_1(\tilde{h})(0)/\sqrt{\lambda} \\ \vdots \\ \mathcal{K}_{n-1}(\tilde{h})(0)/\sqrt{\lambda} \end{pmatrix}, \quad B = \begin{pmatrix} \mathcal{L}_1(\tilde{f})(0)/\sqrt{\lambda} \\ \vdots \\ \mathcal{L}_{n-1}(\tilde{f})(0)/\sqrt{\lambda} \end{pmatrix}.$$

Then (2.3) immediately gives that $A\cdot B^t = Id_{(n-1)\times(n-1)}$. Applying Lemma 2.1 to A and B, we obtain $\tilde{A}$ and $\tilde{B}$ with $\tilde{A}\cdot\tilde{B}^t = Id_{(N-1)\times(N-1)}$.

Next let $\tau(z^*, w^*, \xi^*, \eta^*) = (\frac{1}{\sqrt{\lambda}}z^*\tilde{A}^t, \frac{1}{\lambda}w^*, \frac{1}{\sqrt{\lambda}}\xi^*\tilde{B}^t, \frac{1}{\lambda}\eta^*)$. Obviously $\tau \in Aut_0(\mathcal{H}^N)$. By composing Φ with τ, we obtain the *first normalization* of Φ as follows:

$$\begin{aligned} \Phi^* &= (\tilde{f}^*, g^*, \tilde{h}^*, e^*): \\ &= \tau\circ\Phi = \frac{1}{\sqrt{\lambda}}\,\Phi\cdot\begin{pmatrix} \tilde{A}^t & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} & 0 & 0 \\ 0 & 0 & \tilde{B}^t & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}, \text{ with} \end{aligned}$$

$$\frac{\partial \widetilde{f_j^*}}{\partial z_k}(0) = \frac{\partial \widetilde{h_j^*}}{\partial \xi_k}(0) = \delta_j^k \qquad \text{for } 1\le k\le n-1,\ 1\le j\le N-1, \tag{2.5}$$

$$\frac{\partial g^*}{\partial w}(0) = \frac{\partial e^*}{\partial \eta}(0) = 1. \tag{2.6}$$

Write

$$u = \frac{1}{2}\frac{\partial^2 e^*}{\partial \eta^2}\bigg|_0,$$

$$\vec{a} = (\frac{\partial \tilde{f}_1^*}{\partial w}, \ldots, \frac{\partial \tilde{f}_{N-1}^*}{\partial w})\bigg|_0,$$

$$\vec{s} = 2i(\frac{\partial \tilde{h}_1^*}{\partial \eta}, \ldots, \frac{\partial \tilde{h}_{N-1}^*}{\partial \eta})\bigg|_0$$

and let $G \in Aut_0(\mathcal{H}^N)$ be defined by $G(z^*, w^*, \xi^*, \eta^*) :=$

$$\left(\frac{z^* - \vec{a}w^*}{1 + z^* \cdot \vec{s} + uw^*}, \frac{w^*}{1 + z^* \cdot \vec{s} + uw^*}, \frac{\xi^* + \frac{i}{2}\vec{s}\eta^*}{1 - 2i\xi^* \cdot \vec{a} + (u + \vec{a} \cdot \vec{s})\eta^*}, \frac{\eta^*}{1 - 2i\xi^* \cdot \vec{a} + (u + \vec{a} \cdot \vec{s})\eta^*}\right).$$

(See Theorem 6.3 of [HJ2]).
We then get the *second normalization* Φ^{**} as follows:

$$\Phi^{**} = (\tilde{f}^{**}, g^{**}, \tilde{h}^{**}, e^{**}) = (f^{**}, \phi^{**}, g^{**}, h^{**}, \psi^{**}, e^{**}) := G \circ \Phi^* = G \circ \tau \circ \Phi.$$

Simple computation shows that

$$\frac{\partial \tilde{f}^{**}}{\partial w}(0) = \frac{\partial \tilde{h}^{**}}{\partial \eta}(0) = 0, \qquad \frac{\partial f^{**}}{\partial z}(0) = \frac{\partial h^{**}}{\partial \xi}(0) = Id, \tag{2.7}$$

$$\frac{\partial \phi^{**}}{\partial z}(0) = \frac{\partial \psi^{**}}{\partial \xi}(0) = 0, \qquad \frac{\partial g^{**}}{\partial w}(0) = \frac{\partial e^{**}}{\partial \eta}(0) = 1, \qquad \frac{\partial^2 e^{**}}{\partial \eta^2}(0) = 0. \tag{2.8}$$

Now we assign the weight of (z, ξ) to be 1 and of (w, η) to be 2. In the following, we use $(\cdot)^l$ for a homogeneous polynomial of weighted degree l. For a function f on $\mathcal{H}^n$ we say that $f \in o_{wt}(k)$ if $\lim_{t \to 0} \frac{f(tz, t^2 w, t\xi, t^2 \eta)}{t^k} = 0$ uniformly for (z, w, ξ, η) in any compact subset of $\mathcal{H}^n$.

Next we do the weighted Taylor series expansion of Φ^{**}:

Lemma 2.2: Write Φ for the Φ^{**} above. Then Φ has the following form:

$f_j^{**} = z_j + J_j^{(1)}(z)w + o_{wt}(3),$
$\phi_j^{**} = B_j^{(2)}(z) + o_{wt}(2),$
$g^{**} = w + o_{wt}(4),$
$h_j^{**} = \xi_j + M_j^{(1)}(\xi)\eta + o_{wt}(3),$
$\psi_j^{**} = F_j^{(2)}(\xi) + o_{wt}(2),$
$e^{**} = \eta + o_{wt}(4),$ with

$$2i(J^{(1)}(z) \cdot \xi)(z \cdot \xi) = -B^{(2)}(z) \cdot F^{(2)}(\xi).$$

Proof of Lemma 2.2: First applying (2.7), (2.8) to Φ^{**}, we get the following weighted expansion:

$$
\begin{aligned}
f_j^{**} &= z_j + A_j^{(2)}(z) + I_j^{(3)}(z) + J_j^{(1)}(z)w + o_{wt}(3),\\
\phi_j^{**} &= B_j^{(2)}(z) + o_{wt}(2),\\
g^{**} &= w + Cw^2 + D^{(1)}(z)w + N^{(2)}(z)w + o_{wt}(4),\\
h_j^{**} &= \xi_j + E_j^{(2)}(\xi) + L_j^{(3)}(\xi) + M_j^{(1)}(\xi)\eta + o_{wt}(3),\\
\psi_j^{**} &= F_j^{(2)}(\xi) + o_{wt}(2),\\
e^{**} &= \eta + H^{(1)}(\xi)\eta + P^{(2)}(\xi)\eta + o_{wt}(4)
\end{aligned}
$$

and

$$g^{**} - e^{**} = 2i(f^{**} \cdot h^{**} + \phi^{**} \cdot \psi^{**}). \tag{2.9}$$

Collecting terms of weighted degree 3 in (2.9), we have:

$$D^{(1)}(z)w - H^{(1)}(\xi)\eta = 2i(z \cdot E^{(2)}(\xi) + A^{(2)}(z) \cdot \xi) \quad \text{over} \quad w = \eta + 2iz \cdot \xi.$$

