

CR submanifolds with vanishing second fundamental forms

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Abstract

In this paper, we show that all CR immersions from smooth Levi-nondegenerate hypersurfaces into hyperquadrics with vanishing CR second fundamental forms are necessarily linear fractional.

1 Introduction

Let M be a Levi-nondegenerate smooth real hypersurface of signature ℓ in \mathbb{C}^{n+1} , $0 \leq \ell \leq [n/2]$. Locally, M is defined by

$$M := \{(z, w) = (z_1, \dots, z_n, w) \in \mathbb{C}^n \times \mathbb{C} : \Im w = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^n |z_j|^2 + o_{wt}(2)\}.$$

For simplicity of notation, we write $|z|_{\ell}^2 := \sum_j \delta_{j,\ell} |z_j|^2 := -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^n |z_j|^2$ and the corresponding inner product $\langle z, \xi \rangle_{\ell} := \sum_j \delta_{j,\ell} z_j \bar{\xi}_j$ given $z, \xi \in \mathbb{C}^n$. Here $\delta_{j,\ell}$ is -1 if $j \leq \ell$ and 1 if $j \geq \ell + 1$. The hyperquadric in \mathbb{C}^{n+1} of signature ℓ is defined explicitly by

$$\mathbb{H}_{\ell}^{n+1} := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \Im w = |z|_{\ell}^2\}.$$

The rigidity phenomenon of CR maps has been investigated extensively in literature. See [Al], [CS], [Fa], [For], [Ha], [Hu2-3], [HJ], [HJX] and [HJY] for instance for rigidity or classification between balls, the model CR manifolds. Rigidity between hyperquadrics with positive signatures was first studied in the fundamental work of Baouendi and Huang [BH], where the authors showed the linearity for the CR embeddings between hyperquadrics with same signatures. Later on, Baouendi, Ebenfelt and Huang [BEH2] extended the linearity result to the case when the difference of signatures between the target and the source is small. For CR embeddings of general Levi-nondegenerate hypersurfaces into hyperquadrics, the same three authors in another paper [BEH1] further proved the rigidity when the signature of the source and the target manifolds is the same. A partial linearity result was due to Ebenfelt and Shroff [ES] with small signature difference. See also [Pi], [BER2], [BHR],

*Supported in part by National Science Foundation grant DMS-1412384

†Supported in part by National Science Foundation grant DMS-1501024

[BR], [Eb], [EHZ1-2], [Hu1], [La] and [HZ1-3] for related topics. In general, the rigidity phenomenon is not expected to be true as various examples have shown if the difference in signatures of the target and the source manifolds is large. The reader is referred to [BER1], [Da] et al. for general theory on CR maps.

Another different approach leading to rigidity is to impose appropriate geometric conditions. As the classical second fundamental forms play a crucial role in characterizing the totally geodesic submanifolds in differential geometry and projective geometry, its CR analogue becomes a good candidate on this aspect (see [IL] on projective second fundamental forms of submanifolds in homogeneous spaces). In the joint work with Ji [JY], the first author used the CR second fundamental form to determine totally geodesic CR submanifolds in spheres. More precisely, they considered the CR embedding from a strictly pseudoconvex CR real hypersurface into the sphere, and showed that if the CR second fundamental form of the image vanishes, then the CR embedding has to be equivalent to the linear embedding up to the automorphism group action of the sphere.

The idea of CR second fundamental form appeared firstly in the pioneer work of Chern-Moser [CM] when they investigated the normal forms between Levi-nondegenerate CR hypersurfaces and then in [We] while Webster studied the pseudo-Hermitian structure. In the paper [EHZ1] of Ebenfelt, Huang and Zaitsev, they defined the notion of CR second fundamental form in order to study the rigidity for CR immersions into spheres. The CR second fundamental form was later on rephrased in the framework of Cartan's geometry by using the Maurer-Cartan form for the CR submanifolds of spheres in [JY]. Other related study can also be found in [CJ] and [CJL]. In this paper, we focus on the submanifolds in hyperquadrics with vanishing CR second fundamental forms. Along this line of work, we are able to prove the following geometric rigidity result in the hyperquadrics case.

Theorem 1.1. *Let $M \subset \mathbb{C}^{n+1}$ be a Levi-nondegenerate smooth real hypersurface of signature ℓ and $H : M \rightarrow \mathbb{H}_{\ell'}^{N+1}$ be a smooth CR embedding. Suppose the CR second fundamental form $II_M \equiv 0$. Then there exists a CR map G such that $G : \mathbb{H}_{\ell}^{n+1} \rightarrow M$ is locally a CR diffeomorphism and $H \circ G$ extends to a linear fractional map from \mathbb{H}_{ℓ}^{n+1} to $\mathbb{H}_{\ell'}^{N+1}$.*

The definition of CR second fundamental form will be elaborated in Section 2 for the convenience of the reader. When the CR second fundamental vanishes, M is equivalent to \mathbb{H}_{ℓ}^{n+1} by Lemma 2.2 following the same approach as in [EHZ1] through CR Gauss equation. In this sense, Theorem 1.1 then essentially says that any CR embedding into a hyperquadric is necessarily linear fractional if its CR second fundamental form is 0. We note that there is no restriction on the difference in signatures (ℓ can be 0) or codimension in the assumption. It is also worthwhile to point out that since the map H is CR transversal, $\ell \leq \ell'$ holds necessarily.

Acknowledgement: Both authors would like to thank Professor X. Huang and S. Ji for helpful discussions.

