Optimal Hölder regularity for the $\bar{\partial}$ problem on product domains in $\mathbb{C}^2$

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Abstract

The note concerns the $\bar{\partial}$ problem on product domains in $\mathbb{C}^2$. We show that there exists a bounded solution operator from $C^{k,\alpha}$ into itself, $k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha < 1$. The regularity result is optimal in view of an example of Stein-Kerzman.

1 Introduction

Let $\Omega \subset \mathbb{C}^n$ be the product of planar domains whose boundaries consist of a finite number of non-intersecting rectifiable Jordan curves. Then Ω is weakly pseudoconvex with at most Lipschitz boundary. A natural question is to look for a solution operator to the $\bar{\partial}$ problem on Ω that achieves the optimal regularity.

As indicated by Example 3.2 of Stein-Kerzman [12], the $\bar{\partial}$ problem on product domains does not gain regularity in general. This phenomenon is in sharp contrast with some well-understood domains having nice geometry (such as strict pseudoconvexity, convexity and/or finite type), on which solutions with a gain in regularity always exist. See [4, 7, 8, 10, 12, 13] et al. and the references therein.

Initiated by the work of Henkin [9] on the bidisc, Bertrams [1], Chen-McNeal [2][3], Fassina-Pan [5] and Jin-Yuan [11] etc. investigated uniform C^k and Sobolev norms of solutions on product domains. In the Hölder category, the celebrated work of Nijenhuis and Woolf [14] constructed optimal Hölder solutions in some special *iterated* Hölder spaces for polydiscs. Pan and the author [15] recently proved existence of (the standard) Hölder solutions with an infinitesimal loss of Hölder regularity by analysing the parameter dependence of the Cauchy singular integrals.

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In this note, we prove that for product domains in $\mathbb{C}^2$, the solution operator in [15] must attain the same regularity as that of the Hölder data. Thus the operator achieves the optimal regularity in view of Example 3.2. The proof relies on a careful inspection of the Hölder regularity along each direction.

Theorem 1.1. *Let $\Omega = \Omega_1 \times \Omega_2$, where Ω_1 and Ω_2 are two bounded domains in $\mathbb{C}$ with $C^{k+1,\alpha}$ boundaries, $k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha < 1$. For any $0 \le p \le 2, 1 \le q \le 2$, there exists a linear operator $T_{(p,q)} : C^{k,\alpha}_{(p,q)}(\Omega) \to C^{k,\alpha}_{(p,q-1)}(\Omega)$ such that for any $\bar\partial$-closed (p,q) form $\mathbf{f} \in C^{k,\alpha}_{(p,q)}(\Omega)$ (in the sense of distributions if $k = 0$), $T\mathbf{f}$ solves $\bar\partial u = \mathbf{f}$ on Ω. Moreover, $\|T\mathbf{f}\|_{C^{k,\alpha}_{(p,q-1)}(\Omega)} \le C\|\mathbf{f}\|_{C^{k,\alpha}_{(p,q)}(\Omega)}$, where the constant C depends only on Ω, k and α.*

It is not clear whether the same result extends to general product domains in $\mathbb{C}^n, n \ge 3$, as Example 3.3 demonstrates. As a direct consequence of Theorem 1.1, the following regularity corollary holds for smooth forms up to the boundary.

Corollary 1.2. *Let $\Omega := \Omega_1 \times \Omega_2$, where Ω_1 and Ω_2 are two bounded domains in $\mathbb{C}$ with smooth boundaries. Assume $\mathbf{f} \in C^\infty_{(p,q)}(\overline{\Omega})$ is a $\bar\partial$-closed (p,q) form on Ω, $0 \le p \le 2, 1 \le q \le 2$. Then there exists a solution $u \in C^\infty_{(p,q-1)}(\overline{\Omega})$ to $\bar\partial u = \mathbf{f}$ on Ω such that for each $k \in \mathbb{Z}^+ \cup \{0\}$, $0 < \alpha < 1$, $\|u\|_{C^{k,\alpha}_{(p,q-1)}(\Omega)} \le C\|\mathbf{f}\|_{C^{k,\alpha}_{(p,q)}(\Omega)}$, where the constant C depends only on Ω, k and α.*