Hence, $(D^{(1)}(z) - H^{(1)}(\xi))\eta + 2iz \cdot \xi^{(1)}(z) = 2i(z \cdot E^{(2)}(\xi) + A^{(2)}(z) \cdot \xi)$. Collecting coefficients of terms of the form: η, $z^2\xi$ and $z\xi^2$ in the above, we have:

$$
\begin{aligned}
&D^{(1)}(z) = H^{(1)}(\xi) = 0,\\
&A^{(2)}(z) = 0,\\
&E^{(2)}(\xi) = 0.
\end{aligned}
$$

Collecting terms of weighted degree 4 in (2.9), we have:

$$Cw^2 + N^{(2)}(z)w - P^{(2)}(\xi)\eta = 2i(z \cdot L^{(3)}(\xi) + z \cdot M^{(1)}(\xi)\eta + I^{(3)}(z) \cdot \xi + J^{(1)}(z)w \cdot \xi + B^{(2)}(z) \cdot F^{(2)}(\xi))$$

where $w = \eta + 2iz \cdot \xi$.
Similar arguments then show that the following holds:

$$C = 0, \tag{2.10}$$

$$N^{(2)}(z) = P^{(2)}(\xi) = 0, \tag{2.11}$$

$$L^{(3)}(\xi) = I^{(3)}(z) = 0, \tag{2.12}$$

$$2i(J^{(1)}(z) \cdot \xi)(z \cdot \xi) = -B^{(2)}(z) \cdot F^{(2)}(\xi), \tag{2.13}$$

$$z \cdot M^{(1)}(\xi) + J^{(1)}(z) \cdot \xi = 0. \tag{2.14}$$

This completes the proof of Lemma 2.2. ∎

If we assume $N - n \leq n - 2$, i.e. $N \leq 2n - 2$, then applying [Lemma 3.2, Hua1] to (2.13), we immediately get:

$$J^{(1)}(z) = 0,$$
$$B^{(2)}(z) \cdot F^{(2)}(\xi) = 0.$$

Hence, we have the following:

Lemma 2.3: With the same assumption as above, if we further assume that $N \leq 2n - 2$, then for the Φ in Lemma 2.2, we have the following weighted Taylor expansion:

$$f_j^{**} = z_j + o_{wt}(3), \tag{2.15}$$
$$\phi_j^{**} = B_j^{(2)}(z) + o_{wt}(2), \tag{2.16}$$
$$g^{**} = w + o_{wt}(4), \tag{2.17}$$
$$h_j^{**} = \xi_j + o_{wt}(3), \tag{2.18}$$
$$\psi_j^{**} = F_j^{(2)}(\eta) + o_{wt}(2), \tag{2.19}$$
$$e^{**} = \eta + o_{wt}(4) \tag{2.20}$$

with

$$B^{(2)}(z) \cdot F^{(2)}(\xi) = 0. \tag{2.21}$$

3 A partial linearity for Φ

We assume in this section that $N \leq 2n - 2$. Let Φ satisfies (2.15) through (2.21) in Lemma 2.3. Write

$$\Phi = (z + \hat{f}(z, w), \hat{\phi}(z, w), w + \hat{g}(z, w), \xi + \hat{h}(\xi, \eta), \hat{\psi}(\xi, \eta), \eta + \hat{e}(\xi, \eta)). \tag{3.1}$$

Theorem 3.1: With the above notation, we have $\hat{f} = \hat{h} = 0,\ \hat{g} = 0$ on $\mathcal{M}$ and $\hat{\phi} \cdot \hat{\psi} \equiv 0$ over $\mathcal{U}$. Moreover, after composing an element $\tau_U \in Aut_0(\mathcal{H}^N)$ from the left onto Φ, if necessary, there is a non-negative integer k such that $\phi_j \equiv 0$ for $j > k$ and $\psi_j \equiv 0$ for $j \leq k$.

Proof of Theorem 3.1: We will follow the approach used in [EHZ]. Since $\Phi(\mathcal{M}) \subset \mathcal{H}^N$, from (3.1), we have:

$$w + \hat{g}(z, w) - \eta - \hat{e}(\xi, \eta) = 2i(z + \hat{f}(z, w)) \cdot (\xi + \hat{h}(\xi, \eta)) + 2i\hat{\phi}(z, w) \cdot \hat{\psi}(\xi, \eta),\ w - \eta = 2iz \cdot \xi.$$

Therefore,

$$\hat{g}(z,w) - \hat{e}(\xi,\eta) - 2i\xi \cdot \hat{f}(z,w) - 2iz \cdot \hat{h}(\xi,\eta) = 2i\hat{\phi}(z,w) \cdot \hat{\psi}(\xi,\eta) + 2i\hat{f}(z,w) \cdot \hat{h}(\xi,\eta)$$
$$:= 2iA(z,w,\xi,\eta) \qquad \text{over} \quad w - \eta = 2iz \cdot \xi. \tag{3.2}$$

In view of the normalization obtained in Lemma 2.3, we have the following expansions:

$$\hat{f}(z,w) = \sum_{\mu+\nu\geq 2} f_{\mu\nu}(z)w^{\nu}, \qquad \hat{h}(\xi,\eta) = \sum_{\mu+\nu\geq 2} h_{\mu\nu}(\xi)\eta^{\nu},$$
$$\hat{\phi}(z,w) = \sum_{\mu+\nu\geq 2} \phi_{\mu\nu}(z)w^{\nu}, \qquad \hat{\psi}(\xi,\eta) = \sum_{\mu+\nu\geq 2} \psi_{\mu\nu}(\xi)\eta^{\nu},$$
$$\hat{g}(z,w) = \sum_{\mu+\nu\geq 2} g_{\mu\nu}(z)w^{\nu}, \qquad \hat{e}(\xi,\eta) = \sum_{\mu+\nu\geq 2} e_{\mu\nu}(\xi)\eta^{\nu},$$
$$A(z,w,\xi,\eta) = \sum_{\alpha+\beta+\mu+\nu\geq 4} A_{\alpha\mu\beta\nu}(z,\xi)w^{\mu}\eta^{\nu}$$

where $(\cdot)_{\mu\nu}(z)w^{\nu}$ and $(\cdot)_{\mu\nu}(\xi)\eta^{\nu}$ are homogeneous polynomials of degree (μ,ν) with respect to (z,w) and (ξ,η), respectively, and $(\cdot)_{\alpha\mu\beta\nu}(z,\xi)w^{\mu}\eta^{\nu}$ is a homogeneous polynomial of degree (α,μ,β,ν) with respect to (z,w,ξ,η). Letting $w=0$, i.e. $\eta = -2iz\cdot\xi$ in (3.2) and collecting terms of a fixed bi-degree (α,β) with respect to (z,ξ), we obtain:

$$\beta = 1 \Rightarrow \hat{f}(z,0) = 0, \tag{3.3}$$
$$\beta = 0 \Rightarrow \hat{g}(z,0) = 0. \tag{3.4}$$