2 On CR second fundamental forms

2.1 CR second fundamental forms

We start by reviewing the CR second fundamental forms defined in [EHZ1] and we follow the terminology as in [EHZ1].

Let (M, p) and (\tilde{M}, \tilde{p}) be two germs of real smooth Levi-nondegenerate hypersurfaces in \mathbb{C}^{n+1} and \mathbb{C}^{N+1} defined by two smooth functions ρ and $\tilde{\rho}$, respectively. The corresponding CR tangent spaces of M and \tilde{M} at p and \tilde{p} are denoted by $T_p^{1,0}M$ and $T_{\tilde{p}}^{1,0}\tilde{M}$. Assume F is a local smooth CR-embedding from M to \tilde{M} sending $p \in M$ to $\tilde{p} \in \tilde{M}$. Consider the following vector space

$$E_k(p) := \text{span}_{\mathbb{C}}\{L^{\bar{J}}(\tilde{\rho}_{Z'} \circ F)(p) : J \in (Z_+)^n, |J| \leq k, L \in T_p^{1,0}M\} \subset T_{\tilde{p}}^{1,0'}\mathbb{C}^{N+1}.$$

Here $\tilde{\rho}_{Z'} := \partial\tilde{\rho}$ is the complex gradient realized as an element in the space $T_{\tilde{p}}^{1,0'}\mathbb{C}^{N+1}$ of $(1,0)$ covector in \mathbb{C}^{N+1} in local coordinate system Z' near \tilde{p} , and the multi-index notation $L^{\bar{J}} := L_1^{\bar{J}_1} \cdots L_n^{\bar{J}_n}$ and $|J| = J_1 + \dots + J_n$. Then the CR second fundamental form is defined in [EHZ1].

Definition 2.1. *Given any two vectors $X_p, Y_p \in T_p^{1,0}M$, the CR second fundamental form II_M of M is defined by*

$$II_M(X_p, Y_p) := \overline{\pi(XY(\tilde{\rho}_{Z'} \circ F)(p))} \in \overline{T_{\tilde{p}}'\tilde{M}/E_1(p)}, \quad (1)$$

where $\tilde{\rho}_{Z'} = \bar{\partial}\tilde{\rho}$, X, Y are two $(1,0)$ vector fields on M extending the given vectors $X_p, Y_p \in T_p^{1,0}(M)$, and $\pi : T_{\tilde{p}}'\tilde{M} \rightarrow \overline{T_{\tilde{p}}'\tilde{M}/E_1(p)}$ is the projection map.

Since \tilde{M} and M are Levi-nondegenerate, the Levi form of \tilde{M} (at \tilde{p}) with respect to $\tilde{\rho}$ defines an isomorphism

$$\overline{T_{\tilde{p}}'\tilde{M}/E_1(p)} \cong T_{\tilde{p}}^{1,0}\tilde{M}/F_*(T_p^{1,0}M).$$

Therefore, the CR second fundamental form can be viewed as an \mathbb{C} -linear symmetric form

$$II_{M,p} : T_p^{1,0}M \times T_p^{1,0}M \rightarrow \overline{T_{\tilde{p}}^{1,0}\tilde{M}/F_*(T_p^{1,0}M)} \quad (2)$$

independent on the choice of $\tilde{\rho}$ (cf. [EHZ1], §2). By choosing appropriate admissible coframes (θ, θ^α) for M , we denote by $(\omega_\alpha^\mu)_\beta$ the CR second fundamental form matrix, i.e.,

$$II(L_\alpha, L_\beta) = \omega_\alpha^\mu{}_\beta L_\mu, \quad n+1 \leq \mu \leq N,$$

where $\{L_\alpha\}_{\alpha=1}^n$ are the CR tangent vectors in M dual to the admissible coframes. For the consistency throughout the paper, we use Greek letters α and β to represent integer numbers $1, \dots, n$; use μ and ν to represent integer numbers $n+1, \dots, N$; and use capital letters A and B for integer numbers $1, \dots, N$.

The following pseudoconformal analogue of the Gauss equation is derived in [EHZ1] and plays a crucial role in the proof of Lemma 2.2.

$$\begin{aligned}
S_{\alpha\bar{\beta}\mu\bar{\nu}} &= \tilde{S}_{\alpha\bar{\beta}\mu\bar{\nu}} - \frac{\tilde{S}_{\gamma\alpha\bar{\beta}}^{\gamma}g_{\mu\bar{\nu}} + \tilde{S}_{\gamma\mu\bar{\beta}}^{\gamma}g_{\alpha\bar{\nu}} + \tilde{S}_{\gamma\alpha\bar{\nu}}^{\gamma}g_{\mu\bar{\beta}} + \tilde{S}_{\gamma\mu\bar{\nu}}^{\gamma}g_{\alpha\bar{\beta}}}{n+2} \\
&+ \frac{\tilde{S}_{\gamma\delta}^{\gamma\delta}(g_{\alpha\bar{\beta}}g_{\mu\bar{\nu}} + g_{\alpha\bar{\nu}}g_{\mu\bar{\beta}})}{(n+1)(n+2)} - g_{a\bar{b}}\omega_{\alpha}^a\omega_{\mu}^{\bar{b}} \\
&+ \frac{\omega_{\gamma\alpha}^a\omega_{a\bar{\beta}}^{\gamma}g_{\mu\bar{\nu}} + \omega_{\gamma\mu}^a\omega_{a\bar{\beta}}^{\gamma}g_{\alpha\bar{\nu}} + \omega_{\gamma\alpha}^a\omega_{a\bar{\nu}}^{\gamma}g_{\mu\bar{\beta}} + \omega_{\gamma\mu}^a\omega_{a\bar{\nu}}^{\gamma}g_{\alpha\bar{\beta}}}{n+2} \\
&- \frac{\omega_{\gamma\delta}^a\omega_a^{\gamma\delta}(g_{\alpha\bar{\beta}}g_{\mu\bar{\nu}} + g_{\alpha\bar{\nu}}g_{\mu\bar{\beta}})}{(n+1)(n+2)}
\end{aligned} \tag{3}$$

Here S and \tilde{S} are the Chern-Moser-Weyl curvature tensors of M and \tilde{M} , respectively; g is the pseudo-Hermitian metric of M induced by the Levi-form of M ; and ω is the second fundamental form.