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2 Notations and preliminaries

Let Ω be an open subset of $\mathbb{C}^n$. For $0 < \alpha < 1$, define the (α-)Hölder semi-norm of a function f on Ω to be

$$H^\alpha[f] := \sup_{z,z' \in \Omega, z \ne z'} \frac{|f(z) - f(z')|}{|z - z'|^\alpha}.$$

Given any $f \in C^k(\Omega), k \in \mathbb{Z}^+ \cup \{0\}$, its C^k norm is denoted by $\|f\|_{C^k(\Omega)} := \sum_{|\beta|=0}^k \sup_{z\in\Omega} |D^\beta f(z)|$, where D^β represents any $|\beta|$-th derivative operator. A function $f \in C^k(\Omega)$ is said to be in $C^{k,\alpha}(\Omega)$ if

$$\|f\|_{C^{k,\alpha}(\Omega)} := \|f\|_{C^k(\Omega)} + \sum_{|\beta|=k} H^\alpha[D^\beta f] < \infty.$$

We say a (p,q) form is in $C^{k,\alpha}_{(p,q)}(\Omega)$ (or simply $C^{k,\alpha}(\Omega)$ when the context is clear) if all its coefficients are in $C^{k,\alpha}(\Omega)$. When $k=0$, we suppress k in the notations by writing $C^{0,\alpha}(\Omega)$ as $C^{\alpha}(\Omega)$, and $C^0(\Omega)$ as $C(\Omega)$.

Assume that $\Omega := \Omega_1 \times \ldots \times \Omega_n$ is a product of planar domains $\Omega_j, 1 \le j \le n$. Fixing $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in \Omega_1 \times \ldots \times \Omega_{j-1} \times \Omega_{j+1} \times \ldots \times \Omega_n$, denote the Hölder semi-norm of a function f on Ω along the z_j variable by

$$\begin{aligned} & H^{\alpha}_j[f](z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) : \\ = & \sup_{\zeta,\zeta' \in \Omega_j, \zeta \ne \zeta'} \frac{|f(z_1, \ldots, z_{j-1}, \zeta', z_{j+1}, \ldots, z_n) - f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n)|}{|\zeta' - \zeta|^{\alpha}}. \end{aligned}$$

Then one has by the triangle inequality that

$$H^{\alpha}[f] \le \sum_{j=1}^{n} \sup_{\substack{z_l \in \Omega_l \\ 1 \le l(\ne j) \le n}} H^{\alpha}_j[f](z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n). \tag{1}$$

Suppose in addition that each slice Ω_j of Ω is bounded with $C^{k+1,\alpha}$ boundary, $1 \le j \le n$. We define the solid and boundary Cauchy integral of a function $f \in C^{k,\alpha}(\Omega)$ along the z_j variable to be

$$\begin{aligned} T_j f(z) &:= -\frac{1}{2\pi i} \int_{\Omega_j} \frac{f(z_1, \ldots, z_{j-1}, \zeta_j, z_{j+1}, \ldots, z_n)}{\zeta_j - z_j} d\bar{\zeta}_j \wedge d\zeta_j, \quad z \in \Omega; \\ S_j f(z) &:= \frac{1}{2\pi i} \int_{b\Omega_j} \frac{f(z_1, \ldots, z_{j-1}, \zeta_j, z_{j+1}, \ldots, z_n)}{\zeta_j - z_j} d\zeta_j, \quad z \in \Omega. \end{aligned}$$

The classical one-dimensional singular integral theory (see [18], or [15, Lemma 4.1]) states that for each $1 \le j \le n$,

$$\sup_{\substack{z_l \in \Omega_l \\ 1 \le l(\ne j) \le n}} H^{\alpha}_j[D^k_j T_j f](z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \lesssim \left\{ \begin{array}{ll} \|f\|_{C(\Omega)}, & k=0 \\ \|f\|_{C^{k-1,\alpha}(\Omega)}, & k \ge 1 \end{array} \right. ; \tag{2}$$

$$\sup_{\substack{z_l \in \Omega_l \\ 1 \le l(\ne j) \le n}} H^{\alpha}_j[D^k_j S_j f](z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \lesssim \|f\|_{C^{k,\alpha}(\Omega)}. \tag{3}$$

Here D^k_j represents a k-th order derivative operator with respect to the z_j variable, and two quantities a and b are said to satisfy $a \lesssim b$ if there exists a constant C dependent only on Ω, k and α, such that $a \le Cb$.