Similarly, we also have

$$\hat{h}(\xi,0) = 0, \;\; \hat{e}(\xi,0) = 0. \tag{3.5}$$

We will use an induction argument to prove that $\hat{f}(z,w) = \hat{h}(\xi,\eta) = 0$ and $\hat{g}(z,w) = \hat{e}(\xi,\eta) = 0$. First by Lemma 2.3, we have $\hat{f}(z,w) = o_{wt}(3)$, $\hat{h}(\xi,\eta) = o_{wt}(3)$, $\hat{g}(z,w) = o_{wt}(4)$, $\hat{e}(\xi,\eta) = o_{wt}(4)$. Now, suppose that for l with $k-1 \geq l \geq 3$, $\hat{f}^{(l)}(z,w) = \hat{h}^{(l)}(\xi,\eta) = 0$ and $\hat{g}^{(l+1)}(z,w) = \hat{e}^{(l+1)}(\xi,\eta) = 0$. We then need to show that $\hat{f}^{(k)}(z,w) = \hat{h}^{(k)}(\xi,\eta) = 0$ and $\hat{g}^{(k+1)}(z,w) = \hat{e}^{(k+1)}(\xi,\eta) = 0$. Collecting terms of a fixed bi-degree (α,β) with respect to (z,ξ) with $\beta \geq 2$ and $\alpha+\beta \leq k+1$ and letting $w=0$ in (3.2), we get

$$-e_{\beta-\alpha,\alpha}(\xi)\eta^{\alpha} - 2iz\cdot h_{\beta-\alpha+1,\alpha-1}(\xi)\eta^{\alpha-1}$$
$$= 2i\sum_{p=0}^{\alpha-2} A_{\alpha-p,0,\beta-p,p}(z,\xi)\eta^{p}$$

$$= 2i\sum_{p=0}^{\alpha-2}\phi_{\alpha-p,0}(z)\psi_{\beta-p,p}(\xi)\eta^p + 2i\sum_{p=0}^{\alpha-2}f_{\alpha-p,0}(z)h_{\beta-p,p}(\xi)\eta^p. \tag{3.6}$$

By the induction assumption, we have for $k-1\geq l\geq 3$, $\hat{f}^{(l)}(z,w)=\hat{h}^{(l)}(\xi,\eta)=0$. We thus obtain $\sum_{p=0}^{\alpha-2}f_{\alpha-p,0}(z)h_{\beta-p,p}(\xi)\eta^p=0$ for $\alpha+\beta\leq k+1$. Then (3.6) becomes:

$$-e_{\beta-\alpha,\alpha}(\xi)\eta^\alpha - 2i\ z\cdot h_{\beta-\alpha+1,\alpha-1}(\xi)\eta^{\alpha-1}$$

$$= 2i\sum_{p=0}^{\alpha-2}A_{\alpha-p,0,\beta-p,p}(z,\xi)\eta^p$$

$$= 2i\sum_{p=0}^{\alpha-2}\phi_{\alpha-p,0}(z)\psi_{\beta-p,p}(\xi)\eta^p \tag{3.7}$$

where $\eta=-2iz\cdot\xi$.
Since we assumed that $N\leq 2n-2$, i.e. $N-n\leq n-2$. Applying [Lemma 3.2, EHZ] to (3.7), we have

$$A_{\mu 0\nu\delta}=0 \quad \text{for} \ \ \mu+\gamma+2\delta\leq k+1. \tag{3.8}$$

Hence

$$-e_{\beta-\alpha,\alpha}(\xi)\eta^\alpha - 2iz\cdot h_{\beta-\alpha+1,\alpha-1}(\xi)\eta^{\alpha-1}=0 \qquad \text{where} \quad \eta=-2iz\cdot\xi. \tag{3.9}$$

Similarly

$$A_{\mu\delta\nu 0}=0 \quad \text{for} \ \ \mu+\gamma+2\delta\leq k+1 \tag{3.10}$$

and

$$g_{\beta-\alpha,\alpha}(z)w^\alpha - 2i\xi\cdot f_{\beta-\alpha+1,\alpha-1}(z)w^{\alpha-1}=0 \qquad \text{where} \ \ w=2iz\cdot\xi. \tag{3.11}$$

Applying $\mathcal{L}_j$ and $\mathcal{K}_j$ to (3.2), we have

$$\mathcal{L}_j\hat{g}(z,w)-2i\xi\cdot\mathcal{L}_j\hat{f}(z,w)-2i\hat{h}_j(\xi,\eta)=2i\mathcal{L}_jA(z,w,\xi,\eta), \tag{3.12}$$

$$-\mathcal{K}_j\hat{e}(\xi,\eta)-2iz\cdot\mathcal{K}_j\hat{h}(\xi,\eta)-2i\hat{f}_j(z,w)=2i\mathcal{K}_jA(z,w,\xi,\eta) \tag{3.13}$$

over $w - \eta = 2iz \cdot \xi$.
By (3.3),(3.4),(3.5),(3.8) and (3.10), we have

$$\mathcal{L}_j(\hat{f}, \hat{g})(z, 0) = 2i\xi_j \sum_{\mu \geq 2} (f_{\mu 1}, g_{\mu 1})(z), \tag{3.14}$$

$$(\mathcal{L}_j(A))^{(l)}(z, 0, \xi, \eta) = 2i\xi_j \sum_{\mu+\nu+2\delta=l-1} A_{\mu 1 \nu \delta}(z, \xi)\eta^\delta \quad \text{for} \quad 0 \leq l \leq k, \tag{3.15}$$

$$\mathcal{K}_j(\hat{h}, \hat{e})(\xi, 0) = -2iz_j \sum_{\mu \geq 2} (h_{\mu 1}, e_{\mu 1})(\xi), \tag{3.16}$$

$$(\mathcal{K}_j(A))^{(l)}(z, w, \xi, 0) = -2iz_j \sum_{\mu+\nu+2\delta=l-1} A_{\mu \delta \nu 1}(z, \xi)w^\delta \quad \text{for} \quad 0 \leq l \leq k. \tag{3.17}$$

where in (3.15), $\eta = -2iz \cdot \xi$; in (3.17), $w = 2iz \cdot \xi$.
Substituting (3.14),(3.15),(3.16) and (3.17) to (3.12) and (3.13), we have

$$\beta = 2 \Rightarrow 2i\xi_j \xi \cdot f_{\alpha 1}(z) + h_{j;2-\alpha,\alpha}(\xi)\eta^\alpha = 0 \quad \text{for} \quad \alpha \leq 2 \quad \text{over} \quad \eta = -2iz \cdot \xi, \tag{3.18}$$

$$\beta = 1 \Rightarrow g_{\mu 1}(z) = 0 \quad \text{for} \quad \mu \leq k-1, \tag{3.19}$$

$\beta \geq 3$ and $\alpha + \beta = k \Rightarrow$

$$\begin{aligned} -h_{j;\beta-\alpha,\alpha}(\xi)\eta^\alpha &= \xi_j \sum_{p=0}^{\alpha-1} A_{\alpha-p,1,\beta-p-1,p}(z, \xi)\eta^p \\ &= \xi_j \sum_{p=0}^{\alpha-1} \phi_{\alpha-p,1}(z)\psi_{\beta-p-1,p}(\xi)\eta^p \quad \text{where} \quad \eta = -2iz \cdot \xi. \end{aligned} \tag{3.20}$$

Applying [Lemma 3.2, EHZ] again, we conclude:

$$h_{\mu\nu}(\xi) = 0 \quad \text{for} \quad \mu + \nu \geq 3 \quad \text{and} \quad \mu + 2\nu = k.$$

Back to (3.9), we get

$$e_{\mu\nu}(\xi) = 0 \quad \text{for} \quad \mu + \nu \geq 3 \quad \text{and} \quad \mu + 2\nu = k + 1.$$