The classical theory in Riemannian geometry states that the vanishing of the second fundamental forms implies the local isometry. Making use of the definition of the CR second fundamental form and (3), the following lemma shows the similar phenomenon in the CR geometry.

Lemma 2.2. *Let $F : M \rightarrow \mathbb{H}_{\ell'}^{N+1}$ be a smooth CR embedding of a Levi-nondegenerate smooth real hypersurface $M \subset \mathbb{C}^{n+1}$ of signature ℓ . Denote by $(\omega_{\alpha}^{\mu}{}_{\beta})$ the CR second fundamental form of M relative to an admissible coframe (θ, θ^A) on $\mathbb{H}_{\ell'}^{N+1}$ adapted to M . If $\omega_{\alpha}^{\mu}{}_{\beta} \equiv 0$ for all α, β and μ , then M is locally CR-equivalent to \mathbb{H}_{ℓ}^{n+1} .*

Proof of Lemma 2.2: The argument essentially follows from [EHZ1]. Letting \tilde{S} and ω equal to 0 on the right hand side of (3), we have $S \equiv 0$. According to a result in [CM], M is locally CR equivalent to a hyperquadric \mathbb{H}_{ℓ}^{n+1} . ■

2.2 Projective CR second fundamental forms

We will review two definitions for the projective CR second fundamental form of CR submanifolds in the homogeneous space $\mathbb{H}_{\ell'}^{N+1}$, following Cartan's language [IL][JY]. These two definitions are equivalent to Definition 2.1, whereas we will use the projective geometry to simplify the calculation of the CR second fundamental forms for CR embeddings between Levi-nondegenerate hypersurfaces and thus prove Theorem 1.1. The idea follows essentially from [JY] and the interesting reader can refer to [IL] and [JY] for the detailed study on the projective CR second fundamental form.

Consider a real hypersurface Q in \mathbb{C}^{N+2} defined by the homogeneous equation

$$\langle Z, Z \rangle := |Z^A|_{\ell'}^2 + \frac{i}{2}(\overline{Z^0}Z^{N+1} - Z^0\overline{Z^{N+1}}) = 0, \tag{4}$$

where $Z^A \in \mathbb{C}^N$, $Z = (Z^0, Z^A, Z^{N+1})^t \in \mathbb{C}^{N+2}$. Let

$$\pi_0 : \mathbb{C}^{N+2} - \{0\} \rightarrow \mathbb{C}\mathbb{P}^{N+1}, \quad (Z_0, \dots, Z_{N+1}) \mapsto [Z_0 : \dots : Z_{N+1}], \tag{5}$$

be the standard projection. Then the image $\pi_0(Q - \{0\})$ is the hyperquadrics $\mathbb{H}_{\ell'}^{N+1} \subset \mathbb{C}\mathbb{P}^{N+1}$. For any $A \in GL(\mathbb{C}^{N+2})$, the space of $(N+2) \times (N+2)$ invertible matrices, there is a natural self-map of $\mathbb{C}\mathbb{P}^{N+1}$, still denoted by A , given by the following matrix multiplication

$$A\left([Z_0 : Z_1 : \dots : Z_{N+1}]\right) = \left[\sum_{j=0}^{N+1} a_j^{(0)} Z_j : \sum_{j=0}^{N+1} a_j^{(1)} Z_j : \dots : \sum_{j=0}^{N+1} a_j^{(N+1)} Z_j\right], \quad (6)$$

where

$$A = (a_0, \dots, a_{N+1}) = \begin{bmatrix} a_0^{(0)} & a_1^{(0)} & \dots & a_{N+1}^{(0)} \\ a_0^{(1)} & a_1^{(1)} & \dots & a_{N+1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_0^{(N+1)} & a_1^{(N+1)} & \dots & a_{N+1}^{(N+1)} \end{bmatrix} \in GL(\mathbb{C}^{N+2}). \quad (7)$$

In fact, $A \in Aut(\mathbb{C}\mathbb{P}^{N+1})$. One can further focus on a subspace of $GL(\mathbb{C}^{N+2})$ as a linear transformation of \mathbb{C}^{N+2} .

Definition 2.3. $A \in GL^Q(\mathbb{C}^{N+2})$ if $A(Q) \subseteq Q$.

