Rigidity for local holomorphic isometric embeddings from $\mathbb{B}^n$ into $\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}$

Yuan Yuan and Yuan Zhang

Abstract

In this article, we study local holomorphic isometric embeddings from $\mathbb{B}^n$ into $\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}$ with respect to the normalized Bergman metrics up to conformal factors. Assume that each conformal factor is smooth Nash algebraic. Then each component of the map is a multi-valued holomorphic map between complex Euclidean spaces by the algebraic extension theorem derived along the lines of Mok and Mok-Ng. Applying holomorphic continuation and analyzing real analytic subvarieties carefully, we show that each component is either a constant map or a proper holomorphic map between balls. Applying a linearity criterion of Huang, we conclude the total geodesy of non-constant components.

1 Introduction

Write $\mathbb{B}^n := \{z \in \mathbb{C}^n : |z| < 1\}$ for the unit ball in $\mathbb{C}^n$. Denote by ds_n^2 the normalized Bergman metric on $\mathbb{B}^n$ defined as follows:

$$ds_n^2 = \sum_{j,k \leq n} \frac{1}{(1-|z|^2)^2}\big((1-|z|^2)\delta_{jk} + \bar{z}_j z_k\big) dz_j \otimes d\bar{z}_k. \tag{1}$$

Let $U \subset \mathbb{B}^n$ be a connected open subset. Consider a holomorphic embedding

$$F = (F_1, \ldots, F_m) : (U, ds_n^2) \to (\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}, \oplus_{j=1}^m ds_{N_j}^2) \tag{2}$$

with conformal factors $\{\lambda(z, \bar{z}); \lambda_1(z, \bar{z}), \cdots, \lambda_m(z, \bar{z})\}$ in the sense that

$$\lambda(z, \bar{z}) ds_n^2 = \sum_{j=1}^m \lambda_j(z, \bar{z}) F_j^*(ds_{N_j}^2).$$

Here and in what follows, the conformal factors $\lambda(z, \bar{z}),\ \lambda_j(z, \bar{z})$ $(j = 1, \cdots, m)$ are assumed to be positively real-valued smooth Nash algebraic functions over $\mathbb{C}^n$. One can in fact assume that $\lambda(z, \bar{z}) = 1$, and replace $\lambda_j(z, \bar{z})$ by $\frac{\lambda_j(z,\bar{z})}{\lambda(z,\bar{z})}$. Under such notation, $\lambda_j(z, \bar{z})$ is assumed to be positive, smooth and Nash algebraic. Moreover, for each j with $1 \le j \le m$, $ds^2_{N_j}$ denotes the corresponding normalized Bergman metric of $\mathbb{B}^{N_j}$ and F_j is a holomorphic map from U to $\mathbb{B}^{N_j}$. We write $F_j = (f_{j,1}, \ldots, f_{j,l}, \ldots, f_{j,N_j})$, where $f_{j,l}$ is the l-th component of F_j. In this paper, we prove the following rigidity theorem:

Theorem 1.1. *Suppose $n \ge 2$. Under the above notation and assumption, we then have, for each j with $1 \le j \le m$, that either F_j is a constant map or F_j extends to a totally geodesic holomorphic embedding from $(\mathbb{B}^n, ds^2_n)$ into $(\mathbb{B}^{N_j}, ds^2_{N_j})$. Moreover, we have the following identity*

$$\sum_{F_j \text{ is not a constant}} \lambda_j(z,\bar{z}) = \lambda(z,\bar{z}).$$

In particular, when $\lambda_j(z,\bar{z}), \lambda(z,\bar{z})$ are positive constant functions, we have the following rigidity result for local isometric embeddings:

Corollary 1.2. *Let*

$$F = (F_1, \ldots, F_m) : (U \subset \mathbb{B}^n, \lambda ds^2_n) \to (\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}, \oplus_{j=1}^m \lambda_j ds^2_{N_j}) \tag{3}$$

be a local holomorphic isometric embedding in the sense that $\lambda ds^2_n = \sum_{j=1}^m \lambda_j F_j^*(ds^2_{N_j})$. Assume that $n \ge 2$ and λ, λ_j are positive constants. We then have, for each j with $1 \le j \le m$, that either F_j is a constant map or F_j extends to a totally geodesic holomorphic embedding from $(\mathbb{B}^n, ds^2_n)$ into $(\mathbb{B}^{N_j}, ds^2_{N_j})$. Moreover, we have the following identity

$$\sum_{F_j \text{ is not a constant}} \lambda_j = \lambda.$$

Recall that a function $h(z,\bar{z})$ is called a Nash algebraic function over $\mathbb{C}^n$ if there is an irreducible polynomial $P(z,\xi,X)$ in $(z,\xi,X) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$ with $P(z,\bar{z},h(z,\bar{z})) \equiv 0$ over $\mathbb{C}^n$. We mention that a holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ is a totally geodesic embedding with respect to the normalized Bergman metric if and only if there are a (holomorphic) automorphism $\sigma \in Aut(\mathbb{B}^n)$ and an automorphism $\tau \in Aut(\mathbb{B}^N)$ such that $\tau \circ F \circ \sigma(z) \equiv (z, 0)$. Also, we mention that by the work of Mok [Mo1], the result in Corollary 1.2 does not hold anymore when $n = 1$. (See also many examples and related classification results in the work of Ng ([Ng1]).

The study of the global extension and rigidity problem for local isometric embedding was first carried out in a paper of Calabi [Ca]. After [Ca], there appeared quite a few papers along these lines of research (see [Um], for instance). In 2003, motivated by problems from Arithmetic Algebraic Geometry, Clozel and Ullmo [CU] took up again the problem by considering the rigidity problem for a local isometric embedding with a certain symmetry from $\mathbb{B}^1$ into $\mathbb{B}^1 \times \cdots \times \mathbb{B}^1$. More recently, Mok carried out a systematic study of this problem in a very general setting. Many far reaching deep results have been obtained by Mok and later by Ng and Mok-Ng. (See [Mo1] [Mo2] [MN] [Ng1-3] and the references therein). Here, we would like to mention that our result was already included in the papers by Calabi when $m = 1$ [Ca], by Mok [Mo1] [Mo2] when $N_1 = \cdots = N_m$, and by Ng [Ng1] [Ng3] when $m = 2$ and $N_1, N_2 < 2n$.

As in the work of Mok [Mo1], our proof of the theorem is also based on the similar algebraic extension theorem derived in [Mo2] and Mok-Ng [MN]. However, different from the case considered in [Mo1] [Ng2], the properness of a factor of F does not immediately imply the linearity of that factor; for the classical linearity theorem does not hold anymore for proper rational mappings from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N > 2n-2$. (See [Hu1]). Hence, the cancelation argument as in [Mok1] [Ng3] seems to be difficult to apply in our setting.

In our proof of Theorem 1.1, a major step is to prove that a non-constant component F_j of F must be proper from $\mathbb{B}^n$ into $\mathbb{B}^{N_j}$, using the multi-valued holomorphic continuation technique. This then reduces the proof of Theorem 1.1 to the case when all components are proper. Unfortunately, due to the non-constancy for the conformal factors $\lambda_j(z, \bar{z})$ and $\lambda(z, \bar{z})$, it is not immediate that each component must also be conformal (and thus λ_j must be a constant multiple of λ) with respect to the normalized Bergman metric. However, we observe that the blowing-up rate for the Bergman metric of $\mathbb{B}^n$ with $n \geq 2$ in the complex normal direction is twice of that along the complex tangential direction, when approaching the boundary. From this, we will be able to derive an equation regarding the CR invariants associated to the map at the boundary of the ball. Lastly, a linearity criterion of Huang in [Hu1] can be applied to simultaneously conclude the linearity of all components.

We mention that in the context of Corollary 1.2, namely, when each conformal factor is assumed to be constant, the proof used to prove Theorem 1.1 can be further simplified as told us by Mok and Ng in their private communications. In this case, one can work directly on the Kähler potential functions instead of the hyperbolic metrics. However, when the conformal factors are not constant, and $\partial\bar{\partial}$-lemma cannot be applied and the metric equation (which can be regarded as differential equations on the map) does not lead to the functional equation on the components of the map. We appreciate very much many valuable comments of Mok and Ng to the earlier version of this paper, especially, for telling us how to essentially simplify the proof of a key lemma (Lemma 2.2) through the consideration of the metric potential functions. Their very helpful comments lead to the present version.

Acknowledgement: The present work is written under the guidance of Professor Xiaojun Huang. The authors are deeply indebted to Professor Huang, especially, for formulating the problem and for suggesting the method used in the paper. The authors also would like to thank very much N. Mok and, in particular, S. Ng, for many stimulating discussions and conversations which inspired the present work as well as many valuable suggestions and comments. In fact, this work was originated by reading the work of Mok ([Mo1]) and Ng ([Ng1-3]). The authors acknowledge the partial financial support of NSF-0801056 for the summer research project (through Professor Huang). Part of the work was done when the first author was visiting Erwin Schrödinger Institute in Vienna, Austria in the fall semester of 2009. He also would like to thank members in the institute, especially, Professor Lamel for the invitation and hospitality. Finally, the authors would like to thank the referee for the valuable comments.