Similarly, we have

$$f_{\mu\nu}(z) = 0 \quad \text{for} \quad \mu + \nu \geq 3 \quad \text{and} \quad \mu + 2\nu = k,$$

$$g_{\mu\nu}(z) = 0 \quad \text{for} \quad \mu + \nu \geq 3 \quad \text{and} \quad \mu + 2\nu = k + 1.$$

Notice that for $k \geq 4$, $\{(\mu, \nu) : \mu+\nu \geq 3 \text{ and } \mu+2\nu = k\} = \{(\mu, \nu) : \mu+2\nu = k\} - \{(0, 2)\}$.
On the other hand, substituting $f_{21}(z) = 0$ into (3.18) yields $h_{02} = 0$. Similarly we have

$f_{02}=0$. Hence, we proved that $\hat{f}^{(k)}(z,w)=\hat{h}^{(k)}(\xi,\eta)=0,\ \hat{g}^{(k+1)}(z,w)=\hat{e}^{(k+1)}(\xi,\eta)=0$. By induction, we conclude that

$$\tau\circ\Phi(z,w,\xi,\eta)=(z,\hat{\phi}(z,w),w,\xi,\hat{\psi}(\xi,\eta),\eta)$$

with

$$\sum_{j=1}^{N-n}\hat{\phi}_j(z,w)\hat{\psi}_j(\xi,\eta)=0\ \text{ over } \mathcal{M}.$$

Next we prove that $\sum_{j=1}^{N-n}\hat{\phi}_j(z,w)\hat{\psi}_j(\xi,\eta)\equiv 0$. To this aim, we need only to prove the following lemma, whose proof follows the same line as in [Lemma 4.2, EHZ]:

Lemma 3.2: If $A(z,w,\xi,\eta):=\sum_{j=1}^{k_0}\phi_j(z,w)\psi_j(\xi,\eta)=0$ over $w-\eta=2iz\cdot\xi,\ k_0\le n-2$. then $A(z,w,\xi,\eta)\equiv 0$ as a formal power series in (z,w,ξ,η).

Proof of Lemma 3.2: Write

$$\phi_j(z,w)=\sum_{\mu,\nu}\phi_{j;\mu\nu}(z)w^\nu,\quad \psi_j(\xi,\eta)=\sum_{\mu,\nu}\psi_{j;\mu\nu}(\xi)\eta^\nu$$

where $(\cdot)_{\mu\nu}(z)w^\nu,\ (\cdot)_{\mu\nu}(\xi)\eta^\nu$ are homogeneous polynomials of degree (μ,ν) with respect to (z,w) and (ξ,η). Denote $A_{\alpha\nu_1\beta\nu_2}(z,\xi)w^{\nu_1}\eta^{\nu_2}$ to be terms of degree $(\alpha,\nu_1,\beta,\nu_2)$ with respect to (z,w,ξ,η), respectively, for A. Assume $A(z,w,\xi,\eta)\neq 0$. Then there is a smallest nonnegative l_0 such that $A_{\alpha\nu_1\beta l_0}(z,\xi)\neq 0$ for some α,β,ν_1. We are going to reach a contradiction. In fact, since $A(z,w,\xi,\eta)=0$ on $w-\eta=2iz\cdot\xi$ or equivalently $\sum_{\alpha,\beta,\nu_1,\nu_2}A_{\alpha\nu_1\beta\nu_2}(z,\xi)(\eta+2iz\cdot\xi)^{\nu_1}\eta^{\nu_2}=0$. By factoring out η^{l_0} and setting $\eta=0$, we have

$$\sum_{\alpha,\beta,\nu_1}A_{\alpha\nu_1\beta l_0}(z,\xi)(2iz\cdot\xi)^{\nu_1}\equiv 0. \tag{3.21}$$

Collecting terms of bi-degree (μ_1,μ_2) with respect to (z,ξ) in (3.21), we obtain

$$\sum_{\nu_1}A_{\mu_1-\nu_1,\nu_1,\mu_2-\nu_1,l_0}(z,\xi)(2iz\cdot\xi)^{\nu_1}\equiv 0.$$

On the other hand

$$A_{\mu_1-\nu_1,\nu_1,\mu_2-\nu_1,l_0}(z,\xi)=\sum_{j=1}^{k_0}\phi_{j;\mu_1-\nu_1,\nu_1}(z)\psi_{j;\mu_2-\nu_1,l_0}(\xi)$$

where $k_0\le n-2$. It now follows from [Lemma 3.2, EHZ] that for any α,β,ν_1,

$$A_{\alpha\nu_1\beta l_0}(z,\xi)=0,$$

which contradicts the choice of l_0. This completes the proof of the lemma. ∎

Now, for simplicity of notation, assume that $\{\phi_1, \cdots, \phi_k\}$ is basis for the vector space spanned by $\{\phi_j\}_{j=1}^{N-n}$ over $\mathbf{C}$ with $k > 0$. Then there is an $(N-n) \times (N-n)$ invertible matrix U such that

$$(\phi_1, \cdots, \phi_k, 0, \cdots, 0) = (\phi_1, \cdots, \phi_{N-n}) \cdot U.$$

Now, define $\tau_U \in Aut_0(\mathcal{H}^N)$ by mapping $(z_1, \cdots, z_{N-1}, w, \xi_1, \cdots, \xi_{N-1}, \eta)$ to

$$\left(z_1, \cdots, z_{n-1}, (z_n, \cdots, z_{N-1}) \cdot U, \xi_1, \cdots, \xi_{n-1}, (\xi_n, \cdots, \xi_{N-1}) \cdot (U^{-1})^t\right).$$

Then $\tau_U \circ \Phi$ has the same form as above but with the extra property: $\sum_{j=1}^{k} \phi_j \psi_j \equiv 0$. Since $\{\phi_j\}_{j=1}^{k}$ is a linearly independent system, we get that $\psi_j \equiv 0$ for $j = 1, \cdots, k$. The proof of Theorem 3.1 is complete. ∎

4 Holomorphic Segre Transversality

For any $p \in \mathcal{H}^n$ close to the origin, recall in the introduction:

$$\Phi_p = (\tilde{f}_p, g_p, \tilde{h}_p, e_p) = (f_p, \phi_p, g_p, h_p, \psi_p, e_p) := (\widetilde{\sigma}^0_{\Phi(p)})^{-1} \circ \Phi \circ \sigma^0_p,$$

where for each $p = (z_0, w_0, \xi_0, \eta_0) \in \mathcal{H}^n$, $\sigma^0_p \in Aut(\mathcal{H}^n)$ and

$$\sigma^0_p(z, w, \xi, \eta) = (z + z_0, w + w_0 + 2iz \cdot \xi_0, \xi + \xi_0, \eta + \eta_0 - 2i\xi \cdot z_0)$$

for any $(z, w, \xi, \eta) \in \mathcal{H}^n$. Easy computation tells that for any $(z^*, w^*, \xi^*, \eta^*) \in \mathcal{H}^N$,

$$\begin{aligned}(\widetilde{\sigma}^0_{\Phi(p)})^{-1}(z^*, w^*, \xi^*, \eta^*) : = & (z^* - \tilde{f}(z_0, w_0), w^* - e(\xi_0, \eta_0) - 2iz^* \cdot \tilde{h}(\xi_0, \eta_0), \\ & \xi^* - \tilde{h}(\xi_0, \eta_0), \eta^* - g(z_0, w_0) + 2i\xi^* \cdot \tilde{f}(z_0, w_0)).\end{aligned}$$

Obviously $\Phi_p(0) = 0$. Without loss of generality, we may assume in this section $p_0 = 0, \widetilde{p}_0 = 0$ in Theorem 1.2.