It is straightforward to check that $A \in GL^Q(\mathbb{C}^{N+2})$ if and only if $A \in Aut(\mathbb{H}_{\ell'}^{N+1})$. Moreover one considers a map

$$\begin{aligned} \pi : \quad GL^Q(\mathbb{C}^{N+2}) &\rightarrow \mathbb{H}_{\ell'}^{N+1} \\ A = (a_0, a_1, \dots, a_{N+1}) &\mapsto \pi_0(a_0). \end{aligned} \quad (8)$$

We get $\mathbb{H}_{\ell'}^{N+1} \simeq GL^Q(\mathbb{C}^{N+2})/P_1$, where P_1 is the isotropy subgroup of $GL^Q(\mathbb{C}^{N+2})$. Let $M \subset \mathbb{H}_{\ell'}^{N+1}$ be a CR submanifold of CR dimension n , and let $T^{1,0}M, R_M$ be the CR tangent bundle and the Reeb vector field with respect to the canonical contact form respectively. For any subset $X \subset \mathbb{H}_{\ell'}^{N+1}$, we denote $\hat{X} := \pi_0^{-1}(X)$. In particular, for any $x \in M$, \hat{x} is a complex line. For any $x \in M$, we let $v \in \hat{x} = \pi_0^{-1}(x) \subset \mathbb{C}^{N+2} - \{0\}$, and define

$$\hat{T}_x^{1,0}M = T_v^{1,0}\hat{M}, \quad \hat{R}_{M,x} = R_{\hat{M},v}.$$

Definition 2.4. A smooth map $e = (e_0, e_\alpha, e_\mu, e_{N+1}) : M \rightarrow GL^Q(\mathbb{C}^{N+2})$, where $1 \leq \alpha \leq n$ and $n+1 \leq \mu \leq N$, is called a first-order adapted lift if $\pi \circ e = Id$ and

$$e_0(x) \in \pi_0^{-1}(x), \quad span_{\mathbb{C}}(e_0, e_\alpha)(x) = \hat{T}_x^{1,0}M, \quad span(e_0, e_\alpha, e_{N+1})(x) = \hat{T}_x^{1,0}M \oplus \hat{R}_{M,x} \quad (9)$$

where

$$span(e_0, e_\alpha, e_{N+1})(x) := \{c_0 e_0 + c_\alpha e_\alpha + c_{N+1} e_{N+1} \mid c_0, c_\alpha \in \mathbb{C}, c_{N+1} \in \mathbb{R}\}. \quad (10)$$

Denote the Maurer-Cartan form from $GL^Q(\mathbb{C}^{N+2})$ pulled back by a first-order adapted lift e on M by

$$\omega = \begin{pmatrix} \omega_0^0 & \omega_\beta^0 & \omega_\nu^0 & \omega_{N+1}^0 \\ \omega_0^\alpha & \omega_\beta^\alpha & \omega_\nu^\alpha & \omega_{N+1}^\alpha \\ \omega_0^\mu & \omega_\beta^\mu & \omega_\nu^\mu & \omega_{N+1}^\mu \\ \omega_0^{N+1} & \omega_\beta^{N+1} & \omega_\nu^{N+1} & \omega_{N+1}^{N+1} \end{pmatrix}.$$

The standard argument (see [JY]) yields that $\omega_0^\mu = 0$, for all $n + 1 \leq \mu \leq N$. By the Maurer-Cartan equation and Cartan's lemma, it holds that

$$\omega_\beta^\nu = q_{\alpha\beta}^\nu \omega_0^\alpha \quad \text{mod}(\omega_0^{N+1}),$$

for some functions $q_{\alpha\beta}^\nu = q_{\beta\alpha}^\nu$.

Definition 2.5. *The projective CR second fundamental form II_M^{GL} with respect to $GL^Q(\mathbb{C}^{N+2})$ is defined by $II_M^{GL} = II_M^{GL,e} = q_{\alpha\beta}^\mu \omega_0^\alpha \omega_0^\beta \otimes \underline{e}_\mu \quad \text{mod}(\omega_0^{N+1})$, where $\underline{e}_\mu = e^0 \otimes (e_\mu \text{ mod } \hat{T}_x^{1,0}M) \in N_x^{1,0}M$.*

The bilinear form $\langle \cdot, \cdot \rangle$ is naturally extended to the scalar product

$$\langle Z, Z' \rangle := - \sum_{1 \leq A \leq \ell'} Z^A \overline{Z'^A} + \sum_{\ell' < A \leq N} Z^A \overline{Z'^A} + \frac{i}{2} (Z^{N+1} \overline{Z'^0} - Z^0 \overline{Z'^{N+1}}), \quad (11)$$

for any $Z = (Z^0, Z^A, Z^{N+1})^t, Z' = (Z'^0, Z'^A, Z'^{N+1})^t \in \mathbb{C}^{N+2}$. Let $SU(N + 1, \ell' + 1)$ be the group of unimodular linear transformations of \mathbb{C}^{N+2} that leave the form $\langle Z, Z \rangle$ invariant [CM] [JY]. An element $E = (E_0, E_A, E_{N+1}) \in GL(\mathbb{C}^{N+2})$ is called a Q -frame if E satisfies

$$\left\{ \begin{array}{l} \det(E) = 1, \quad \langle E_A, E_B \rangle = \delta_{AB} \delta_{A,\ell'}, \\ \langle E_0, E_{N+1} \rangle = -\langle E_{N+1}, E_0 \rangle = -\frac{i}{2}, \\ \text{all other products are zero.} \end{array} \right. \quad (12)$$

Here $\delta_{AB} = 1$ if $A = B$, and 0 otherwise, $\delta_{A,\ell'} = 1$ if $A \leq \ell'$, and -1 otherwise. Note that $SU(N + 1, \ell' + 1) \subset GL^Q(\mathbb{C}^{N+2})$ is a subgroup and $\pi : SU(N + 1, \ell' + 1) \rightarrow \mathbb{H}_{\ell'}^{N+1}$ is induced by (8). In fact, $\mathbb{H}_{\ell'}^{N+1} \simeq SU(N + 1, \ell' + 1)/P_2$, where P_2 is the isotropy subgroup of $SU(N + 1, \ell' + 1)$. By choosing a first-order adapted lifts e from M into $SU(N + 1, \ell' + 1)$, as in Definition 2.5, we can define the projective CR second fundamental form II_M^{SU} with respect to $SU(N + 1, \ell' + 1)$.