2 Bergman metric and proper rational maps

Let $\mathbb{B}^n$ and ds_n^2 be the unit ball and its normalized Bergman metric, respectively, as defined before. Denote by $\mathbb{H}^n \subset \mathbb{C}^n$ the Siegel upper half space. Namely, $\mathbb{H}^n = \{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \Im w - |z|^2 > 0\}$. Here, for m-tuples a, b, we write dot product $a \cdot b = \sum_{j=1}^m a_j b_j$ and $|a|^2 = a \cdot \bar{a}$. Recall the following Cayley transformation

$$\rho_n(z,w) = \left(\frac{2z}{1-iw}, \frac{1+iw}{1-iw}\right). \tag{4}$$

Then ρ_n biholomorphically maps $\mathbb{H}^n$ to $\mathbb{B}^n$, and biholomorphically maps $\partial\mathbb{H}^n$, the Heisenberg hypersurface, to $\partial\mathbb{B}^n \backslash \{(0,1)\}$. Applying the Cayley transformation, one can compute the normalized Bergman metric on $\mathbb{H}^n$ as follows:

$$\begin{aligned} ds^2_{\mathbb{H}^n} = &\sum_{j,k<n} \frac{\delta_{jk}(\Im w - |z|^2) + \bar{z}_j z_k}{(\Im w - |z|^2)^2} dz_j \otimes d\bar{z}_k + \frac{dw \otimes d\bar{w}}{4(\Im w - |z|^2)^2} \\ &+ \sum_{j<n} \frac{\bar{z}_j dz_j \otimes d\bar{w}}{2i(\Im w - |z|^2)^2} - \sum_{j<n} \frac{z_j dw \otimes d\bar{z}_j}{2i(\Im w - |z|^2)^2}. \end{aligned} \tag{5}$$

One can easily check that

$$\begin{aligned} L_j &= \frac{\partial}{\partial z_j} + 2i\bar{z}_j \frac{\partial}{\partial w}, j = 1, \ldots, n-1. \\ \overline{L_j} &= \frac{\partial}{\partial \bar{z}_j} - 2iz_j \frac{\partial}{\partial \bar{w}}, j = 1, \ldots, n-1. \\ T &= 2(\frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}}) \end{aligned}$$

span the complexified tangent vector bundle of $\partial\mathbb{H}^n$. (See, for instance, [BER], [Hu2], [Hu3] [HJX] [JX] [Mi].)

Let F be a rational proper holomorphic map from $\mathbb{H}^n$ to $\mathbb{H}^N$. By a result of Cima-Suffridge [CS], F is holomorphic in a neighborhood of $\partial\mathbb{H}^n$. Assign the weight of w to be 2 and that of z to be 1. Denote by $o_{wt}(k)$ terms with weighted degree higher than k and by $P^{(k)}$ a function of weighted degree k. For $p_0 = (z_0, w_0) \in \partial\mathbb{H}^n$, write $\sigma^0_{p_0} : (z,w) \to (z + z_0, w + w_0 + 2iz \cdot \bar{z}_0)$ for the standard Heisenberg translation. The following normalization lemma will be used here:

Lemma 2.1. *[Hu2-3] For any $p \in \partial\mathbb{H}^n$, there is an element $\tau \in Aut(\mathbb{H}^{N+1})$ such that the map $F_p^{**} = ((f_p^{**})_1(z), \cdots, (f_p^{**})_{n-1}(z), \phi_p^{**}, g_p^{**}) = (f_p^{**}, \phi_p^{**}, g_p^{**}) = \tau \circ F \circ \sigma_p^0$ takes the following normal form:*

$$\begin{aligned} f_p^{**}(z,w) &= z + \frac{i}{2}a^{(1)}(z)w + o_{wt}(3), \\ \phi_p^{**}(z,w) &= \phi^{(2)}(z) + o_{wt}(2), \\ g_p^{**}(z,w) &= w + o_{wt}(4) \end{aligned}$$

with

$$(\bar{z} \cdot a^{(1)}(z))|z|^2 = |\phi^{(2)}(z)|^2. \tag{6}$$

In particular, write $(f_p^{**})_l(z) = z_j + \frac{i}{2}\sum_{k=1}^{n-1} a_{lk} z_k w + o_{wt}(3)$. Then, $(a_{lk})_{1\le l,k\le n-1}$ is an $(n-1)\times(n-1)$ semi-positive Hermitian metrix. We next present the following key lemma for our proof of Theorem 1.1:

Lemma 2.2. *Let F be a proper rational map from $\mathbb{B}^n$ to $\mathbb{B}^N$. Then*

$$X := ds_n^2 - F^*(ds_N^2), \tag{7}$$

is a semi-positive real analytic symmetric (1,1)-tensor over $\mathbb{B}^n$ that extends also to a real analytic (1,1)-tensor in a small neighborhood of $\partial\mathbb{B}^n$ in $\mathbb{C}^n$.

Proof of lemma 2.2: Our original proof was largely simplified by Ng [Ng4] and Mok [Mo4] by considering the potential $-\log(1-\|F(z)\|^2)$ of the pull-back metric $F^*(ds_N^2)$ as follows: Since $1-\|F(z)\|^2$ vanishes identically on $\partial\mathbb{B}^n$ and since $1-\|z\|^2$ is a defining equation for $\partial\mathbb{B}^n$, one obtains

$$1-\|F(z)\|^2 = (1-\|z\|^2)\varphi(z)$$

for a real analytic function $\varphi(z)$.

Since $\rho := \|F(z)\|^2 - 1$ is subharmonic over $\mathbb{B}^n$ and has maximum value 0 on the boundary, applying the classical Hopf lemma, we conclude that $\varphi(z)$ cannot vanish at any boundary point of $\mathbb{B}^n$. Apparently, $\varphi(z)$ cannot vanish inside $\mathbb{B}^n$. Therefore, $X = \sqrt{-1}\partial\bar{\partial}\log\varphi(z)$ is real analytic on an open neighborhood of $\overline{\mathbb{B}^n}$. The semi-positivity of X over $\mathbb{B}^n$ is an easy consequence of the Schwarz lemma. ■

Applying the Cayley transformation (and also a rotation transformation when handling the regularity near $(0,1)$), we have the following corollary:

Corollary 2.3. *Let F be a rational proper holomorphic map from $\mathbb{H}^n$ to $\mathbb{H}^N$. Then*

$$X := ds_{\mathbb{H}^n}^2 - F^*(ds_{\mathbb{H}^N}^2), \tag{8}$$

is a semi-positive real analytic symmetric (1,1)-tensor over $\mathbb{H}^n$ that extends also to a real analytic (1,1)-tensor in a small neighborhood of $\partial\mathbb{H}^n$ in $\mathbb{C}^n$.

The boundary value of X is an intrinsic CR invariant associated with the equivalence class of the map F. Next, we compute X in the normal coordinates at the boundary point.

Write $t = \Im w - |z|^2$ and $H = \Im g - |\tilde{f}|^2$. Here $(\tilde{f}, g)$ denotes the map between Heisenberg hypersurfaces. Write $o(k)$ for terms whose degrees with respect to t are higher than k. For a real analytic function h in (z, w), we use h_z, h_w to denote the derivatives of h with respect to z, w. By replacing w by $u + i(t + |z|^2)$, H can also be regarded as an analytic function on $z, \bar{z}, u, t$. The following lemma gives an asymptotic behavior of H with respect to t:

Lemma 2.4. $H(z,\bar z,u,t)=(g_w-2i\tilde f_w\cdot\bar{\tilde f})|_{t=0}t-(2|\tilde f_w|^2)|_{t=0}t^2+\frac13(-\frac12 g_{w^3}+3i\tilde f_w\cdot\overline{\tilde f_{w^2}}+i\tilde f_{w^3}\cdot\bar{\tilde f})|_{t=0}t^3+o(3)$.

Proof of Lemma 2.4: Notice that $H=H(z,\bar z,u+i(t+|z|^2),u-i(t+|z|^2))$. Since F is proper, H, as a function of t with parameters $\{z,u\}$, can be written as $P_1t+P_2t^2+P_3t^3+o(3)$, where P_1,P_2,P_3 are analytic in $(z,\bar z,u)$. Then

$$\begin{aligned}P_1&=\frac{\partial H(z,\bar z,u+i(t+|z|^2),u-i(t+|z|^2))}{\partial t}\Big|_{t=0}\\&=iH_w-iH_{\bar w}\Big|_{t=0}\\&=\frac12(g_w+\overline{g_w})+i(\tilde f\cdot\overline{\tilde f_w}-\bar{\tilde f}\cdot\tilde f_w)\Big|_{t=0},\end{aligned}\tag{9}$$

$$\begin{aligned}P_2&=\frac12\frac{\partial^2 H(z,\bar z,u+i(t+|z|^2),u-i(t+|z|^2))}{\partial t^2}\Big|_{t=0}\\&=\frac12(-H_{w^2}+2H_{w\bar w}-H_{\bar w^2})\Big|_{t=0}\\&=\frac12(\frac i2 g_{w^2}-\frac i2\overline{g_{w^2}}-2|\tilde f_w|^2+\tilde f_{w^2}\cdot\bar{\tilde f}+\tilde f\cdot\overline{\tilde f_{w^2}})\Big|_{t=0},\end{aligned}\tag{10}$$

and

$$\begin{aligned}P_3&=\frac16\frac{\partial^3 H(z,\bar z,u+i(t+|z|^2),u-i(t+|z|^2))}{\partial t^3}\Big|_{t=0}\\&=\frac16(-iH_{w^3}+3iH_{w^2\bar w}-3iH_{\bar w^2 w}+iH_{\bar w^3})\Big|_{t=0}\\&=\frac16(-\frac12 g_{w^3}-\frac12\overline{g_{w^3}}+i\tilde f_{w^3}\cdot\bar{\tilde f}-i\tilde f\cdot\overline{\tilde f_{w^3}}-3i\tilde f_{w^2}\cdot\overline{\tilde f_w}+3i\tilde f_w\cdot\overline{\tilde f_{w^2}})\Big|_{t=0}.\end{aligned}\tag{11}$$

On the other hand, applying T,T^2,T^3 to the defining equation $g-\bar g=2i\tilde f\cdot\bar{\tilde f}$, we have

$$g_w-\overline{g_w}-2i(\tilde f_w\cdot\bar{\tilde f}+\tilde f\cdot\overline{\tilde f_w})=0,\tag{12}$$
$$g_{w^2}-\overline{g_{w^2}}-2i(\tilde f_{w^2}\cdot\bar{\tilde f}+\overline{\tilde f_{w^2}}\cdot\tilde f+2|\tilde f_w|^2)=0,\tag{13}$$
$$g_{w^3}-\overline{g_{w^3}}-2i(\tilde f_{w^3}\cdot\bar{\tilde f}+\overline{\tilde f_{w^3}}\cdot\tilde f+3\tilde f_{w^2}\cdot\overline{\tilde f_w}+3\tilde f_w\cdot\overline{\tilde f_{w^2}})=0.\tag{14}$$

over $\Im w=|z|^2$.