Lemma 4.1: Φ is defined as in Theorem 1.2. Then Φ is holomorphic Segre transversal at the origin if and only if $\frac{\partial g}{\partial w}(0) \neq 0$.

Proof of Lemma 4.1: Write the coordinate of $\mathcal{H}^N$ to be $(\widetilde{z}, \widetilde{w}, \widetilde{\xi}, \widetilde{\eta})$. It is straightforward then $\{\frac{\partial}{\partial \widetilde{z}_j}\Big|_0\}_{j=1}^{N-1}$ and $\{\frac{\partial}{\partial \widetilde{\xi}_j}\Big|_0\}_{j=1}^{N-1}$ span $T_0^{(1,0)}\widetilde{Q}_0$ and $T_0^{(1,0)}\hat{\widetilde{Q}}_0$, respectively. On the other hand, $d\Phi((\frac{\partial}{\partial w} + \frac{\partial}{\partial \eta})\Big|_0) = \frac{\partial g}{\partial w}(0) \cdot (\frac{\partial}{\partial \widetilde{w}} + \frac{\partial}{\partial \widetilde{\eta}})\Big|_0 \mathrm{mod}(\frac{\partial}{\partial \widetilde{\xi}_j}\Big|_0, \frac{\partial}{\partial \widetilde{z}_j}\Big|_0, j = 1, \ldots, N-1)$ by the second equality of (2.4). The proof is thus complete by the definition of holomorphic Segre transversality. ∎

Since holomorphic Segre transversality is invariant under the composition of holomorphic automorphisms, we see that Φ is holomorphic Segre transversal at p if and only if Φ_p is holomorphic Segre transversal at 0. This is equivalent to $(g_p)_w(0) = g_w(z_0, w_0) - 2i\widetilde{f}_w(z_0, w_0) \cdot \tilde{h}(\xi_0, \eta_0) \neq 0$ where $(\cdot)_w := \frac{\partial(\cdot)}{\partial w}$ by the above Lemma. Hence, write $\mathcal{E}_\Phi$ for the set of points where Φ fails to be holomorphic Segre transversal. Then we have

$$\mathcal{E}_\Phi = \{(z, w, \xi, \eta) \in \mathcal{M} : g_w(z, w) - 2i\widetilde{f}_w(z, w) \cdot \tilde{h}(\xi, \eta) = 0\}.$$

In particular if $\mathcal{E}_\Phi \neq \mathcal{M}$, then we conclude $\mathcal{E}_\Phi$ is either empty or a complex analytic variety of codimension one in $\mathcal{M}$.

Proof of Proposition 1.6: We keep the same notation as in Proposition 1.6. Without loss of generality, we assume that $\mathcal{E}$ is defined by $\chi_1(z, w) = 0$ with $\chi_1(z, w)$ holomorphic over $\mathcal{M}$ and $\chi_1(z, w) \neq -K_0$ for any $(z, w, \xi, \eta) \in \mathcal{M}$ for some constant $K_0 \in \mathbf{C}$. Define

$$\Phi(z, w, \xi, \eta) = \left(\frac{\chi_1(z, w)z}{K_0 + \chi_1(z, w)}, \frac{K_0}{K_0 + \chi_1(z, w)}, \frac{\chi_1(z, w)w}{K_0 + \chi_1(z, w)}, \xi, \frac{i}{2}\eta, \eta \right).$$

Then, one can verify that Φ is a holomorphic Segre map from $\mathcal{M}$ into $\mathcal{H}^N$ with $N = n + 1$. Also, we can verify that $\mathcal{E}_\Phi$ is precisely the complex analytic variety defined by $\chi_1(z, w) = 0$.

Notice that when $\mathcal{E}$ is defined by $\chi_2(\xi, \eta) = 0$ where $\chi_2(\xi, \eta)$ is holomorphic over $\mathcal{M}$ and $\chi_2(\xi, \eta) \neq -K_0$ for any $(z, w, \xi, \eta) \in \mathcal{M}$ for some constant $K_0 \in \mathbf{C}$, then the holomorphic Segre map Φ with $\mathcal{E}_\Phi = \mathcal{E}$ is given as follows:

$$\Phi(z, w, \xi, \eta) = \left(z, -\frac{i}{2}w, w, \frac{\chi_2(\xi, \eta)\xi}{K_0 + \chi_2(\xi, \eta)}, \frac{K_0}{K_0 + \chi_2(\xi, \eta)}, \frac{\chi_2(\xi, \eta)\eta}{K_0 + \chi_2(\xi, \eta)} \right).$$

This proves Proposition 1.6. ∎

Proof of Proposition 1.7: If we assume $\chi_1(z, w) \neq K_1$ and $\chi_2(\xi, \eta) \neq K_2$ for any $(z, w, \xi, \eta) \in \mathcal{M}$ for some constants $K_1, K_2 \in \mathbf{C}$, then the holomorphic Segre map defined below meets the requirement:

$$\begin{aligned}\Phi(z, w, \xi, \eta) = \Bigg(& \frac{\chi_1(z, w)z}{K_1 + \chi_1(z, w)}, \frac{K_1}{K_1 + \chi_1(z, w)}, -\frac{i}{2}\frac{\chi_1(z, w)w}{(K_1 + \chi_1(z, w))}, \frac{\chi_1(z, w)w}{K_1 + \chi_1(z, w)}, \\ & \frac{\chi_2(\xi, \eta)\xi}{K_2 + \chi_2(\xi, \eta)}, \frac{i}{2}\frac{\chi_2(\xi, \eta)\eta}{(K_2 + \chi_2(\xi, \eta))}, \frac{K_2}{K_2 + \chi_2(\xi, \eta)}, \frac{\chi_2(\xi, \eta)\eta}{K_2 + \chi_2(\xi, \eta)} \Bigg)\end{aligned}$$

This proves Proposition 1.7. ∎

Proof of Theorem 1.5: Notice that a holomorphic Segre-related set is mapped to a holomorphic Segre-related set by a holomorphic automorphism of the complexification of the Heisenberg hypersurface. Without loss of generality, we can assume that $0 \in \mathcal{M}$ and Φ is holomorphic Segre transversal at 0 and Φ satisfies first normalization condition (2.5) and (2.6). Consider the Φ_p defined above. Assume that $\mathcal{E}_\Phi \neq \emptyset$. Then it is of codimension one. By Theorem 3.1, there exists $\tau \in Aut_0(\mathcal{H}^N)$ such that $\Phi^{**} = \tau \circ \Phi$ and

$$\Phi^{**}(z, w, \xi, \eta) = (z, \phi^{**}(z, w), w, \xi, \psi^{**}(\xi, \eta), \eta) \text{ with } \phi^{**}(z, w) \cdot \psi^{**}(\xi, \eta) \equiv 0 \text{ over } \mathcal{U}. \tag{4.1}$$

Notice that for any $(N - n) \times (N - n)$ invertible matrix U, $\Phi = (f, \phi, g, h, \psi, e)$, $\hat{\Phi} = (f, \phi, 0, g, h, \psi, 0, e)$ and $\hat{\hat{\Phi}} := (f, \phi \cdot U, g, h, \psi \cdot (U^{-1})^t, e)$ all have the same set of non-holomorphic Segre transversal points. Making use of the same argument as in the last paragraph in the proof of Theorem 3.1, we can assume, without loss of generality, that both $\{\phi\}$ and $\{\psi\}$ are linearly independent over $\mathbf{C}$ and $\phi_j\psi_j \equiv 0$ for $1 \leq j \leq N - n$.