Definition 2.6.

$$II_M^{SU} = II_M^{SU,e} = q_{\alpha\beta}^\mu \omega_0^\alpha \omega_0^\beta \otimes \underline{e}_\mu \text{ mod}(\omega_0^{N+1}). \quad (13)$$

We remark that the projective CR second fundamental forms in Definition 2.5, 2.6 are both independent of the first order adapted lifts chosen ([IL][JY]).

Let $M \subset \mathbb{H}_{\ell'}^{N+1}$ be a CR submanifold of CR dimension n . The existence of the first order adapted lift from M into $SU(N + 1, \ell' + 1)$, thus into $GL^Q(N + 2)$, is obtained in [JY]. In particular, if $M = F(\mathbb{H}_\ell^{n+1})$, where $F : \mathbb{H}_\ell^{n+1} \rightarrow \mathbb{H}_{\ell'}^{N+1}$ is a smooth CR embedding, one can write down a first order adapted lift in an explicit way. Without loss of generality, we assume that $F : \mathbb{H}_\ell^{n+1} \rightarrow \mathbb{H}_{\ell'}^{N+1}$ is normalized such that $F = (f, \phi, g)$, with $F(0) = 0$. For any $p \in M$, we define a lift $e = (e_0, \dots, e_{N+1})$ of M into $SU(N + 1, \ell' + 1)$, such that

$$e_0(p) := [1 : f : \phi : g]^t (F^{-1}(p)), \quad (14)$$

$$e_\alpha(p) := \left(\frac{1}{\sqrt{|L_\alpha f|^2 + |L_\alpha \phi|^2}} [0 : L_\alpha f : L_\alpha \phi : L_\alpha g]^t \right) (F^{-1}(p)), \quad (15)$$

$$e_{N+1}(p) := A(p)e_0(p) + \sum B(p)_\alpha e_\alpha(p) + C(p) ([0, Tf, T\phi, Tg]^t) (F^{-1}(p)). \quad (16)$$

Here $L_\alpha = \frac{\partial}{\partial z^\alpha} + 2i\bar{z}^\alpha \frac{\partial}{\partial z^{N+1}}$, $T = \frac{\partial}{\partial \Re(z^{N+1})}$, the coefficient functions A, B_α and C are chosen such that $\langle e_0, e_{N+1} \rangle = \frac{i}{2}$, $\langle e_\alpha, e_{N+1} \rangle = 0$, $\langle e_{N+1}, e_{N+1} \rangle = 0$.

It follows from the argument in [JY] that the CR second fundamental forms defined in Definition 2.1, 2.5, 2.6 are equivalent in the sense that if the form in one definition vanishes, then so does the other two.

3 Normalization of CR maps between hyperquadrics

In this section, we focus our attention on CR embeddings between hyperquadrics (of possibly different signatures).

Let $F : (\mathbb{H}_\ell^{n+1}, 0) \rightarrow (\mathbb{H}_{\ell'}^{N+1}, 0)$. As in [BH], we define

$$\mathbb{H}_{\ell, \ell', n}^{N+1} := \{(z, w) \in \mathbb{C}^N \times \mathbb{C} : \Im w = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^n |z_j|^2 - \sum_{j=n+1}^{n+\ell'-\ell} |z_j|^2 + \sum_{j=n+\ell'-\ell+1}^N |z_j|^2\}.$$

For simplicity, we write $|z|_{\ell, \ell', n}^2 := \sum_{j=1}^N \delta_{j, \ell, \ell', n} |z_j|^2$ and the corresponding inner product $\langle z, \xi \rangle_{\ell, \ell', n} := \sum_{j=1}^N \delta_{j, \ell, \ell', n} z_j \bar{\xi}_j$ for any $z, \xi \in \mathbb{C}^N$. Here $\delta_{j, \ell, \ell', n}$ is -1 when $1 \leq j \leq \ell$ or $n+1 \leq j \leq n+\ell'-\ell$; 1 when $\ell+1 \leq j \leq n$ or $n+\ell'-\ell+1 \leq j \leq N$. Since $\mathbb{H}_{\ell'}^{N+1}$ is holomorphically equivalent to $\mathbb{H}_{\ell, \ell', n}^{N+1}$ through the linear map

$$\sigma_{\ell, \ell', n}(z^*, w^*) := (z_1^*, \dots, z_\ell^*, z_{\ell'+1}^*, \dots, z_{\ell'+n-\ell}^*, z_{\ell'+1}^*, \dots, z_{\ell'}^*, z_{\ell'+n-\ell+1}^*, \dots, z_N^*, w^*),$$

we obtain $\sigma_{\ell, \ell', n} \circ F$ as a CR immersion from $(\mathbb{H}_\ell^{n+1}, 0)$ into $(\mathbb{H}_{\ell, \ell', n}^{N+1}, 0)$. Without loss of generality, we still denote it by F . It is straightforward to see that the CR tangent vector fields of $\mathbb{H}_{\ell, \ell', n}^{N+1}$ is spanned by

$$L_j = \frac{\partial}{\partial z_j} + 2i\delta_{j, \ell, \ell', n} \bar{z}_j \frac{\partial}{\partial w}, \quad j = 1, \dots, N. \quad (17)$$

Let $T = \frac{\partial}{\partial \Re(w)}$ be the Reeb vector field. It is easy to verify that the commutator

$$[\bar{L}_j, L_k] = 2i\delta_{j, \ell, \ell', n} \delta_{jk} T. \quad (18)$$

To suit our purpose, we need the following notation. Let $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. For a holomorphic function $h(z, w)$, we denote by $h^{(k_1, k_2)}$ the homogenous terms in its power series expansion at 0 whose degrees with respect to z and w are k_1 and k_2 , respectively. Furthermore, we assign 1 to be the weight of z and 2 that of w . $h^{(k)}$ will be used to denote the terms in its power series expansion of weighted degree k and write $o_{wt}(k)$ for terms of weighted degree larger than k .