Substituting (12), (13) and (14) into (9), (10) and (11), we get

$$\begin{aligned}
P_1 &= g_w - 2i\tilde{f}_w \cdot \bar{\tilde{f}}\Big|_{t=0}, \\
P_2 &= -2|\tilde{f}_w|^2\Big|_{t=0}, \\
P_3 &= \frac{1}{3}\Big(-\frac{1}{2}g_{w^3} + 3i\tilde{f}_w \cdot \overline{\tilde{f}_{w^2}} + i\tilde{f}_{w^3} \cdot \bar{\tilde{f}}\Big)\Big|_{t=0}.
\end{aligned} \tag{15}$$

■

We remark that by the Hopf Lemma, it follows easily that $P_1 \neq 0$ along $\partial\mathbb{H}^n$.

We next write $X = X_{jk}dz_j \otimes d\bar{z}_k + X_{jn}dz_j \otimes d\bar{w} + X_{nj}dw \otimes d\bar{z}_j + X_{nn}dw \otimes d\bar{w}$. By making use of Lemma 2.1, we shall compute in the next proposition the values of X at the origin. The proposition might be of independent interest, as the CR invariants in the study of proper holomorphic maps between Siegel upper half spaces are related to the CR geometry of the map.

Proposition 2.5. *Assume that $F = (\tilde{f}, g) = (f_1, \ldots, f_{N-1}, g) : \mathbb{H}^n \to \mathbb{H}^N$ is a proper rational holomorphic map, that satisfies the normalization (at the origin) stated in Lemma 2.1. Then*

$$\begin{aligned}
X_{jk}(0) &= -2i(f_k)_{z_jw}(0) = a_{kj}, \\
X_{jn}(0) &= \overline{X_{nj}}(0) = \frac{3i}{4}\overline{(f_j)_{w^2}}(0) + \frac{1}{8}g_{z_jw^2}(0), \\
X_{nn}(0) &= \frac{1}{6}g_{w^3}(0).
\end{aligned}$$

Proof of Proposition 2.5: Along the direction of $dz_j \otimes d\bar{z}_k$, collecting the coefficient of t^2 in the Taylor expansion of H^2X with respect to t, we get

$$\begin{aligned}
P_1^2X_{jk}(0) = \Big[&(2P_1P_2\delta_{jk} + (P_2^2 + 2P_1P_3)\bar{z}_jz_k) - \frac{1}{2}\Big\{2iP_1(\tilde{f}_{wz_j} \cdot \overline{\tilde{f}_{z_k}} - \tilde{f}_{z_j} \cdot \overline{\tilde{f}_{wz_k}}) + 2P_2\tilde{f}_{z_j} \cdot \overline{\tilde{f}_{z_k}} \\
&- (\overline{\tilde{f}_{w^2}} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \overline{\tilde{f}_{z_k}}) + 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j})(\tilde{f}_w \cdot \overline{\tilde{f}_{z_k}}) + 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{wz_j})(\tilde{f} \cdot \overline{\tilde{f}_{z_k}}) \\
&- 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \overline{\tilde{f}_{z_kw}}) - (\bar{\tilde{f}} \cdot \tilde{f}_{z_j})(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_{z_k}}) - 2(\bar{\tilde{f}} \cdot \tilde{f}_{z_jw})(\tilde{f}_w \cdot \overline{\tilde{f}_{z_k}}) \\
&+ 2(\bar{\tilde{f}} \cdot \tilde{f}_{z_j})(\tilde{f}_w \cdot \overline{\tilde{f}_{z_kw}}) - (\tilde{f}_{z_jw^2} \cdot \bar{\tilde{f}})(\tilde{f} \cdot \overline{\tilde{f}_{z_k}}) + 2(\bar{\tilde{f}} \cdot \tilde{f}_{z_jw})(\tilde{f} \cdot \overline{\tilde{f}_{z_kw}}) \\
&- (\bar{\tilde{f}} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \overline{\tilde{f}_{z_kw^2}}) - \frac{1}{4}g_{z_jw^2}\overline{g_{z_k}} + \frac{1}{2}g_{z_jw}\overline{g_{z_kw}} - \frac{1}{4}g_{z_j}\overline{g_{z_kw^2}} + \frac{i}{2}(\overline{\tilde{f}_{w^2}} \cdot \tilde{f}_{z_j})\bar{g}_{z_k} \\
&- i(\overline{\tilde{f}_w} \cdot \tilde{f}_{wz_j})\overline{g_{z_k}} + i(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j})\overline{g_{z_kw}} + \frac{i}{2}(\bar{\tilde{f}} \cdot \tilde{f}_{z_jw^2})\overline{g_{z_k}} - i(\bar{\tilde{f}} \cdot \tilde{f}_{z_jw})\overline{g_{z_kw}} \\
&+ \frac{i}{2}(\bar{\tilde{f}} \cdot \tilde{f}_{z_j})\overline{g_{z_kw^2}} - \frac{i}{2}(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_{z_k}})g_{z_j} - i(\tilde{f}_w \cdot \overline{\tilde{f}_{z_k}})g_{z_jw} + i(\tilde{f}_w \cdot \overline{\tilde{f}_{z_kw}})g_{z_j} \\
&- \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_{z_k}})g_{z_jw^2} + i(\tilde{f} \cdot \overline{\tilde{f}_{z_kw}})g_{z_jw} - \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_{z_kw^2}})g_{z_j}\Big\}\Big]\Big|_{t=0}.
\end{aligned}$$

Letting $(z, w) = 0$ and applying the normalization condition as stated in Lemma 2.1, we have

$$X_{jk}(0) = \frac{\partial a_k^{(1)}(z)}{\partial z_j} = a_{kj}.$$

Similarly, considering the coefficient of t^2 along $dz_j \otimes d\bar{w}$ and $dw \otimes d\bar{w}$, respectively, we have

$$\begin{aligned}
P_1^2 X_{jn}(0) = \Big[& (-iP_1P_3 - \frac{i}{2}P_2^2)\bar{z}_j - \frac{1}{2}\{2iP_1(\tilde{f}_{wz_j} \cdot \overline{\tilde{f}_w} - \tilde{f}_{z_j} \cdot \overline{\tilde{f}_{w^2}}) + 2P_2\tilde{f}_{z_j} \cdot \overline{\tilde{f}_w} \\
& - (\overline{\tilde{f}_{w^2}} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \overline{\tilde{f}_w}) + 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j})(\tilde{f}_w \cdot \overline{\tilde{f}_w}) + 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{wz_j})(\tilde{f} \cdot \overline{\tilde{f}_w}) \\
& - 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \overline{\tilde{f}_{w^2}}) - (\bar{\tilde{f}} \cdot \tilde{f}_{z_j})(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_w}) - 2(\bar{\tilde{f}} \cdot \tilde{f}_{z_jw})(\tilde{f}_w \cdot \overline{\tilde{f}_w}) \\
& + 2(\bar{\tilde{f}} \cdot \tilde{f}_{z_j})(\tilde{f}_w \cdot \overline{\tilde{f}_{w^2}}) - (\tilde{f}_{z_jw^2} \cdot \bar{\tilde{f}})(\tilde{f} \cdot \overline{\tilde{f}_w}) + 2(\bar{\tilde{f}} \cdot \tilde{f}_{z_jw})(\tilde{f} \cdot \overline{\tilde{f}_{w^2}}) - (\bar{\tilde{f}} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \overline{\tilde{f}_{w^3}}) \\
& - \frac{1}{4}g_{z_jw^2}\overline{g_w} + \frac{1}{2}g_{z_jw}\overline{g_{w^2}} - \frac{1}{4}g_{z_j}\overline{g_{w^3}} + \frac{i}{2}(\overline{\tilde{f}_{w^2}} \cdot \tilde{f}_{z_j})\overline{g_w} - i(\overline{\tilde{f}_w} \cdot \tilde{f}_{wz_j})\overline{g_w} + i(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j})\overline{g_{w^2}} \\
& + \frac{i}{2}(\bar{\tilde{f}} \cdot \tilde{f}_{z_jw^2})\overline{g_w} - i(\bar{\tilde{f}} \cdot \tilde{f}_{z_jw})\overline{g_{w^2}} + \frac{i}{2}(\bar{\tilde{f}} \cdot \tilde{f}_{z_j})\overline{g_{w^3}} - \frac{i}{2}(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_w})g_{z_j} - i(\tilde{f}_w \cdot \overline{\tilde{f}_w})g_{z_jw} \\
& + i(\tilde{f}_w \cdot \overline{\tilde{f}_{w^2}})g_{z_j} - \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_w})g_{z_jw^2} + i(\tilde{f} \cdot \overline{\tilde{f}_{w^2}})g_{z_jw} - \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_{w^3}})g_{z_j}\}\Big]\Big|_{t=0}
\end{aligned}$$