Write E for the set of points in $\mathcal{M}$, whose image under Φ is contained in the pole of τ. Suppose that $p = (z_0, w_0, \xi_0, \eta_0) \notin E$. We have $(g^{**})_p = g^{**} \circ \sigma_p^0 - e^{**}(\xi_0, \eta_0) - 2i(\tilde{f}^{**} \circ \sigma_p^0) \cdot \tilde{h}^{**}(\xi_0, \eta_0)$. Hence

$$((g^{**})_p)_w(0) = (g_w^{**})(z_0, w_0) - 2i\tilde{f}_w^{**}(z_0, w_0) \cdot \tilde{h}^{**}(\xi_0, \eta_0). \tag{4.2}$$

Since ${\phi_j}_w^{**}(z, w) \cdot \psi_j^{**}(\xi, \eta) \equiv 0$ on $\mathcal{M}$, we thus get:

$$((g^{**})_p)_w(0) = 1 \neq 0 \quad \text{for any} \quad p \in \mathcal{M}.$$

This shows that Φ^{**} and thus Φ are holomorphic Segre transversal at p. Therefore, we get that $\mathcal{E}_\Phi \subset E$.

From the construction in §2, we notice that $\Phi^{**} = \tau \circ \Phi$ is given by the following expression:

$$\left(\frac{\tilde{f} - \vec{a}g}{1 + \tilde{f} \cdot \vec{s} + ug}, \frac{g}{1 + \tilde{f} \cdot \vec{s} + ug}, \frac{\tilde{h} + \frac{i}{2}\vec{s}e}{1 - 2i\tilde{h} \cdot \vec{a} + (u + \vec{a} \cdot \vec{s})e}, \frac{e}{1 - 2i\tilde{h} \cdot \vec{a} + (u + \vec{a} \cdot \vec{s})e} \right) \tag{4.3}$$

for certain $\vec{a}, \vec{s} \in \mathbf{C}^{N-1}$.

Now write $\chi_1 := 1 + \tilde{f} \cdot \vec{s} + ug$, $\chi_2 := 1 - 2i\tilde{h} \cdot \vec{a} + (u + \vec{a} \cdot \vec{s})e$, $A := \phi - \vec{a}g$ and $B := \psi - \frac{i}{2}\vec{s}e$ in (4.3). Then $\phi^{**}(z, w) = \frac{A(z,w)}{\chi_1(z,w)}$, $\psi^{**}(\xi, \eta) = \frac{B(\xi,\eta)}{\chi_2(\xi,\eta)}$ for any $(z, w, \xi, \eta) \in \mathcal{M}$.

Write $E_1 := \{(z,w,\xi,\eta)\in\mathcal{M}:\ \chi_1(z,w)=0\}$, $E_2 := \{(z,w,\xi,\eta)\in\mathcal{M}:\ \chi_2(\xi,\eta)=0\}$. Then $E=E_1\cup E_2$.

Claim 4.2: $\{E_1-E_2\}\cup\{E_2-E_1\}\subset\mathcal{E}_\Phi$.

Since $E_1\cap E_2$ is of codimension 2 in $\mathcal{M}$, together with what we obtained above, Claim 4.2 leads to the following:

Corollary 4.3: $E=\mathcal{E}_\Phi$.

Assuming Claim 4.2 for the moment, we next complete the proof of Theorem 1.5. Since each irreducible component of E is obviously a holomorphic Segre-related set of codimension one, we see that $\mathcal{E}_\Phi$ must be a locally finite union of holomorphic Segre-related sets of codimension one.

Back to (4.3), for any $p\in\mathcal{E}_\Phi$, we have either $p\in E_1$ or $p\in E_2$. Without loss of generality, assume the first one. Then $g(p)=0$. Now replacing Φ by Φ_{p_0} for any $p_0\in\mathcal{M}$ near the origin, then the holomorphic Segre transversality breaks down for Φ_{p_0} over $(\sigma^0_{p_0})^{-1}(E_1)$. By a similar argument as above, we may also assume, without loss of generality, that $g_{p_0}(p)=0$ for $p\in(\sigma^0_{p_0})^{-1}(E_1)$ and p_0 is in a certain connected open subset U of $\mathcal{M}$ with 0 in its closure.

Since $g_{p_0}=g\circ\sigma^0_{p_0}-e(p_0)-2i\widetilde{f}\circ\sigma^0_{p_0}\cdot\widetilde{h}(p_0)$, we have $g_{p_0}((\sigma^0_{p_0})^{-1}(p))=g(p)-e(p_0)-2i\widetilde{f}(p)\cdot\widetilde{h}(p_0)$. Let $p\in E_1$, then we have

$$e(p_0)=-2i\widetilde{f}(p)\cdot\widetilde{h}(p_0)\qquad\text{for any } p\in E_1,\ p_0\in U. \tag{4.4}$$

Notice that Φ satisfies the first normalization condition. Hence we have

$$h_j=\xi_j+o_{wt}(1),\quad \psi_j=o_{wt}(1),\quad e=o_{wt}(1).$$

Substituting the above into (4.4), we get

$$f_j(p)=0\qquad\text{for } j=1,\ldots,n-1,\ p\in E_1.$$

Therefore (4.4) becomes

$$e(p_0)=-2i\sum_{j=1}^{N-n}\phi_j(p)\psi_j(p_0)\qquad\text{for}\ \ p\in E_1,\ p_0\in U.$$

Since $\{\psi_k\}_{l=1}^{N-n}$ is a linearly independent system and U is a uniqueness set for holomorphic functions over $\mathcal{M}$, we immediately get $\phi_k\equiv$ constant over E_1. This proves that $\Phi(E_1)\subset\{q\}\times\hat{\widetilde{Q}}_q$ with $\Phi_1(E_1)\equiv q$. Similarly we can prove that $\Phi(E_2)\subset\widetilde{Q}_{\Phi_2(E_2)}\times\{\Phi_2(E_2)\}$, where $\Phi_2\equiv$ constant on E_2. Hence, to complete the proof of Theorem 1.5, we need only to give a proof for Claim 4.2.

Proof of Claim 4.2: We write $\mathcal{H}^N_{proj}$ for the compactification of $\mathcal{H}^N$ in $\mathbf{CP}^N \times \mathbf{CP}^N$ with the following homogeneous coordinates and defining equation:

$$[z_1, \cdots, z_N, t; \xi_1, \cdots, \xi_N, \gamma], \quad \gamma z_N - t\xi_N = 2i \sum_{j=1}^{N-1} z_j \xi_j.$$

Then the τ in (4.1) extends naturally to a new mapping, denoted by $\hat{\tau}$, from $\mathcal{H}^N$ to $\mathcal{H}^N_{proj}$. Write $\hat{\Phi} = \hat{\tau} \circ \Phi$. Then $\hat{\Phi}$ is a holomorphic map from $\mathcal{M}$ into $\mathcal{H}^N_{proj}$. Apparently, Φ is holomorphic Segre transversal at $p \in \mathcal{M}$ if and only if $\hat{\Phi}$ is holomorphic Segre transversal in a similar way.