Write $F = (\tilde{f}, g)$ with \tilde{f} an N -tuple vector-valued holomorphic function and g a holomorphic function. Since $F(0) = 0$, we have

$$\begin{aligned} \tilde{f} &= zA + \mathbf{a}w + O(|(z, w)|^2) \\ g &= \lambda w + O(|(z, w)|^2) \end{aligned} \quad (19)$$

with A an $n \times N$ matrix, \mathbf{a} an $N \times 1$ vector and $\lambda \in \mathbb{R} \setminus \{0\}$ satisfying

$$\lambda E_{\ell,n} = A E_{\ell,\ell',n} \bar{A}^t.$$

Here $E_{\ell,n}$ is the $n \times n$ diagonal matrix whose j th diagonal entry is -1 when $j \leq \ell$; 1 when $j \geq \ell + 1$, $E_{\ell,\ell',n}$ is the $N \times N$ diagonal matrix whose j th diagonal entry is $\delta_{j,\ell,\ell',n}$.

Extending A to an $N \times N$ matrix \tilde{A} by adding $N - n$ rows so that

$$\lambda E_{\ell,\ell',n} = \tilde{A} E_{\ell,\ell',n} \tilde{A}^t. \quad (20)$$

Now compose F from the left by the following two automorphisms of $\mathbb{H}_{\ell,\ell',n}^{N+1}$ mentioned in [BH]:

$$\begin{aligned} F_1(z^*, w^*) &:= (z^* \tilde{A}^{-1}, \frac{1}{\lambda} w^*), \\ F_2(z^*, w^*) &:= \left(\frac{z^* - \mathbf{a} w^*}{\Delta(z^*, w^*)}, \frac{w^*}{\Delta(z^*, w^*)} \right), \end{aligned} \quad (21)$$

where $\Delta(z^*, w^*) := 1 + 2i \sum_j \delta_{j,\ell,\ell',n} z_j^* \bar{\mathbf{a}}_j + (r - i \sum_j \delta_{j,\ell,\ell',n} |z_j^*|^2) w^*$ and $r = \frac{1}{\lambda} \Re(\frac{\partial^2 g}{\partial w^2}(0))$. One can then further obtain a CR embedding F^* at 0 from \mathbb{H}_ℓ^{n+1} into $\mathbb{H}_{\ell,\ell',n}^{N+1}$ in the following normal form:

$$\begin{aligned} F^* &= (f^*, \phi^*, g^*) := F_2 \circ F_1 \circ \sigma_{\ell,\ell',n} \circ F, \quad \text{with} \\ f^*(z, w) &= z + \frac{i}{2} a^{(1,0)}(z) w + o_{wt}(3), \\ \phi^*(z, w) &= \phi^{(2,0)}(z) + o_{wt}(2), \\ g^*(z, w) &= w + o_{wt}(4), \end{aligned} \quad (22)$$

with

$$\langle a^{(1,0)}(z), \bar{z} \rangle_\ell |z|_\ell^2 = |\phi^{(2,0)}(z)|_{\ell'-\ell}^2. \quad (23)$$

Here f^*, ϕ^* are vector valued holomorphic functions of n tuples and $N - n$ tuples, respectively, and g^* is a holomorphic function.

Similar to the definition of geometric rank given by Huang in [Hu3] for CR maps between spheres, we define for CR maps between hyperquadrics based on the above formulation.

Definition 3.1. *Let $F : (\mathbb{H}_\ell^{n+1}, 0) \rightarrow (\mathbb{H}_{\ell'}^{N+1}, 0)$ be a local CR embedding. The geometric rank of F at 0 is defined to be the rank of the matrix $\frac{\partial^2 f^*}{\partial z \partial w}|_0$, denoted by $Rk_F(0)$.*

In the next section, we will show under the above normalization applied onto the CR map, the CR second fundamental form can be expressed in terms of a simpler form. Moreover, if the CR second fundamental form of F vanishes at the origin, we have $Rk_F(0) = 0$.

4 Proof of Theorem 1.1

Under the assumption in Theorem 1.1, let $M' := H(M) \subset \mathbb{H}_{\ell'}^{N+1}$. Then M' is a Levi-nondegenerate hypersurface of signature ℓ with $II_{M'} \equiv 0$. It follows from Lemma 2.2 that

M' is locally CR equivalent to \mathbb{H}_ℓ^{n+1} . Let G be a smooth CR map defined in an open neighborhood U of 0 in \mathbb{H}_ℓ^{n+1} to M' and thus $H \circ G : U \subset \mathbb{H}_\ell^{n+1} \rightarrow \mathbb{H}_{\ell'}^{N+1}$ and we denote the map $H \circ G$ by F .