and

$$\begin{aligned}
P_1^2 X_{nn}(0) = \Big[& \frac{1}{4}(2P_1P_3 + P_2^2) - \frac{1}{2}\{2iP_1(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_w} - \tilde{f}_w \cdot \overline{\tilde{f}_{w^2}}) + 2P_2\tilde{f}_w \cdot \overline{\tilde{f}_w} - (\overline{\tilde{f}_{w^2}} \cdot \tilde{f}_w)(\tilde{f} \cdot \overline{\tilde{f}_w}) \\
& + 2(\overline{\tilde{f}_w} \cdot \tilde{f}_w)(\tilde{f}_w \cdot \overline{\tilde{f}_w}) + 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{w^2})(\tilde{f} \cdot \overline{\tilde{f}_w}) - 2(\overline{\tilde{f}_w} \cdot \tilde{f}_w)(\tilde{f} \cdot \overline{\tilde{f}_{w^2}}) \\
& - (\bar{\tilde{f}} \cdot \tilde{f}_w)(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_w}) - 2(\bar{\tilde{f}} \cdot \tilde{f}_{w^2})(\tilde{f}_w \cdot \overline{\tilde{f}_w}) + 2(\bar{\tilde{f}} \cdot \tilde{f}_w)(\tilde{f}_w \cdot \overline{\tilde{f}_{w^2}}) - (\tilde{f}_{w^3} \cdot \bar{\tilde{f}})(\tilde{f} \cdot \overline{\tilde{f}_w}) \\
& + 2(\bar{\tilde{f}} \cdot \tilde{f}_{w^2})(\tilde{f} \cdot \overline{\tilde{f}_{w^2}}) - (\bar{\tilde{f}} \cdot \tilde{f}_w)(\tilde{f} \cdot \overline{\tilde{f}_{w^3}}) - \frac{1}{4}g_{w^3}\overline{g_w} + \frac{1}{2}g_{w^2}\overline{g_{w^2}} - \frac{1}{4}g_w\overline{g_{w^3}} \\
& + \frac{i}{2}(\overline{\tilde{f}_{w^2}} \cdot \tilde{f}_w)\bar{g}_w - i(\overline{\tilde{f}_w} \cdot \tilde{f}_{w^2})\overline{g_w} + i(\overline{\tilde{f}_w} \cdot \tilde{f}_w)\overline{g_{w^2}} + \frac{i}{2}(\bar{\tilde{f}} \cdot \tilde{f}_{w^3})\overline{g_w} - i(\bar{\tilde{f}} \cdot \tilde{f}_{w^2})\overline{g_{w^2}} \\
& + \frac{i}{2}(\bar{\tilde{f}} \cdot \tilde{f}_w)\overline{g_{w^3}} - \frac{i}{2}(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_w})g_w - i(\tilde{f}_w \cdot \overline{\tilde{f}_w})g_{w^2} \\
& + i(\tilde{f}_w \cdot \overline{\tilde{f}_{w^2}})g_w - \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_w})g_{w^3} + i(\tilde{f} \cdot \overline{\tilde{f}_{w^2}})g_{w^2} - \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_{w^3}})g_w\}\Big]\Big|_{t=0}.
\end{aligned}$$

Let $(z, w) = 0$. It follows that

$$\begin{aligned}
X_{jn}(0) &= \frac{3i}{4}\overline{(f_j)_{w^2}}(0) + \frac{1}{8}g_{z_jw^2}(0), \\
X_{nn}(0) &= \frac{1}{6}g_{w^3}(0),
\end{aligned}$$

for $g_{w^3}(0) = \overline{g_{w^3}(0)}$ by (14). ∎

Making use of the computation in Proposition 2.5, we give a proof of Theorem 1.1 in the case when each component extends as a proper holomorphic map. Indeed, we prove a slightly more general result than what is needed later as following:

Proposition 2.6. *Let*

$$F = (F_1, \dots, F_m) : (\mathbb{B}^n, ds_n^2) \to (\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}, \oplus_{j=1}^m ds_{N_j}^2)$$

be a holomorphic isometric embedding up to conformal factors $\{\lambda(z,\bar{z}); \lambda_1(z,\bar{z}), \cdots, \lambda_m(z,\bar{z})\}$ *in the sense that*

$$\lambda(z,\bar{z}) ds_n^2 = \sum_{j=1}^m \lambda_j(z,\overline{z}) F_j^*(ds_{N_j}^2).$$

Here for each j, $\lambda(z,z), \lambda_j(z,\overline{z})$ *are positively real-valued* C^2*-smooth functions over* $\overline{\mathbb{B}^n}$*, and* F_j *is a proper rational map from* $\mathbb{B}^n$ *into* $\mathbb{B}^{N_j}$ *for each* j*. Then* $\lambda(z,\bar{z}) \equiv \sum_{j=1}^m \lambda_j(z,\overline{z})$ *over* $\overline{\mathbb{B}^n}$*, and for any* j*,* F_j *is a totally geodesic embedding from* $\mathbb{B}^n$ *to* $\mathbb{B}^{N_j}$*.*

Proof of Proposition 2.6: After applying the Cayley transformation and considering $((\rho_{N_1})^{-1}, \cdots, (\rho_{N_m})^{-1}) \circ F \circ \rho_n$ instead of F, we can assume, without loss of generality, that

$$F = (F_1, \dots, F_m) : (\mathbb{H}^n, ds_{\mathbb{H}^n}^2) \to (\mathbb{H}^{N_1} \times \cdots \times \mathbb{H}^{N_m}, \oplus_{j=1}^m ds_{\mathbb{H}^{N_j}}^2)$$

is an isometric map up to conformal factors $\{\lambda(Z,\bar{Z}); \lambda_1(Z,\bar{Z}), \cdots, \lambda_m(Z,\bar{Z})\}$. Also, each F_j a proper rational map from $\mathbb{H}^n$ into $\mathbb{H}^{N_j}$, respectively. Here we write $Z = (z, w)$. Moreover, we can assume, without loss of generality, that each component F_j of F satisfies the normalization condition as in Lemma 2.1. Since F is an isometry, we have

$$\lambda(Z,\bar{Z}) ds_{\mathbb{H}^n}^2 = \sum_{j=1}^m \lambda_j(Z,\bar{Z}) F_j^*(ds_{\mathbb{H}^{N_j}}^2), \text{ or } (\lambda(Z,\bar{Z}) - \sum_{j=1}^m \lambda_j(Z,\bar{Z})) ds_{\mathbb{H}^n}^2 + \sum_{j=1}^m \lambda_j(Z,\bar{Z}) X(F_j) = 0. \tag{16}$$

Here, we write $X(F_j) = ds_{\mathbb{H}^n}^2 - F_j^*(ds_{\mathbb{H}^{N_j}}^2)$. Collecting the coefficient of $dw \otimes d\bar{w}$, one has

$$\frac{\lambda(Z,\bar{Z}) - \sum_{j=1}^m \lambda_j(Z,\bar{Z})}{4(\Im w - |z|^2)^2} + \sum_{j=1}^m \lambda_j(Z,\bar{Z})(X(F_j))_{nn} = 0. \tag{17}$$

Since $X(F_j)$ is smooth up to $\partial\mathbb{H}^n$, we see that $\lambda(Z,\bar{Z}) - \sum_{j=1}^m \lambda_j(Z,\bar{Z}) = O(t^2)$ as $Z = (z, w)(\in \mathbb{H}^n) \to 0$, where $t = \Im w - |z|^2$. However, since the $dz_l \otimes d\bar{z}_k$-component of $ds_{\mathbb{H}^n}^2$ blows up at the rate of $o(\frac{1}{t^2})$ as $(z,w)(\in \mathbb{H}^n) \to 0$, collecting the coefficients of the $dz_l \otimes d\bar{z}_k$-component in (16) and then letting $(z,w)(\in \mathbb{H}^n) \to 0$, we conclude that, for any $1 \le l,k \le n-1$,

$$\sum_{j=1}^m \lambda_j(0,0)(X(F_j))_{kl}(0) = 0.$$

By Proposition 2.5, we have $\sum_{j=1}^m \lambda_j(0,0)a^j_{lk}(0) = 0$, where a^j_{kl} is associated with F_j in the expansion of F_j at 0 as in Lemma 2.1. Since $(a^j_{kl})_{1\le l,k\le n-1}$ is a semi-positive matrix and $\lambda_j(0,0) > 0$, it follows immediately that $a^j_{lk}(0) = 0$ for all j,k,l. Namely, $F_j = (z,w) + O_{wt}(3)$ for each j.