It was further proved in [15, Theorem 1.1] that for each $1 \le j \le n$, the operator T_j sends $C^{k,\alpha}(\Omega)$ into $C^{k,\alpha}(\Omega)$ with

$$\|T_j f\|_{C^{k,\alpha}(\Omega)} \lesssim \|f\|_{C^{k,\alpha}(\Omega)} \tag{4}$$

for any $f \in C^{k,\alpha}(\Omega)$; and for any small ϵ with $0 < \epsilon < \alpha$, the operator S_j sends $C^{k,\alpha}(\Omega)$ into $C^{k,\alpha-\epsilon}(\Omega)$ with

$$\|S_j f\|_{C^{k,\alpha-\epsilon}(\Omega)} \lesssim \|f\|_{C^{k,\alpha}(\Omega)} \tag{5}$$

for any $f \in C^{k,\alpha}(\Omega)$. It is worth mentioning that both (4) and (5) are sharp estimates (see Example 4.2-4.3 in [15]), in the sense that the Hölder regularity in neither inequality can be further improved.

Finally, given any $\bar\partial$ closed (0,1) form $\mathbf{f} = \sum_{j=1}^n f_j d\bar z_j \in C^{k,\alpha}(\Omega)$, define as in [14]

$$T\mathbf{f} := T_1 f_1 + T_2 S_1 f_2 + \cdots + T_n S_1 \ldots S_{n-1} f_n. \tag{6}$$

It is not hard to verify that T is a solution operator to $\bar\partial$ on Ω (in the sense of distributions if $k = 0$), using the identities $\bar\partial_j T_j = S_j + T_j \bar\partial_j = id$ and $\bar\partial_j S_k = 0, j \ne k$. Here $\bar\partial_j := \frac{\partial}{\partial \bar z_j}$ (and similarly denote $\frac{\partial}{\partial z_j}$ by ∂_j). In fact, employing the closedness of $\mathbf{f}$ and Fubini's Theorem, we can compute as follows.

$$\begin{aligned}
\bar\partial_1 T\mathbf{f} &= \bar\partial_1 T_1 f_1 + \bar\partial_1 T_2 S_1 f_2 + \cdots + \bar\partial_1 T_n S_1 \cdots S_{n-1} f_n = f_1;\\
\bar\partial_2 T\mathbf{f} &= \bar\partial_2 T_1 f_1 + \bar\partial_2 T_2 S_1 f_2 + \cdots + \bar\partial_2 T_n S_1 \cdots S_{n-1} f_n\\
&= T_1(\bar\partial_2 f_1) + S_1 f_2 = T_1(\bar\partial_1 f_2) + S_1 f_2 = f_2;\\
&\cdots\\
\bar\partial_n T\mathbf{f} &= \bar\partial_n T_1 f_1 + \bar\partial_n T_2 S_1 f_2 + \cdots + \bar\partial_n T_n S_1 \cdots S_{n-1} f_n\\
&= T_1(\bar\partial_n f_1) + T_2 S_1 \bar\partial_n f_2 + \cdots + S_1 \cdots S_{n-1} f_n\\
&= T_1(\bar\partial_1 f_n) + S_1 T_2 \bar\partial_2 f_n + \cdots + S_1 \cdots S_{n-1} f_n\\
&= f_n - S_1 f_n + S_1(f_n - S_2 f_n) + \cdots + S_1 \cdots S_{n-1} f_n = f_n.
\end{aligned}$$

As a consequence of (4) and (5), the solution operator T achieves the Hölder regularity with at most an infinitesimal loss from that of the data.