At any point $p = (z_0, w_0) \in \mathbb{H}_\ell^{n+1}$ near 0, we introduce two translation maps $\sigma_p^0 \in \text{Aut}(\mathbb{H}_\ell^{n+1})$ and $\tau_0^{F(p)} \in \text{Aut}(\mathbb{H}_{\ell'}^{N+1})$ as follows:

$$\begin{aligned} \sigma_p^0(z, w) &= (z + z_0, w + w_0 + 2i \sum_j \delta_{j,\ell} z_j \bar{z}_{0;j}), \\ \tau_0^{F(p)}(z^*, w^*) &= (z^* - \tilde{f}(p), w^* - \overline{g(p)} - 2i \sum_j \delta_{j,\ell,\ell',n} z_j^* \overline{\tilde{f}_j(p)}). \end{aligned} \tag{24}$$

It is easy to verify that $\sigma_p^0(0) = p$ and $\tau_0^{F(p)}(f(p)) = 0$. Composing F by the above two translations from the left and the right, we obtain a new map $F_p := \tau_0^{F(p)} \circ F \circ \sigma_p^0$ sending 0 to 0. Furthermore, we apply the normalization procedure in Section 3 to F_p and get $F_p^* = (f_p^*, \phi_p^*, g_p^*)$ with the normal form (22).

The following lemma reveals the relationship between the CR second fundamental form and the geometric rank of F .

Lemma 4.1. *If $II_{M'}(F(p)) = 0$, then $\frac{\partial^2(\phi_p^*)_\mu}{\partial z_k \partial z_j}|_0 = 0$, $n + 1 \leq \mu \leq N$.*

Proof of Lemma 4.1: Let W be an open set in \mathbb{H}_ℓ^{n+1} containing 0 such that $\sigma_p^0(W) \subset U$. Let $F_p^* = \tau_0^{F(p)} \circ F \circ \sigma_p^0$ be the normalization of F . Hence $F \circ \sigma_p^0(W) \subset M'$. There exists a first order adapted lift $e : M' \rightarrow SU(N + 1, \ell' + 1)$ such that $II_{M'}^{SU,e}(F(p)) = 0$ according to Definition 2.6. Let $e' = \tau_0^{F(p)} \circ (e \circ (\tau_0^{F(p)})^{-1}) \in GL^Q(\mathbb{C}^{N+2})$. It is straightforward to check that $e' : \tau_0^{F(p)} \rightarrow GL^Q(\mathbb{C}^{N+2})$ is a first order adapted lift and $II_{\tau_0^{F(p)}(M')}^{GL,e'} \circ \tau_0^{F(p)} = II_{M'}^{SU,e}$ (cf. [JY]). By the assumption $II_{M'}(F(p)) = 0$, it follows that $II_{\tau_0^{F(p)}(M')}^{GL,e'}(0) = 0$.

On the other hand, we construct the first order adapted lift

$$E = (e_0, e_\alpha, e_\mu, e_{N+1}) : F_p^*(W) \subset \tau_0^{F(p)}(M') \rightarrow SU(N + 1, \ell' + 1)$$

as in [JY], where e_j 's are defined in (14). Since $E|_0 = Id$, we have

$$\omega|_0 = (E^{-1}|_0)(dE)|_0 = dE|_0$$

so that

$$\omega|_0 = \begin{bmatrix} 0 & * & \dots & * \\ dz_1 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ dz_n & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & & \vdots \\ * & * & \dots & * \\ dz_{N+1} & * & \dots & * \end{bmatrix} \Big|_0.$$

Hence $\omega_0^1|_0 = dz_1, \dots, \omega_0^n|_0 = dz_n, \omega_0^{N+1}|_0 = dz_{N+1}$. Applying the chain rule, we obtain

$$\omega_\alpha^\mu|_0 = dE_\alpha^\mu|_0 = d((L_\alpha(\phi_p^*)_\mu))|_0 = \frac{\partial}{\partial z_k}((L_j(\phi_p^*)_\mu))|_0 dz_k = \frac{\partial^2(\phi_p^*)_\mu}{\partial z_k \partial z_j}|_0 \omega_0^k|_0,$$

where $j, k \in \{1, 2, \dots, n, N+1\}$, $n+1 \leq \mu \leq N$ and E_j^μ is the (μ, j) -th entry in E . The second equality holds because

$$dE_j^\mu|_0 = d\left(\frac{L_j(\phi_p^*)_\mu}{\sqrt{|L_j f_p^*|^2 + |L_j \phi_p^*|^2}}\right)|_0 = d(L_j(\phi_p^*)_\mu)|_0$$

by the product rule. Since $II_{\tau_0^{F(p)}(M')}^{GL, e'}(0) = 0$, we have $\frac{\partial^2(\phi_p^*)_\mu}{\partial z_k \partial z_j}|_0 = 0$, $n+1 \leq \mu \leq N$. ■

Remark 4.2. Under the same assumption as in Lemma 4.1, we have $\frac{\partial^2 f_p^*}{\partial z \partial w}|_0 = 0$ by (23). By the definition, this implies the geometric rank $Rk_F(p) = 0$.

The idea of the remaining proof of the rigidity originates from [Hu2] and [BH], where $\ell = \ell' = 0$ in [Hu2] and $\ell = \ell' > 0$ in [BH]. In those cases, the vanishing of geometric rank, which is equivalent to the vanishing of $\frac{\partial^2(\phi_p^*)_\mu}{\partial z_k \partial z_j}|_0 = 0$, implies the linearity of the map. However in our case, since $\ell \neq \ell'$, the geometric rank being zero does not seem to imply the linearity necessarily.

Proposition 4.3. Let $F : (\mathbb{H}_\ell^{n+1}, 0) \rightarrow (\mathbb{H}_{\ell'}^{N+1}, 0)$ be a local smooth CR embedding. If $\frac{\partial^2(\phi_p^*)_\mu}{\partial z_k \partial z_j}|_0 = 0$ for any p near 0 with $n+1 \leq \mu \leq N$, then F is linear fractional.