Next, for each $p \in \partial\mathbb{H}^n$, let $\tau_j \in Aut(\mathbb{H}^N)$ be such that $(F_j)^{**}_p = \tau_j \circ F_j \circ \sigma^0_p$ has the normalization as in Lemma 2.1. Let $\tau = (\tau_1, \cdots, \tau_m)$. Note that $F^{**}_p := ((F_1)^{**}_p, \cdots, (F_m)^{**}_p) = \tau \circ F \circ \sigma^0_p$ is still an isometric map satisfing the condition as in the proposition. Applying the just presented argument to F^{**}_p, we conclude that $(F_j)^{**}_p = (z,0,w) + O_{wt}(3)$. By Theorem 4.2 of [Hu2], this implies that $F_j = (Z,0)$. Namely, F_j is a totally geodesic embedding. In particular, we have $X(F_j) \equiv 0$. This also implies that $\lambda \equiv \sum_{j=1}^m \lambda_j$ over $\mathbb{B}^n$. The proof of Proposition 2.6 is complete. ∎

3 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. As in the theorem, we let $U \subset \mathbb{B}^n$ be a connected open subset. Let

$$F = (F_1, \ldots, F_m) : (U, ds^2_n) \to (\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}, \oplus_{j=1}^m ds^2_{N_j})$$

be a holomorphic isometric embedding up to conformal factors $\{\lambda(z,\bar{z}); \lambda_1(z,\bar{z}), \cdots, \lambda_m(z,\bar{z})\}$ in the sense that

$$\lambda(z,\bar{z})ds^2_n = \sum_{j=1}^m \lambda_j(z,\bar{z})F^*_j(ds^2_{N_j}).$$

Here $\lambda_j(z,\bar{z}), \lambda(z,\bar{z}) > 0$ are smooth Nash algebraic functions; ds^2_n and $ds^2_{N_j}$ are the Bergman metrics of $\mathbb{B}^n$ and $\mathbb{B}^{N_j}$, respectively; and F_j is a holomorphic map from U to $\mathbb{B}^{N_j}$ for each j. For the proof of Theorem 1.1, we can assume without loss of generality that none of the F_j's is a constant map. Following the idea in [MN], we can show that F extends to an algebraic map over $\mathbb{C}^n$. (For the convenience of readers, we include the detailed argument in the appendix.) Namely, for each (non-constant) component $f_{j,l}$ of F_j, there is an irreducible polynomial $P_{j,l}(z,X) = a_{j,l}(z)X^{m_{jl}} + \ldots$ in $(z,X) \in \mathbb{C}^n \times \mathbb{C}$ of degree $m_{jl} \ge 1$ in X such that $P_{j,l}(z, f_{j,l}) \equiv 0$ for $z \in U$.

We will proceed to show that, for each j, F_j extends to a proper rational map from $\mathbb{B}^n$ into $\mathbb{B}^{N_j}$. For this purpose, we let $R_{j,l}(z)$ be the resultant of $P_{j,l}$ in X and let $E_{j,l} = \{R_{j,l} \equiv 0, a_{j,l} \equiv 0\}$, $E = \cup E_{j,l}$. Then E defines a proper affine-algebraic subvariety of $\mathbb{C}^n$. For any continuous curve $\gamma : [0,1] \to \mathbb{C}^n \setminus E$ where $\gamma(0) \in U$, F can be continued holomorphically along γ to get a germ of holomorphic map at $\gamma(1)$. Also, if γ_1 is homotopic to γ_2 in $\mathbb{C}^n \setminus E$, $\gamma_1(0) = \gamma_2(0) \in U$ and $\gamma_1(1) = \gamma_2(1)$, then continuations of F along γ_1 and γ_2 are the same at $\gamma_1(1) = \gamma_2(1)$. Now let $p_0 \in U$ and $p_1 \in \partial\mathbb{B}^n \setminus E$. Let $\gamma(t)$ be a smooth simple curve connecting p_0 to p_1 and $\gamma(t) \notin \partial\mathbb{B}^n$ for $t \in (0,1)$. Then each F_j defines a holomorphic map in a connected neighborhood V_γ of γ by continuing along γ the initial germ of F_j at p_0. (We can also assume that $V_\gamma \cap \mathbb{B}^n$ is connected.) Let

$$S_\gamma = \big\{p \in V_\gamma : \ \|F_j(p)\| = 1 \text{ for some } j\big\}.$$

Then S_γ is a real analytic (proper) subvariety of V_γ. We first prove

Claim 3.1. *When V_γ is sufficiently close to γ, $dim(S_\gamma \cap \mathbb{B}^n) \le 2n-2$.*

Proof of Claim 3.1: Supposing otherwise we are going to deduce a contradiction. Assume that $t_0 \in (0,1]$ is the first point such that for a certain j, the local variety defined by $\|F_j(z)\|^2 = 1$ near $p^* = \gamma(t_0)$ has real dimension $2n-1$ at p^*. Let Σ_0 be an irreducible component of the germ of the real analytic subvariety S_γ at p^* of real codimension 1 and let Σ be a connected locally closed subvariety of $\mathbb{B}^n$ representing the germ Σ_0 at p^*. Since any real analytic subset of real codimension two inside a connected open set does not affect the connectivity, by slightly changing γ without changing its homotopy type and terminal point, we can assume that $\gamma(t) \notin S_\gamma$ for any $t < t_0$. Hence, p^* also lies on the boundary of the connected component $\hat{V}$ of $(V_\gamma \cap \mathbb{B}^n) \setminus S_\gamma$, that contains $\gamma(t)$ for $t < t_0$ and Σ also lies in the boundary of $\hat{V}$. Now, for any $p \in \Sigma$, let $q (\in \hat{V}) \to p$, we have along $\{q\}$,

$$\lambda(z,\bar{z}) ds_n^2 = \sum_j \lambda_j(z,\bar{z}) F_j^*(ds_{N_j}^2).$$

Suppose that $j^\sharp$ is such that $\|F_{j^\sharp}(p)\| = 1$ and $\|F_j(z)\| < 1$ for any j, $p \in \Sigma$ and $z \in \hat{V}$. Since $p \in \Sigma \subset \mathbb{B}^n$, ds_n^2 is a smooth Hermitian metric in an open neighborhood of p. For any $v = (v_1, \ldots, v_\xi, \ldots, v_n) \in \mathbb{C}^n$ with $\|v\| = 1$, it follows that

$$\left| \overline{\lim}_{q \to p} F_{j^\sharp}^*(ds_{N_{j^\sharp}}^2)(v,v)(q) \right| < \infty.$$

On the other hand,

$$F_{j^\sharp}^*(ds_{N_{j^\sharp}}^2) = \frac{\sum_{l,k}\{\delta_{lk}(1 - \|F_{j^\sharp}\|^2) + \bar{f}_{j^\sharp,l} f_{j^\sharp,k}\} df_{j^\sharp,l} \otimes d\overline{f}_{j^\sharp,k}}{(1 - \|F_{j^\sharp}\|^2)^2}.$$

it follows that

$$F_{j^\sharp}^*(ds_{N_{j^\sharp}}^2)(v,v)(q) = \frac{\|\sum_\xi \frac{\partial f_{j^\sharp,l}}{\partial z_\xi}(q) v_\xi\|^2}{1 - \|F_{j^\sharp}(q)\|^2} + \frac{|\sum_{l,\xi} \overline{f_{j^\sharp,l}(q)} \frac{\partial f_{j^\sharp,l}}{\partial z_\xi}(q) v_\xi|^2}{(1 - \|F_{j^\sharp}(q)\|^2)^2}. \tag{18}$$

Letting $q \to p$, since $1 - \|F_{j^\sharp}(q)\|^2 \to 0^+$, we get

$$\|\sum_{\xi} \frac{\partial f_{j^\sharp,l}(p)}{\partial z_\xi} v_\xi\|^2 = 0.$$

Thus

$$\frac{\partial f_{j^\sharp,l}(p)}{\partial z_\xi} = 0, \quad \text{for} \quad l = 1, \ldots, N_{j^\sharp}.$$

Hence, we see $dF_{j^\sharp} = 0$ in a certain open subset of Σ. Since Σ is of real codimension 1 in $\mathbb{B}^n$, any non-empty open subset of Σ is a uniqueness set for holomorphic functions. Hence $F_{j^\sharp} \equiv const.$ This is a contradiction. ∎

Now, since $dim(S_\gamma \cap \mathbb{B}^n) \leq 2n-2$, we can always slightly change γ without changing the homotopy type of γ in $V_\gamma \setminus E$ and the end point of γ so that $\gamma(t) \notin S_\gamma$ for any $t \in (0,1)$. Since $\lambda(z,\bar{z})ds_n^2 = \sum \lambda_j(z,\bar{z})F_j^*(ds_{N_j}^2)$ in $(V_\gamma \cap \mathbb{B}^n) \setminus S_\gamma$ and since ds_n^2 blows up when $q \in V_\gamma \cap \mathbb{B}^n$ approaches to $\partial\mathbb{B}^n$, we see that for each $q \in V_\gamma \cap \partial\mathbb{B}^n$, $\|F_{j_q}(q)\| = 1$ for some j_q. Hence, we can assume without loss of generality, that there is a $j_0 \geq 1$ such that each of $F_1, \ldots, F_{j_0}$ maps a certain open piece of $\partial\mathbb{B}^n$ into $\partial\mathbb{B}^{N_1}, \ldots, \partial\mathbb{B}^{N_{j_0}}$, but for $j > j_0$,

$$dim\{q \in \partial\mathbb{B}^n \cap V_\gamma : \|F_j(q)\| = 1\} \leq 2n-2.$$

It follows from the Hopf lemma that $N_j \geq n$ for $j \leq j_0$. By the results of Forstneric [Fo] and Cima-Suffridge [CS], F_j extends to a rational proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{N_j}$ for $j \leq j_0$. Now, we must have

$$\lambda(z,\bar{z})ds_n^2 - \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})F_j^*(ds_{N_j}^2) = \sum_{j=j_0+1}^{m} \lambda_j(z,\bar{z})F_j^*(ds_{N_j}^2)$$

in $(V_\gamma \cap \mathbb{B}^n) \setminus S_\gamma$, that is a connected set by Claim 3.1. Let $q \in (V_\gamma \cap \mathbb{B}^n) \setminus S_\gamma \to p \in \partial\mathbb{B}^n \cap V_\gamma$. Note that

$$(\lambda(z,\bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z,\bar{z}))ds_n^2\Big|_q + \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})(ds_n^2 - F_j^*(ds_{N_j}^2))\Big|_q = \sum_{j=j_0+1}^{m} \lambda_j(z,\bar{z})F_j^*(ds_{N_j}^2)\Big|_q.$$