3 The optimal Hölder estimates

Let $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_j \subset \mathbb{C}$ is a bounded domain with $C^{k+1,\alpha}$ boundary, $j = 1, 2$, $k \in \mathbb{Z}^+ \cup \{0\}, 0 < \alpha < 1$. Despite a loss of Hölder regularity of S_j in $C^{k,\alpha}(\Omega)$ as in (5), the following proposition shows that the composition operator $S_j T_l, j \ne l$, preserves exactly the same Hölder regularity. The key observation of the proof is that the loss of Hölder regularity of S_j only occurs along the z_l direction, which is compensated by a gain of Hölder regularity of T_l in this same direction.

Proposition 3.1. *For each $k \in \mathbb{Z}^+ \cup \{0\}$ and $0 < \alpha < 1$, $1 \le j \ne l \le 2$, there exists some constant C dependent only on Ω, k and α, such that for any $f \in C^{k,\alpha}(\Omega)$,*

$$\|S_jT_lf\|_{C^{k,\alpha}(\Omega)} \le C\|f\|_{C^{k,\alpha}(\Omega)}.$$

Proof. Without loss of generality, assume $j = 1$ and $l = 2$. Let $\gamma := (\gamma_1, \gamma_2)$ with $|\gamma| \le k$. Since S_1T_2f is holomorphic with respect to the z_1 variable, we only need to estimate $\|D_2^{\gamma_2}\partial_1^{\gamma_1}S_1T_2f\|_{C^\alpha(\Omega)}$.

Write $b\Omega_1 = \cup_{m=1}^N \Gamma_m$, where each Jordan curve Γ_m is connected, positively oriented with respect to Ω_1, and of length s_m. Let $\zeta_1|_{s\in[\sum_{j=1}^{m-1} s_j, \sum_{j=1}^m s_j)}$ be a $C^{k+1,\alpha}$ parametrization of Γ_m with respect to the arclength variable s, and $\tilde{s} = \sum_{m=1}^N s_m$ is the total length of $b\Omega_1$. In particular, $\zeta_1' = 1/\bar{\zeta}_1'$ on the interval $(\sum_{j=1}^{m-1} s_j, \sum_{j=1}^m s_j)$ for each $1 \le m \le N$. For any $(z_1, z_2) \in \Omega$, integration by parts on $(\sum_{j=1}^{m-1} s_j, \sum_{j=1}^m s_j)$ for each $1 \le m \le N$ gives

$$\begin{aligned}
\partial_1 S_1T_2f(z_1, z_2) =& \frac{1}{2\pi i}\int_{b\Omega_1} \partial_{z_1}\left(\frac{1}{\zeta_1(s) - z_1}\right) T_2f(\zeta_1(s), z_2)\zeta_1'(s)ds \\
=& -\frac{1}{2\pi i}\sum_{m=1}^N \int_{\sum_{j=1}^{m-1} s_j}^{\sum_{j=1}^m s_j} \partial_s\left(\frac{1}{\zeta_1(s) - z_1}\right) T_2f(\zeta_1(s), z_2)ds \\
=& \frac{1}{2\pi i}\sum_{m=1}^N \int_{\sum_{j=1}^{m-1} s_j}^{\sum_{j=1}^m s_j} \frac{\partial_s\left(T_2f(\zeta_1(s), z_2)\right)}{\zeta_1(s) - z_1}ds \\
=& \frac{1}{2\pi i}\sum_{m=1}^N \int_{\sum_{j=1}^{m-1} s_j}^{\sum_{j=1}^m s_j} \frac{T_2\left(\partial_1 f(\zeta_1(s), z_2)\zeta_1'(s) + \bar{\partial}_1 f(\zeta_1(s), z_2)\bar{\zeta}_1'(s)\right)}{\zeta_1(s) - z_1}ds \\
=& \frac{1}{2\pi i}\sum_{m=1}^N \int_{\sum_{j=1}^{m-1} s_j}^{\sum_{j=1}^m s_j} \frac{T_2\left(\partial_1 f(\zeta_1(s), z_2) + \bar{\partial}_1 f(\zeta_1(s), z_2)(\bar{\zeta}_1'(s))^2\right)}{\zeta_1(s) - z_1}\zeta_1'(s)ds \\
=:& \frac{1}{2\pi i}\int_{b\Omega_1} \frac{T_2\tilde{f}(\zeta_1, z_2)}{\zeta_1 - z_1}d\zeta_1 = S_1T_2\tilde{f}(z_1, z_2),
\end{aligned}$$