For a fixed $p_0 \in E_1 - E_2$, there exists a $j \in \{1, \ldots, N-n\}$ such that $A_j(p) \neq 0$ by (4.3) and the definition of E_1. Without loss of generality, assume that $j = 1$. We next use the following local holomorphic coordinates chart of $\mathbf{CP}^N \times \mathbf{CP}^N$ near $\hat{\Phi}(p_0)$:

$$\sigma^n_N([z_1, \ldots, z_{N-1}, w, t; \xi_1, \cdots, \xi_{N-1}, \eta, 1]) :=$$
$$(\frac{z_1}{z_n}, \frac{z_2}{z_n}, \ldots, \frac{z_{n-1}}{z_n}, \frac{z_{n+1}}{z_n}, \frac{z_{N-1}}{z_n}, \frac{w}{z_n}, \frac{t}{z_n}, \xi_1, \ldots, \xi_{N-1}, \eta).$$

Then, with the new coordinates, the image of $\mathcal{H}^N_{proj}$ under this coordinate transformation is locally defined by the following equation:

$$\hat{\mathcal{H}}^N = \{(\hat{z}, \hat{w}, \hat{t}, \hat{\xi}, \hat{\eta}) \in \mathbf{C}^{N-2} \times \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{N-1} \times \mathbf{C} : \ \hat{w} - \hat{t}\hat{\eta} = 2i(\sum_{j=1}^{n-1} \hat{z}_j \hat{\xi}_j + \hat{\xi}_n + \sum_{j=n+1}^{N-1} \hat{z}_{j-1} \hat{\xi}_j)\}.$$

An easy computation shows that $\{\hat{\mathcal{L}}_j, \hat{\mathcal{K}}_j, \hat{\mathcal{T}}\}_{j=1}^{N-1}$ forms a global basis of the sections of holomorphic tangent bundle $T^{(1,0)}\hat{\mathcal{H}}^N$ of $\hat{\mathcal{H}}^N$ and that

$$< \hat{\theta}, \hat{\mathcal{T}} >= 1, \quad < \hat{\theta}, \hat{\mathcal{L}}_j >=< \hat{\theta}, \hat{\mathcal{K}}_j >= 0,$$

where

$$\hat{\mathcal{L}}_j = \frac{\partial}{\partial \hat{z}_j} + 2i\hat{\xi}_j \frac{\partial}{\partial \hat{w}}, \qquad \hat{\mathcal{K}}_j = -\frac{\partial}{\partial \hat{\xi}_j} + \hat{z}_j \frac{\partial}{\partial \hat{\xi}_n}, \qquad j = 1, \ldots, n-1,$$
$$\hat{\mathcal{L}}_n = \frac{\partial}{\partial \hat{t}} + \hat{\eta} \frac{\partial}{\partial \hat{w}}, \qquad \hat{\mathcal{K}}_n = -\hat{t} \frac{\partial}{\partial \hat{\xi}_n} + 2i \frac{\partial}{\partial \hat{\eta}},$$
$$\hat{\mathcal{L}}_j = \frac{\partial}{\partial \hat{z}_{j-1}} + 2i\hat{\xi}_j \frac{\partial}{\partial \hat{w}}, \qquad \hat{\mathcal{K}}_j = -\frac{\partial}{\partial \hat{\xi}_j} + \hat{z}_{j-1} \frac{\partial}{\partial \hat{\xi}_n}, \qquad j = n+1, \ldots, N-1,$$
$$\hat{\mathcal{T}} = 2i \frac{\partial}{\partial \hat{w}} + \frac{\partial}{\partial \hat{\xi}_n}, \qquad \hat{\theta} = \sum_{j=1}^{n-1} \hat{z}_j d\hat{\xi}_j + d\hat{\xi}_n + \sum_{j=n+1}^{N-1} \hat{z}_{j-1} d\hat{\xi}_j + \frac{\hat{t}}{2i} d\hat{\eta}.$$

Write $\mathcal{F}(z,w,\xi,\eta) := \sigma_N^n \circ \hat{\Phi}(z,w,\xi,\eta) = \sigma_N^n \circ \hat{\tau} \circ \Phi(z,w,\xi,\eta) = (\frac{\chi_1(z,w)z}{A_1(z,w)}, \frac{A_2(z,w)}{A_1(z,w)}, \ldots, \frac{A_{N-n}(z,w)}{A_1(z,w)}, \frac{\chi_1(z,w)w}{A_1(z,w)}, \frac{\chi_1(z,w)}{A_1(z,w)}, \xi, \psi^{**}(\xi,\eta), \eta)$ for $(z,w,\xi,\eta) \in \mathcal{M}$. Notice that σ_N^n is biholomorphic near $\hat{\Phi}(p_0)$. Hence Φ is holomorphic Segre transversal at p_0 if and only if $\mathcal{F}$ is holomorphic Segre transversal at p_0. However,

$$< \hat{\theta}, d\mathcal{F}(\mathcal{T}) > \Big|_{p_0} = (\frac{\partial \psi_1^{**}}{\partial \eta} + \sum_{j=2}^{N-n} \frac{A_j}{A_1} \frac{\partial \psi_j^{**}}{\partial \eta}) \Big|_{p_0} = 0,$$

where we used the facts that $\chi_1(p_0) = 0, \psi_1^{**} \equiv 0$ and $A_j \cdot \psi_j^{**} \equiv 0$ for $j \geq 2$. This yields that $\mathcal{F}$ and thus Φ are not holomorphic Segre transversal at p_0. Hence, $p_0 \in \mathcal{E}_\Phi$.

Similarly we can prove that $E_2 - E_1 \in \mathcal{E}_\Phi$. this completes the proof of Claim 4.2 and thus the proof of Theorem 1.5. ∎

As we pointed out before, Φ is holomorphic Segre transversal at p if Φ_p is holomorphic Segre transversal at 0 or equivalently $(g_p)_w(0) \neq 0$. It then follows in Example 1.8 that Φ is not holomorphic Segre transversal at $p = (z,w,\xi,\eta)$ iff $w + \eta = 0$ by using (4.2). Note that the submanifold $\mathcal{G} = \{(z,w,\xi,\eta) \in \mathcal{H}^n : \ w + \eta = 0\}$ is not a holomorphic Segre-related set of $\mathcal{H}^n$. In fact, for any point $(z_0, w_0, \xi_0, \eta_0) \in \mathcal{G}$, $\{(z_0,w_0)\} \times \hat{Q}_{(z_0,w_0)} = \{(z_0, w_0, \xi, w_0 - 2iz_0 \cdot \xi) : \ \xi \in \mathbf{C}^{n-1}\}$, which is not contained in $\mathcal{G}$. Similarly one can show that $Q_{(\xi_0,\eta_0)} \times \{(\xi_0,\eta_0)\}$ is not totally contained in $\mathcal{G}$. Thus $\mathcal{G}$ is not a holomorphic Segre-related set by definition. This example shows that the condition $N \leq 2n-2$ is critical for Theorem 1.5.