By Lemma 2.2, $X_j := ds_n^2 - F_j^*(ds_{N_j}^2)$ is smooth up to $\partial\mathbb{B}^n$ for $j \leq j_0$. We also see, by the choice of j_0 and Claim 3.1, that for a generic point p in $\partial\mathbb{B}^n \cap V_\gamma$, $F_j^*(ds_{N_j}^2)$ are real analytic in a small neighborhood of p for $j \geq j_0 + 1$. Thus by considering the normal component as before in the above equation, we see that $\lambda(z,\bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})$ vanishes to the order ≥ 2 in an open set of the unit sphere. Since $\lambda(z,\bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})$ is real analytic over $\mathbb{C}^n$, we obtain

$$\lambda(z,\bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z,\bar{z}) = (1 - |z|^2)^2 \psi(z,\bar{z}). \tag{19}$$

Here $\psi(z,\bar{z})$ is a certain real analytic function over $\mathbb{C}^n$. Let

$$Y = (\lambda(z,\bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z,\bar{z}))ds_n^2.$$

Then Y extends real analytically to $\mathbb{C}^n$. Write $X = \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})X_j$. From what we argued above, we easily see that there is a certain small neighborhood $\mathcal{O}$ of $q \in \partial\mathbb{B}^n$ in $\mathbb{C}^n$ such that (1): we can holomorphically continue the initial germ of F in U through a certain simple curve γ with $\gamma(t) \in \mathbb{B}^n$ for $t \in (0,1)$ to get a holomorpic map, still denoted by F, over $\mathcal{O}$; (2): $\|F_j\| < 1$ for $j > j_0$ and $\|F_j\| > 1$ for $j \leq j_0$ over $\mathcal{O} \setminus \mathbb{B}^n$; and (3):

$$X = \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})(ds_n^2 - F_j^*(ds_{N_j}^2)) = \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})X_j = \sum_{j=j_0+1}^{m} \lambda_j(z,\bar{z})F_j^*(ds_{N_j}^2) - Y. \quad (20)$$

We mention that we are able to make $\|F_j\| < 1$ for any $z \in \mathcal{O}$ and $j > j_0$ in the above due to the fact that $(V_\gamma \cap \mathbb{B}^n) \setminus S_\gamma$, as defined before, is connected.

Now, let $\mathcal{P}$ be the union of the poles of $F_1, \ldots, F_{j_0}$. Fix a certain $p^* \in \mathcal{O} \cap \partial\mathbb{B}^n$ and let $\widetilde{E} = E \cup \mathcal{P}$. Then for any $\gamma : [0,1] \to \mathbb{C}^n \setminus (\mathbb{B}^n \cup \widetilde{E})$ with $\gamma(0) = p^*$ and $\gamma(t) \notin \partial\mathbb{B}^n$ for $t > 0$, F_j extends holomorphically to a small neighborhood U_γ of γ that contracts to γ. Still denote the holomorphic continuation of F_j (from the initial germ of F_j at $p^* \in \mathcal{O}$) over U_γ by F_j. If for some $t \in (0,1)$, $\|F_j(\gamma(t))\| = 1$, then we similarly have

Claim 3.2. *Shrinking U_γ if necessary, then dim$\{p \in U_\gamma : \|F_j(p)\| = 1$ for some $j\} \leq 2n - 2$.*

Proof of the Claim 3.2: Supposing otherwise we are going to deduce a contradiction. Define S_γ in a similar way. Without loss of generality, we assume that $t_0 \in (0,1)$ is the first point such that for a certain j_{t_0}, the local variety defined by $\|F_{j_{t_0}}(z)\|^2 = 1$ near $\gamma(t_0)$ has real dimension $2n-1$ at $\gamma(t_0)$. Then, as before, we have

$$X = \sum_{j=1}^{j_0} \lambda_j(z,\bar{z})(ds_n^2 - F_j^*(ds_{N_j}^2)) = \sum_{j=j_0+1}^{m} \lambda_j(z,\bar{z})F_j^*(ds_{N_j}^2) - Y \quad (21)$$

in a connected component W of $U_\gamma \setminus S_\gamma$ that contains $\gamma(t)$ for $t << 1$ with $\gamma(t_0) \in \partial W$. Now, for any $q (\in W) \to p \in \partial W$ near $p_0 = \gamma(t_0)$ and $v \in \mathbb{C}^n$ with $\|v\| = 1$, we have the following:

$$\begin{aligned}&\sum_{j=1}^{j_0} \lambda_j(q,\bar{q})\left(ds_n^2(v,v)\Big|_q - \frac{\|\sum_\xi \frac{\partial f_{j,l}}{\partial z_\xi}(q)v_\xi\|^2}{1-\|F_j(q)\|^2} - \frac{|\sum_{l,\xi} \overline{f_{j,l}(q)}\frac{\partial f_{j,l}}{\partial z_\xi}(q)v_\xi|^2}{(1-\|F_j(q)\|^2)^2}\right) \\ &= \sum_{j=j_0+1}^{m} \lambda_j(q,\bar{q})\left(\frac{\|\sum_\xi \frac{\partial f_{j,l}}{\partial z_\xi}(q)v_\xi\|^2}{1-\|F_j(q)\|^2} + \frac{|\sum_{l,\xi} \overline{f_{j,l}(q)}\frac{\partial f_{j,l}}{\partial z_\xi}(q)v_\xi|^2}{(1-\|F_j(q)\|^2)^2}\right) - Y(v,v)\Big|_q.\end{aligned} \quad (22)$$

Now, if the local variety defined by $\|F_j(z)\|^2 = 1$ is not of real codimension one at p_0 for $j \leq j_0$, then the local variety $S_{j'}$ defined by $\|F_{j'}(z)\|^2 = 1$ has to be of real codimension one at p_0 for $j' > j_0$. Let J be the collection of all such j'. Let S^0 be a small open piece of ∂W near p_0. Then for a generic $p \in S^0$, the left hand side of (22) remains bounded as $q \to p \in S^0$. For a term on the right hand side with index $j \in J$, if $S^0 \cap S_j$ contains a germ of an irreducible component of ∂W of real codimension 1 containing p_0, then it approaches to $+\infty$ for a generic p unless $F_j = \textit{constant}$ as argued in the proof of Claim 3.1. The other terms on the right hand side remain bounded as $q \to p$ for a generic p. This is a contradiction to the assumption that none of F_j's for $j > j_0$ is constant. Hence, we can assume that a local variety defined by $\|F_j(z)\|^2 = 1$ near p_0 is of real codimension one for a certain $j \leq j_0$. Let J be the set of indices such that for $j' \in J$, we have $j' \leq j_0$ and $S_{j'} := \{\|F_{j'}\| = 1\}$ is a local real analytic variety of real codimension one near p_0. For $j > j_0$, since $\|F_j(z)\| < 1$ for $z(\in U_\gamma) \approx p_0$ and since t_0 is the first point such that $\|F_{j^*}\| = 1$ defines a variety of real codimension one for some j^*, we see that $\|F_j(z)\| < 1$ for $z(\in W) \approx p_0$. Define S^0 similarly, as an small open piece of ∂W. Hence, as $q(\in W) \to p \in S^0$, the right hand side of (22) is uniformly bounded from below. On the other hand, in the left hand side of (22) , for any $j' \in J$ with $S_{j'} \cap S^0$ containing an irreducible component of ∂W of real codimension 1 near p_0, if the numerator $|\sum_{l,\xi} \overline{f_{j',l}(q)} \frac{\partial f_{j',l}}{\partial z_\xi}(q) v_\xi|^2$ of the last term does not go to 0 for some vectors v, then the term with index j' on the left hand side would go to $-\infty$ for a generic $p \in S^0$. If this happens to such j', the left hand side would approach to $-\infty$. Notice that all other terms on the right hand side remain bounded as $q \to p \in S^0$ for a generic p. This is impossible. Therefore we must have for some $j' \in J$ that $|\sum_l \overline{f_{j',l}(q)} \frac{\partial f_{j',l}}{\partial z_\xi}(q)|^2 = \frac{\partial \sum_l |f_{j',l}|^2}{\partial z_\xi}(q) = \frac{\partial \|F_{j'}\|^2}{\partial z_\xi}(q) \to 0$ and thus $\frac{\partial \|F_{j'}\|^2}{\partial z_\xi}(p) = 0$ for all ξ and $p \in S_{j'}$. This immediately gives the equality $d(\|F_{j'}\|^2) = 0$ along $S_{j'}$. Assume, without loss of generality, that p_0 is also a smooth point of $S_{j'}$. If $S_{j'}$ has no complex hypersurface passing through p_0, by a result of Trepreau [Tr], the union of the image of local holomorphic disks attached to $S_{j'}$ passing through p_0 fills in an open subset. Since $F_{j'}$ is not constant, there is a small holomorphic disk smooth up to the boundary $\phi(\tau) : \mathbb{B}^1 \to \mathbb{C}^n$ such that $\phi(\partial \mathbb{B}^1) \subset S_{j'}, \phi(1) = p_0$ and $F_{j'}$ is not constant along ϕ. Since $\partial \mathbb{B}^{N_{j'}}$ does not contain any non-trival complex curves, $r = (\|F_{j'}\|^2 - 1) \circ \phi \not\equiv 0$. Applying the maximum principle and then the Hopf lemma to the subharmonic function $r = (\|F_{j'}\|^2 - 1) \circ \phi$, we see that the outward normal derivative of r at $\tau = 1$ is positive. This contradicts to the fact that $d(\|F_{j'}\|^2) = 0$ along $S_{j'}$. We can argue in the same way for points $p \in S_{j'}$ near p_0 to conclude that for any $p \in S_{j'}$ near p_0, there is a complex hypersurface contained in $S_{j'}$ passing through p. Namely, $S_{j'}$ is Levi flat, foliated by a family of smooth complex hypersurfaces denoted by Y_η with real parameter η near p_0. Let Z be a holomorphic vector field along Y_η. We then easily see that $0 = \overline{Z} Z(\|F_{j'}\|^2 - 1) = \sum_{k=1}^{N_{j'}} |Z(f_{j',k})|^2$. Thus, we see that $F_{j'}$ is constant along each Y_η. Hence, $F_{j'}$ cannot be a local embedding at each point of $S_{j'}$. On the other hand, notice that $F_{j'}$ is a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{N'}$, then $F_{j'}$ is a local embedding near $\partial \mathbb{B}^n$. Hence, the set of points where $F_{j'}$ is not a local embedding can be at most of complex codimension one (and thus real codimension two). This is a contradiction. This proves Claim 3.2.