Proof of Proposition 4.3: At any point $p = (z_0, w_0)$, apply the normalization process onto F_p as in Section 3, one has

$$\frac{\partial^2(\phi_p^*)_\mu}{\partial z_k \partial z_j}|_0 = \left\langle \frac{\partial^2 \tilde{f}_p}{\partial z_j \partial z_k}(0), \overline{\alpha_\mu(p)} \right\rangle_{\ell, \ell', n} = 0, \quad n+1 \leq \mu \leq N, \quad 0 \leq j, k \leq n \quad (25)$$

where $\alpha_\mu(p)$ corresponds to μ -th row of the matrix $\tilde{A}(p)$ in (20).

On the other hand, since F_p is obtained from F via the translation map defined by (24), we get

$$\frac{\partial^2 \tilde{f}_p}{\partial z_j \partial z_k}(0) = L_j L_k \tilde{f}(p)$$

for $p \in \mathbb{H}_\ell^{n+1}$ near 0. Combining the above with (25) together with the construction of \tilde{A} , we get

$$L_j L_k \tilde{f}(p) \in \text{Span}\{L_r(\tilde{f})(p), 1 \leq r \leq n\}.$$

Equivalently,

$$L_j L_k \tilde{f}(p) = \sum_{r=1}^n c_{jk,r}(p) L_r(\tilde{f})(p),$$

for some uniquely determined functions $c_{jk,r}$ depending smoothly on $p \in \mathbb{H}_\ell^{n+1}$ near 0. If one in particular restricts only on the ϕ components of the above expression, then we have

$$L_j L_k \phi(p) = \sum_{r=1}^n c_{jk,r}(p) L_r(\phi)(p). \quad (26)$$

on $p \in \mathbb{H}_\ell^{n+1}$ near 0.

Next similar as in [BH], we apply \bar{L} onto (26) to gather second derivatives of ϕ along $L \oplus T$ and $T \oplus T$ directions. Here we use the fact that ϕ is holomorphic and (18). Indeed, applying \bar{L}_j onto (26) once, one gets

$$T L_k \phi(p) = \sum_r d_{k,r}(p) L_r(\phi)(p) + d(p) T(\phi)(p) \quad (27)$$

for some uniquely determined coefficients $d_{jk,r}, d$ depending smoothly on $p \in \mathbb{H}_\ell^{n+1}$ near 0.

Applying \bar{L}_k again onto (27), one has

$$T^2 \phi(p) = \sum_r e_r(p) L_r(\phi)(p) + e(p) T(\phi)(p) \quad (28)$$

for some uniquely determined coefficients e_r, e depending smoothly on $p \in \mathbb{H}_\ell^{n+1}$ near 0.

Since $L_j, 1 \leq j \leq n$ and T spans CTM , equations (26), (27) and (28) then consists of a complete second order differential system for the holomorphic vector-valued function ϕ in the form of

$$D^2 \phi = C \cdot D\phi$$

with some function-valued matrix C depending smoothly on $p \in \mathbb{H}_\ell^{n+1}$ near 0. Here, $D\phi$ and $D^2\phi$ represent, respectively, all the first and the second partial derivatives of ϕ along \mathbb{H}_ℓ^{n+1} near 0. Moreover, $\phi(0) = D\phi(0) = 0$. One hence immediately gets $\phi \equiv 0$ by the uniqueness of the solutions to the complete system.

Now the map $F = (f, \phi, g) : (\mathbb{H}_\ell^{n+1}, 0) \rightarrow (\mathbb{H}_{\ell'}^{N+1}, 0)$ is reduced to a CR automorphism (f, g) of $(\mathbb{H}_\ell^{n+1}, 0)$ after removing all $\phi(\equiv 0)$ components. It is therefore linear fractional by a classical result of Chern-Moser [CM]. ■

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1: Without loss of generality by Lemma 2.2, we assume M is the hyperquadric \mathbb{H}_ℓ^{n+1} and $F := H \circ G : (\mathbb{H}_\ell^{n+1}, 0) \rightarrow (\mathbb{H}_{\ell',n}^{N+1}, 0)$ is a CR embedding near 0 with identically vanishing CR second fundamental form.

At any point $p = (z_0, w_0) \in \mathbb{H}_\ell^{n+1}$ near 0, we consider the new map $F_p := \tau_{F(p)} \circ F \circ \sigma_p$ sending 0 to 0. Now applying the normalization process discussed in Section 3, we get

$$\begin{aligned} F_p^* &= (f_p^*, \phi_p^*, g_p^*) := G_p \circ H_p \circ \sigma_{\ell, \ell', n} \circ F_p, \quad \text{with} \\ f_p^*(z, w) &= z + \frac{i}{2} a_p^{(1,0)}(z) w + o_{wt}(3), \\ \phi_p^*(z, w) &= \phi_p^{(2,0)}(z) + o_{wt}(2), \\ g_p^*(z, w) &= w + o_{wt}(4), \end{aligned}$$

and

$$\langle a_p^{(1,0)}(z), \bar{z} \rangle_\ell |z|_\ell^2 = |\phi_p^{(2,0)}(z)|_{\ell'-\ell}^2. \quad (29)$$

Since the CR second fundamental form of F at p vanishes, this implies $\phi_p^{(2,0)} = 0$ by Lemma 4.1. Combining this with Proposition 4.3, the proof of Theorem 1.1 is thus complete.

■

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