5 Proof of Theorem 1.2

We keep the same notation set up before.

Lemma 5.1: Let $\mathcal{M}$ be a connected neighborhood of 0 in $\mathcal{H}^n$. Suppose that the holomorphic Segre map Φ maps a neighborhood $\mathcal{U}$ of $\mathcal{M}$ in $\mathbf{C}^{2n}$ into $\mathbf{C}^{2N}$ with $\Phi(\mathcal{M}) \subset \mathcal{H}^N, \Phi(0) = 0$ and write $\Phi(z,w,\xi,\eta) = (\Phi_1(z,w), \Phi_2(\xi,\eta)) := (\tilde{f}_1(z,w), \ldots, \tilde{f}_{N-1}(z,w), g(z,w), \tilde{h}_1(\xi,\eta), \ldots, \tilde{h}_{N-1}(\xi,\eta), e(\xi,\eta))$ for $(z,w,\xi,\eta) \in \mathcal{M}$. Assume that $N \leq 2n-2$. If there exists a neighborhood V of 0 in $\mathcal{M}$, such that for every $p \in V$, $(g_p)_w(0) = 0$, then $g \equiv 0, e \equiv 0$ and $\widetilde{f} \cdot \widetilde{h} \equiv 0$ over $\mathcal{U}$.

To prove Lemma 5.1, we need the following:

Lemma 5.2: Suppose that $A, B \in M_{(n-1)\times(N-1)}$ where $N \leq 2n-2$ satisfy that $A \cdot \bar{B}^t = 0$. Then either A or B has rank less than $n-1$.

Proof of Lemma 5.2 : Suppose A has rank $n-1$. Then the linear equation $A \cdot y^t = 0$ has at most $N-1-(n-1) = N-n$ linearly independent solutions, which implies that $rank(B) \leq N-n < n-1$. ∎

Proof of Lemma 5.1: We follows the same approach in [BH] for the proof of the Lemma. By the definition of Φ_p, we have:

$$g_p = g \circ \sigma_p^0 - e(\xi_0, \eta_0) - 2i\tilde{f} \circ \sigma_p^0 \cdot \tilde{h}(\xi_0, \eta_0). \tag{5.1}$$

Hence it follows that

$$(g_p)_w(0) = g_w(z_0, w_0) - 2i\tilde{f}_w(z_0, w_0) \cdot \tilde{h}(\xi_0, \eta_0),$$

where $p = (z_0, w_0, \xi_0, \eta_0) \in \mathcal{M}$. By the assumption, $g_w(z,w) = 2i\tilde{f}_w(z,w) \cdot \tilde{h}(\xi, \eta)$ on V, i.e.,

$$g_w(z, \eta + 2iz \cdot \xi) = 2i\tilde{f}_w(z, \eta + 2iz \cdot \xi) \cdot \tilde{h}(\xi, \eta). \tag{5.2}$$

Let $\xi = 0,\ \eta = 0$, we have

$$g_w(z, 0) = 0.$$

Applying $\frac{\partial}{\partial \xi_j},\ j = 1, \ldots, n-1$ to (5.2) and letting $\xi = 0,\ \eta = 0$, we get

$$z_j g_{w^2}(z, 0) = \tilde{f}_w(z, 0) \cdot \tilde{h}_{\xi_j}(0). \tag{5.3}$$

On the other hand, applying $\mathcal{K}_j \mathcal{L}_k$ to $g_p - e_p = 2i\tilde{f}_p \cdot \tilde{h}_p$ and letting $(z, w, \xi, \eta) = 0$, we have

$$\delta_j^k (g_p)_w(0) = (\tilde{f}_p)_{z_k}(0) \cdot (\tilde{h}_p)_{\xi_j}(0) \qquad \text{for any } j, k = 1, \ldots, n-1. \tag{5.4}$$

Applying lemma 5.2 and making use of the assumption $(g_p)_w(0) = 0$ for $p \in V$ on (5.4), we get that

$$V = A \cup B,$$

where $$A = \{p : \{(\tilde{h}_p)_{\xi_j}(0)\}_{j=1}^{n-1} \text{ are linearly dependent}\};$$

$$B = \{p : \{(\tilde{f}_p)_{z_j}(0)\}_{j=1}^{n-1} \text{ are linearly dependent}\}.$$

Then either A or B contains an open neighborhood of V. Without loss of generality, assume that $V_1 (\ni p_0) \subset V$ is an open piece of $\mathcal{M}$ that is contained in A. By considering Φ_{p_0} instead of Φ, we assume, without loss of generality, that $p_0 = 0$.

Therefore $\{(\tilde{h}_p)_{\xi_j}(0)\}_{j=1}^{n-1}$ are linearly dependent for p in some small neighborhood of $\mathcal{M}$ near 0. Take a non-zero $(n-1)$-tuple $(a_1, \ldots, a_{n-1})$ such that $\sum_{j=1}^{n-1} a_j \tilde{h}_{\xi_j}(0) = 0$. It thus follows from (5.3) that $\sum_{j=1}^{n-1} a_j z_j g_{w^2}(z,0) = 0$. Since $\sum_{j=1}^{n-1} a_j z_j \neq 0$, we conclude $g_{w^2}(z,0) = 0$. Now applying the previous argument to Φ_p, we then get $(g_p)_{w^2}(z,0) = 0$ for p in some small neighborhood in $\mathcal{M}$. An induction argument then shows that $g_{w^k}(z,0) = 0$ for $k \geq 0$. Since we also have $g_{z^k}(0,0) = 0$ for $k \geq 0$, we then proved that $g \equiv 0$. Similarly we have $g_p \equiv 0$. Substituting $g \equiv 0$ and $g_p \equiv 0$ into (5.1), we have

$$e(\xi_0, \eta_0) = -2i\tilde{f} \circ \sigma_p^0(z,w) \cdot \tilde{h}(\xi_0, \eta_0)$$

for any $p = (z_0, w_0, \xi_0, \eta_0)$ in $\mathcal{M}$. Choose $(z,w) = (\sigma_p^0)^{-1}(0)$ in the above, then we have $e \equiv 0$. Again by (5.1) we see $\widetilde{f} \cdot \widetilde{h} \equiv 0$ in $\mathcal{U}$. The proof is complete. ∎

Proof of Theorem 1.2: Theorem 1.2 (1) follows from Theorem 3.1 and Theorem 1.2 (2) is an easy consequence of Lemma 5.1. ∎

Remark 5.3: When $\Phi_2(z,w) = \bar{\Phi}_1(z,w)$ in Theorem 1.2, a super-rigidity can be deduced. Namely, one further concludes that $\phi(z,w)$ and $\psi(\xi,\eta)$ must be identically zero and Theorem 1.2 (2) can not occur. Theorem 1.2 in this case then reduces to a Theorem of Faran and Huang (See [Fa1] [Hua1]). When $\Phi_2(z,w) \neq \bar{\Phi}_1(z,w)$, one can easily write down the following example, showing that Φ does not have to be linear nor rational: $\Phi : \mathcal{H}^3 \to \mathcal{H}^4$, where

$$\Phi(z_1, z_2, w, \xi_1, \xi_2, \eta) = (z_1, z_2, \cos z_1, w, \xi_1, \xi_2, 0, \eta).$$

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