where the function $\tilde{f}$ is in $C^{k-1,\alpha}(\Omega)$ such that $\tilde{f}(\zeta_1(s), z_2) = \partial_1 f(\zeta_1(s), z_2) + \bar{\partial}_1 f(\zeta_1(s), z_2)(\bar{\zeta}_1'(s))^2$ on $[0, \tilde{s}) \times \Omega_2$ and $\|\tilde{f}\|_{C^{k-1,\alpha}(\Omega)} \lesssim \|f\|_{C^{k,\alpha}(\Omega)}$ (see [6, Lemma 6.38] on page 137 for the construction of an extension). Repeating the above process, proving the proposition is reduced to proving for each $\gamma \in \mathbb{Z}^+ \cup \{0\}, \gamma \le k, 0 < \alpha < 1$,

$$\|D_2^\gamma S_1 T_2 f\|_{C^\alpha(\Omega)} \lesssim \|f\|_{C^{\gamma,\alpha}(\Omega)}$$

for all $f \in C^{\gamma,\alpha}(\Omega)$.

Firstly, choose an ϵ such that $0 < \epsilon < \alpha$. Applying the estimates (5) and (4) to $S_1 T_2 f$, we get

$$\|D_2^\gamma S_1 T_2 f\|_{C(\Omega)} \le \|S_1 T_2 f\|_{C^{\gamma,\alpha-\epsilon}(\Omega)} \lesssim \|T_2 f\|_{C^{\gamma,\alpha}(\Omega)} \lesssim \|f\|_{C^{\gamma,\alpha}(\Omega)}.$$

We next verify that $H^\alpha[D_2^\gamma S_1 T_2 f] \lesssim \|f\|_{C^{\gamma,\alpha}(\Omega)}$. Fixing $z_2 \in \Omega_2$, since $D_2^\gamma S_1 T_2 f = S_1 D_2^\gamma T_2 f$,

$$H_1^\alpha[D_2^\gamma S_1 T_2 f](z_2) = H_1^\alpha[S_1 D_2^\gamma T_2 f](z_2) \lesssim \|D_2^\gamma T_2 f\|_{C^\alpha(\Omega)}.$$

Here the last inequality used (3) for the estimate of S_1 on Ω_1. Consequently, applying (4) to the operator T_2 in the last term, we obtain

$$H_1^\alpha[D_2^\gamma S_1 T_2 f](z_2) \lesssim \|T_2 f\|_{C^{\gamma,\alpha}(\Omega)} \lesssim \|f\|_{C^{\gamma,\alpha}(\Omega)}.$$

We further show for each $z_1 \in \Omega_1$, $H_2^\alpha[D_2^\gamma S_1 T_2 f](z_1) \lesssim \|f\|_{C^{\gamma,\alpha}(\Omega)}$. If $\gamma \ge 1$, making use of the identity $D_2^\gamma S_1 T_2 f = D_2^\gamma T_2 S_1 f$ by Fubini's theorem, and the second case of (2) for T_2 along the z_2 direction, one deduces

$$H_2^\alpha[D_2^\gamma S_1 T_2 f](z_1) = H_2^\alpha[D_2^\gamma T_2 S_1 f](z_1) \lesssim \|S_1 f\|_{C^{\gamma-1,\alpha}(\Omega)}.$$

Together with (5) for S_1 on Ω, we infer

$$H_2^\alpha[D_2^\gamma S_1 T_2 f](z_1) \lesssim \|f\|_{C^{\gamma,\alpha}(\Omega)}.$$

When $\gamma = 0$, the first case of (2) for T_2 and (5) for S_1 together give

$$H_2^\alpha[D_2^\gamma S_1 T_2 f](z_1) = H_2^\alpha[T_2 S_1 f](z_1) \lesssim \|S_1 f\|_{C(\Omega)} \lesssim \|f\|_{C^\alpha(\Omega)}.$$

The proof of the proposition is complete in view of (1). ∎

Proof of Theorem 1.1 and Corollary 1.2. We only need to prove the case when $p = 0$. If $q = 2$, for any datum $\mathbf{f} = f d\bar{z}_1 \wedge d\bar{z}_2$, it is easy to verify that $T_1 f d\bar{z}_2$ is a solution to $\bar\partial$ on Ω. The optimal Hölder estimate follows from that of the T_1 operator demonstrated in (4). For $q = 1$, the Hölder estimate of the solution given by (6) is a consequence of (4) and Proposition 3.1, from which the theorem and the corollary follow. ∎

Motivated by an L^∞ example of Stein and Kerzman [12], it was shown in [15] that the following $\bar\partial$ problem on the bidisc does not gain regularity in Hölder spaces, according to which the Hölder regularity in Theoerem 1.1 is optimal.