Hence, we see that $\mathcal{E} = \{p \in \mathbb{C}^n \setminus (\overline{\mathbb{B}^n} \cup \widetilde{E})$: some branch, obtained by the holomorphic continuation through curves described before, of F_j for some j maps p to $\partial\mathbb{B}^{N_j}\}$ is a real analytic variety of real dimension at most $2n-2$. Now, for any $p \in \mathbb{C}^n \setminus (\overline{\mathbb{B}^n} \cup \widetilde{E})$, any curve $\gamma : [0,1] \to \mathbb{C}^n \setminus (\mathbb{B}^n \cup \widetilde{E})$ with $\gamma(0) = p^* \in \mathcal{O} \cap \partial\mathbb{B}^n, \gamma(t) \notin \partial\mathbb{B}^n$ for $t > 0$ and $\gamma(1) = p$, we can homotopically change γ in $\mathbb{C}^n \setminus (\overline{\mathbb{B}^n} \cup \widetilde{E})$ (but without changing the terminal point) such that $\gamma(t) \notin \mathcal{E}$ for $t \in (0,1)$. Now, the holomorphic continuation of the initial germ of F_j from p^* never cuts $\partial\mathbb{B}^{N_j}$ along $\gamma(t)$ $(0 < t < 1)$. We thus see that $\|F_j(p)\| \leq 1$ for $j > j_0$.

Let $\{(f_{j,l})_{k;p}\}_{k=1}^{n_{jl}}$ be all possible (distinct) germs of holomorphic functions that we can get at p by the holomorphic continuation, along curves described above in $\mathbb{C}^n \setminus (\mathbb{B}^n \cup \tilde{E})$, of $f_{j,l}$. Let $\sigma_{jl,\tau}$ be the fundamental symmetric function of $\{(f_{j,l})_{k;p}\}_{k=1}^{n_{jl}}$ of degree τ. Then $\sigma_{jl,\tau}$ well defines a holomorphic function over $\mathbb{C}^n \setminus \overline{\mathbb{B}^n}$. $|\sigma_{jl,\tau}|$ is bounded in $\mathbb{C}^n \setminus (\overline{\mathbb{B}^n} \cup \tilde{E})$. By the Riemann removable singularity theorem, $\sigma_{jl,\tau}$ is holomorphic over $\mathbb{C}^n \setminus \overline{\mathbb{B}^n}$. By the Hartogs extension theorem, $\sigma_{jl,\tau}$ extends to a bounded holomorphic function over $\mathbb{C}^n$. Hence, by the Liouville theorem, $\sigma_{jl,\tau} \equiv const$. This forces $(f_{j,l})_k$ and thus F_j for $j > j_0$ to be constant. We obtain a contradiction. This proves that each F_j extends to a proper rational map from $\mathbb{B}^n$ into $\mathbb{B}^{N_j}$. Together with Proposition 2.6, we complete the proof of the main Theorem. ■

Remark 3.3. *The regularity of λ_j, λ can be reduced to be only real analytic in the complement of a certain real codimension two subset. Also, we need only to assume that they are positive outside a real analytic variety of real codimension two. This is obvious from our proof of Theorem 1.1.*

Remark 3.4. *Assume that λ, λ_j are smooth, positive, Nash algebraic (or more generally, real analytic) functions on $\mathbb{B}^n, \mathbb{B}^{N_j}$ respectively for all j and $F = (F_1, \cdots, F_m) : U \subset \mathbb{B}^n \to \mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}$ is a holomorphic embedding such that*

$$\lambda ds_n^2 = \sum_{j=1}^{m} F_j^*(\lambda_j ds_{N_j}^2).$$

It would be very interesting to prove the total geodesy for non-constant component F_j. However, different from the situation in Theorem 1.1, one cannot prove the algebraic extension of F using the current technique since we do not know yet how to construct a target real algebraic hypersurface associated to F. Once the algebraic extension of F is obtained, the total geodesy should follow from our argument without much modification. The algebraic extension of F in this case is related to the algebraicity problem raised in [HY].

4 Appendix: algebraic extension

In this appendix, we prove the algebraicity of the local holomorphic map. As in the theorem, we let $U \subset \mathbb{B}^n$ be a connected open subset. Let

$$F = (F_1, \ldots, F_m) : (U, ds_n^2) \to (\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}, \oplus_{j=1}^m ds_{N_j}^2)$$

be a holomorphic isometric embedding up to conformal factors $\{\lambda(z,\bar{z});\lambda_1(z,\bar{z}),\cdots,\lambda_m(z,\bar{z})\}$. Here $\lambda_j(z,\bar{z})>0, \lambda(z,\bar{z})>0$ are smooth Nash algebraic functions in $\mathbb{C}^n$, and ds_n^2 and $ds_{N_j}^2$ are the Bergman metrics of $\mathbb{B}^n$ and $\mathbb{B}^{N_j}$, respectively. We further assume without loss of generality that none of the F_j's is a constant map. Our proof uses exactly the same method employed in the paper of Mok-Ng [MN], following a suggestion of Yum-Tong Siu. Namely, we use the Grauert tube technique to reduce the problem to the algebraicity problem for CR mappings. However, different from the consideration in [MN], the Grauert tube constructed by using the unit sphere bundle over $\mathbb{B}^{N_1}\times\cdots\times\mathbb{B}^{N_m}$ with respect to the metric $\oplus_{j=1}^m ds_{N_j}^2$, up to conformal factors, may have complicated geometry and may not even be pseudoconvex anymore in general. To overcome the difficulty, we bend the target hypersurface to make it sufficiently positively curved along the tangential direction of the source domain.

Let $K>0$ be a large constant to be determined. Consider $S_1\subset TU$ and $S_2\subset U\times T\mathbb{B}^{N_1}\times\cdots\times T\mathbb{B}^{N_m}$ as follows:

$$S_1 := \left\{(t,\zeta)\in TU : (1+K|t|^2)\lambda(t,\bar{t})ds_n^2(t)(\zeta,\zeta)=1\right\}, \tag{23}$$

$$\begin{aligned} S_2 := \{&(t,z_1,\xi_1,\cdots,z_m,\xi_m)\in U\times T\mathbb{B}^{N_1}\times\cdots\times T\mathbb{B}^{N_m}: \\ &(1+K|t|^2)[\lambda_1(t,\bar{t})ds_{N_1}^2(z_1)(\xi_1,\xi_1)+\cdots+\lambda_m(t,\bar{t})ds_{N_m}^2(z_m)(\xi_m,\xi_m)]=1\}.\end{aligned} \tag{24}$$

The defining functions ρ_1,ρ_2 of S_1,S_2 are, respectively, as follows:

$$\rho_1=(1+K|t|^2)\lambda(t,\bar{t})ds_n^2(t)(\zeta,\zeta)-1,$$

$$\rho_2=(1+K|t|^2)[\lambda_1(t,\bar{t})ds_{N_1}^2(z_1)(\xi_1,\xi_1)+\cdots+\lambda_m(t,\bar{t})ds_{N_m}^2(z_m)(\xi_m,\xi_m)]-1.$$

Then one can easily check that the map $(id,F_1,dF_1,\cdots,F_m,dF_m)$ maps S_1 to S_2 according to the metric equation

$$\lambda(t,\bar{t})ds_n^2=\lambda_1(t,\bar{t})F_1^*(ds_{N_1}^2)+\cdots+\lambda_m(t,\bar{t})F_m^*(ds_{N_m}^2).$$

Lemma 4.1. *S_1,S_2 are both real algebraic hypersurfaces. Moreover for K sufficiently large, S_1 is smoothly strongly pseudoconvex. For any $\xi_1\neq 0,\cdots,\xi_m\neq 0$, $(0,0,\xi_1,\cdots,0,\xi_m)\in S_2$ is a smooth strongly pseudoconvex point when K is sufficiently large, where K depends on the choice of $\xi_1,\cdots,\xi_m$.*

Proof of Lemma 4.1: It is immediate from the defining functions that S_1,S_2 are smooth real algebraic hypersurfaces. We show the strong pseudoconvexity of S_2 at $(0,0,\xi_1,\cdots,0,\xi_m)$ as follows: (The strong pseudoconvexity of S_1 follows from the same computation.)