Example 3.2. [12] Let $\triangle^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ be the bidisc. For each $k \in \mathbb{Z}^+ \cup \{0\}$ and $0 < \alpha < 1$, consider $\bar\partial u = \mathbf{f} := \bar\partial((z_1 - 1)^{k+\alpha}\bar z_2)$ on $\triangle^2$, $\frac{1}{2}\pi < \arg(z_1 - 1) < \frac{3}{2}\pi$. Then $\mathbf{f} \in C^{k,\alpha}(\triangle^2)$ is $\bar\partial$-closed. However, there does not exist a solution $u \in C^{k,\alpha'}(\triangle^2)$ to $\bar\partial u = \mathbf{f}$ for any α' with $1 > \alpha' > \alpha$.

Unfortunately, our method does not obtain optimal Hölder estimates for product domains of dimension larger than 2. For instance, the solution operator of the $\bar\partial$ problem for $(0,1)$ forms on product domains when $n = 3$ is in the form of $T\mathbf{f} = T_1 f_1 + T_2 S_1 f_2 + T_3 S_1 S_2 f_3$. Yet not all three operators involved on the right hand side of the formula are bounded in $C^\alpha(\Omega)$ space. In fact, in the following we adapt an example of Tumanov [17] to show that $T_2 S_1$ fails to send $C^\alpha(\Omega)$ into itself, due to the unboundedness of its Hölder semi-norm along the z_3 variable. As a result of this, Proposition 3.1 holds only when $n = 2$.

Example 3.3. For $(e^{i\theta}, \lambda) \in b\triangle \times \triangle$, let

$$\tilde h(e^{i\theta}, \lambda) := \begin{cases} |\lambda|^\alpha, & -\pi \le \theta \le -|\lambda|^{\frac{1}{2}}; \\ \theta^{2\alpha}, & -|\lambda|^{\frac{1}{2}} \le \theta \le 0; \\ \theta^\alpha, & 0 \le \theta \le |\lambda|; \\ |\lambda|^\alpha, & |\lambda| \le \theta \le \pi, \end{cases}$$

and h be a C^α extension of $\tilde h$ onto $\triangle^2$. Define $f(z_1, z_2, z_3) := h(z_1, z_3)$ for $(z_1, z_2, z_3) \in \triangle^3$. Then $f \in C^\alpha(\triangle^3)$. However, $T_2 S_1 f \notin C^\alpha(\triangle^3)$.

Proof. Clearly $\tilde h \in C^\alpha(b\triangle \times \triangle)$. For each $z' = (z_1, z_3) \in \triangle^2$, let $h(z') := \inf_{w \in b\triangle\times\triangle}\{\tilde h(w) + M|z' - w|^\alpha\}$, where $M = \|\tilde h\|_{C^\alpha(b\triangle\times\triangle)}$. Then $h \in C^\alpha(\triangle^2)$ is a C^α extension of $\tilde h$ onto $\triangle^2$ and $f \in C^\alpha(\triangle^3)$.

In [16, Section 3], it was verified that $H_3^\alpha[S_1 h](z_1)$ is unbounded near $1 \in b\triangle$, and so $S_1 h \notin C^\alpha(\triangle^2)$. On the other hand, making use of the fact that $T_2 1(z) = \bar z_2$, $z \in \triangle^3$ (see [14, Appendix 6.1b] for instance), we get $T_2 S_1 f(z) = T_2 1(z) \cdot S_1 h(z_1, z_3) = \bar z_2 S_1 h(z_1, z_3)$, which does not belong to $C^\alpha(\triangle^3)$. ∎

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