By applying $\partial\bar{\partial}$ to ρ_2 at $(0,0,\xi_1,\cdots,0,\xi_m)$, we have the following Hessian matrix

$$\begin{bmatrix} A & 0 & D_1 & \cdots & 0 & D_m \\ 0 & B_1 & 0 & \cdots & 0 & 0 \\ \bar{D}_1 & 0 & C_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & B_m & 0 \\ \bar{D}_m & 0 & 0 & \cdots & 0 & C_m \end{bmatrix} \tag{25}$$

where $A, B_j, C_j, D_j, j = 1, 2, \cdots, m$ are function-valued matrices with the following (in)equalities:

$$\begin{aligned} A := \Big(\partial_{t_i}\partial_{t_{\bar{j}}}\rho_2\Big) &= \Big(K\sum_{\nu=1}^{m}\lambda_\nu(0)|\xi_\nu|^2\delta_{ij} + \sum_{\nu=1}^{m}\partial_{t_i}\partial_{t_{\bar{j}}}\lambda_\nu(0)|\xi_\nu|^2\Big) \\ &\geq \delta K(|\xi_1|^2 + \cdots + |\xi_m|^2)I_n, \end{aligned} \tag{26}$$

$$B_j := \Big(\partial_{z_{jk}}\partial_{\bar{z}_{jl}}\rho_2\Big) = \Big(-\lambda_j(0)\sum_{\nu,\mu=1}^{N_j}R_{z_{jk}\bar{z}_{jl}\mu\bar{\nu}}(0)\xi_{j\mu}\bar{\xi}_{j\nu}\Big) \geq \delta|\xi_j|^2 I_{N_j}, \tag{27}$$

$$C_j := \Big(\partial_{\xi_{jk}}\partial_{\bar{\xi}_{jl}}\rho_2\Big) = \Big(\lambda_j(0)\delta_{k\bar{l}}\Big) \geq \delta I_{N_j}, \tag{28}$$

$$D_j := \Big(\partial_{t_i}\partial_{\bar{\xi}_{jl}}\rho_2\Big) = \Big(\partial_{t_i}\lambda_j(0)\xi_{jl}\Big)_{i\leq n}^{l\leq N_j}, \tag{29}$$

at $(0,0,\xi_1,\cdots,0,\xi_m)$ for some $\delta > 0$.

Let $(e, r_1, s_1, \cdots, r_m, s_m) \neq 0$, where $e = (e_1, \cdots, e_n), r_j = (r_{j1}, \cdots r_{jN_j}), s_j = (s_{j1}, \cdots s_{jN_j})$ for all j. It holds that

$$\begin{aligned} &\begin{bmatrix} e & r_1 & s_1 & \cdots & r_m & s_m \end{bmatrix} \begin{bmatrix} A & 0 & D_1 & \cdots & 0 & D_m \\ 0 & B_1 & 0 & \cdots & 0 & 0 \\ \bar{D}_1 & 0 & C_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & B_m & 0 \\ \bar{D}_m & 0 & 0 & \cdots & 0 & C_m \end{bmatrix} \begin{bmatrix} \bar{e}^t \\ \bar{r}_1^t \\ \bar{s}_1^t \\ \cdots \\ \bar{r}_m^t \\ \bar{s}_m^t \end{bmatrix} \\ &\geq \delta K|e|^2\sum_{j=1}^{m}|\xi_j|^2 + \delta\sum_{j=1}^{m}|\xi_j|^2|r_j|^2 + \delta\sum_{j=1}^{m}|s_j|^2 - 2\sum_{j=1}^{m}\left|\sum_{i\leq n,\ l\leq N_j}e_i\partial_{t_i}\lambda_j(0)\xi_{jl}\bar{s}_{jl}\right| \\ &\geq \sum_{j=1}^{m}\big[(\delta K - M)|\xi_j|^2|e|^2 + \delta|\xi_j|^2|r_j|^2 + (\delta-\epsilon)|s_j|^2\big] \\ &> 0. \end{aligned} \tag{30}$$

Here the second inequality holds as

$$\left|\sum_{i,l} e_i \partial_{t_i} \lambda_j(0) \xi_{jl} \bar{s}_{jl}\right| \le M_1 |e||\xi_j \cdot \bar{s}_j| \le M_1 |e||\xi_j||s_j| \le \frac{M}{2}|e|^2|\xi_j|^2 + \frac{\epsilon}{2}|s_j|^2$$

by the standard Cauchy-Schwarz inequality and $M = \frac{M_1^2}{\epsilon}$. The last strict inequality holds as $\xi_j \neq 0$ for all j by letting $\epsilon < \delta$ and raising K sufficiently large. ∎

Theorem 4.2. *Under the assumption of Theorem 1.1, F is Nash algebraic.*

Proof of Theorem 4.2: Without loss of generality, one can assume that $F(0) = 0$ by composing elements from $Aut(\mathbb{B}^n)$ and $Aut(\mathbb{B}^{N_1}) \times \cdots \times Aut(\mathbb{B}^{N_m})$. Furthermore, since $F_1, \cdots, F_m$ are not constant maps, we can assume that $dF_1|_0 \not\equiv 0, \cdots, dF_m|_0 \not\equiv 0$. Therefore, there exists $0 \neq \zeta \in T_0\mathbb{B}^n$, such that $dF_j(\zeta) \neq 0$ for all j. After scaling, we assume that $(0, \zeta) \in S_1$. Notice that both the fiber of S_1 over $0 \in U$ and the fiber of S_2 over $(0, 0, \cdots, 0) \in U \times \mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}$ are independent of the choice of K. Now the theorem follows from the algebracity theorem of Huang [Hu1] and Lemma 4.1 applied to the map $(id, F_1, dF_1, \cdots, F_m, dF_m)$ from S_1 into S_2. ∎

References

[BER] Baouendi, S., Ebenfelt, P. and Rothschild, L.: Real Submanifolds in Complex Space and Their Mappings, Princeton Mathematical Series, **47**, Princeton University Press, Princeton, NJ, 1999.

[Ca] Calabi, E.: *Isometric imbedding of complex manifolds*, Ann. of Math. (2) 58, (1953). 1–23.

[CS] Cima, J. A. and Suffridge, T. J.: *Boundary behavior of rational proper maps*, Duke Math. J. 60 (1990), no. 1, 135–138.

[CU] Clozel, L. and Ullmo, E.: *Correspondances modulaires et mesures invariantes*, J. Reine Angew. Math. **558** (2003), 47–83.

[Fo] Forstneric, F.: *Extending proper holomorphic mappings of positive codimension*, Invent. Math.**95**, 31-62 (1989).

[Hu1] Huang, X.: *On the mapping problem for algebraic real hypersurfaces in the complex spaces of different dimensions*, Ann. Inst. Fourier (Grenoble) **44** (1994), no. 2, 433–463.

[Hu2] Huang, X.: *On a linearity problem of proper holomorphic mappings between balls in complex spaces of different dimensions*, J. Diff. Geom. **51**(1999), 13-33.

[Hu3] Huang, X.: *On a semi-rigidity property for holomorphic maps*, Asian J. Math. **7**(2003), no. 4, 463–492. (A special issue dedicated to Y. T. Siu on the ocassion of his 60th birthday.)

[HJ] Huang, X. and Ji, S.: *On some rigidity problems in Cauchy-Riemann analysis*, Proceedings of the International Conference on Complex Geometry and Related Fields, 89–107, AMS/IP Stud. Adv. Math., 39, Amer. Math. Soc., Providence, RI, 2007.

[HJX] Huang, X., Ji, S. and Xu, D.: *A new gap phenomenon for proper holomorphic mappings from B^n into B^N*, Math. Res. Lett. **13**(2006), no. 4, 515–529.

[HY] Huang, X. and Yuan, Y.: *Local holomorphic conformal maps between Hermitian symmetric spaces of compact type*, preprint.

[JX] Ji, S., and Xu, D.: *Maps between B^n and B^N with geometric rank $k_0 \leq n-2$ and minimum N*, Asian J. Math. 8 (2004), no. 2, 233–257.

[Mi] Mir, N.: *Convergence of formal embeddings between real-analytic hypersurfaces in codimension one*, J. Differential Geom. 62 (2002), no. 1, 163–173.

[Mo1] Mok, N.: *Local holomorphic isometric embeddings arising from correspondences in the rank-1 case*, Contemporary trends in algebraic geometry and algebraic topology (Tianjin, 2000), 155–165, Nankai Tracts Math., 5, World Sci. Publ., River Edge, NJ, 2002.

[Mo2] Mok, N.: *Extension of germs of holomorphic isometries up to normalizing constants with respect to the Bergman metric*, http://www.hku.hk/math/imr/, Preprint.

[Mo3] Mok, N.: Private communications.

[MN] Mok, N. and Ng, S.: *Germs of measure-preserving holomorphic maps from bounded symmetric domains to their Cartesian products*, J. Reine Angew. Math. to appear.

[Ng1] Ng, S.: *On holomorphic isometric embeddings from the unit disk into polydisks and their generalizations*, Ph. D. thesis. 2008.

[Ng2] Ng, S.: *On holomorphic isometric embeddings of the unit disk into polydisks*, Proc. Amer. Math. Soc. 138 (2010), 2907-2922.

[Ng3] Ng, S.: *On holomorphic isometric embeddings of the unit n-ball into products of two unit m-balls*, Math. Z. 268 (2011) no. 1-2, 347-354.

[Ng4] Ng, S.: Private communications.

[Tr] Trépreau, J.-M.: *Sur le prolongement holomorphe des fonctions CR défines sur une hypersurface réelle de classe $\mathbb{C}^2$ dans $\mathbb{C}^n$*, Invent. Math. 83 (1986), no. 3, 583–592.

[Um] Umehara, M.: *Einstein Kähler submanifolds of a complex linear or hyperbolic space*, Tohoku Math. J. (2) 39 (1987), no. 3, 385–389.

Yuan Yuan, yuan@math.jhu.edu, Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA.

Yuan Zhang, yuz009@math.ucsd.edu, Department of Mathematics, University of California at San Diego, La Jolla, CA 92093, USA.