Classification of Rational Holomorphic Maps from $\mathbb{B}^2$ into $\mathbb{B}^N$ with Degree 2

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1 Introduction

Denote by $Rat(\mathbb{B}^2, \mathbb{B}^N)$ the space of all rational proper holomorphic maps from the unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$ into the unit ball $\mathbb{B}^N \subset \mathbb{C}^N$. We recall that F and $G \in Prop(\mathbb{B}^n, \mathbb{B}^N)$ are said to be *equivalent* if there are automorphisms $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Aut(\mathbb{B}^N)$ such that $F = \tau \circ G \circ \sigma$. In this paper, we study the classification problem for elements in $Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree two. For an element F in $Rat(\mathbb{B}^2, \mathbb{B}^N)$, there is a naturally associated invariant $Rk_F \leq 1$, called the geometric rank of the map. Since F is linear if and only if its geometric rank (for the definition, see §2) $Rk_F = 0$, we only need to consider maps with geometric rank $Rk_F = 1$. By using Cayley transformation $\rho_k : \mathbb{H}^k \to \mathbb{B}^k$ where $\mathbb{H}^k$ is the Siegel upper-half space (see § 2), studying $Rat(\mathbb{B}^2, \mathbb{B}^N)$ is equivalent to studying $Rat(\mathbb{H}^2, \mathbb{H}^N)$.

Making use of results obtained in the previous work [HJX06] [CJX06], we give a complete description for the modular space for maps in $Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree ≤ 2 under the above mentioned equivalence relation. Our main result is the following Theorem 1.1. Notice that when $N = 3$, $Rat(\mathbb{B}^2, \mathbb{B}^3)$ has been classified by Faran ([Fa82]); and when $N = 4$, a complete list of monomial maps in $Rat(\mathbb{B}^2, \mathbb{B}^4)$ has been given by D'Angelo ([DA88]):

Theorem 1.1 *(i) Any nonlinear map in $Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2 is equivalent to a map $(F, 0)$ where $F \in Rat(\mathbb{B}^2, \mathbb{B}^5)$ is of one of the following forms:*
(I): $F = (G_t, 0)$ where $G_t \in Rat(\mathbb{B}^2, \mathbb{B}^4)$ is defined by

$$G_t(z, w) = (z^2, \sqrt{1 + \cos^2 t}\; zw, (\cos t)w^2, (\sin t)w), \quad 0 \leq t < \pi/2. \tag{1}$$

(IIA): $F = (F_\theta, 0)$ where $F_\theta \in Rat(\mathbb{B}^2, \mathbb{B}^4)$ is defined by

$$F_\theta(z, w) = (z, (\cos\theta)w, (\sin\theta)zw, (\sin\theta)w^2), \quad 0 < \theta \leq \frac{\pi}{2}. \tag{2}$$

(IIC): $F = F_{c_1,c_3,e_1,e_2} = \rho_5^{-1} \circ F \circ \rho_2 = (f, \phi_1, \phi_2, \phi_3, g) \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ *is of the form:*

$$f = \frac{z + (\frac{i}{2} + ie_1)zw}{1 + ie_1 w + e_2 w^2}, \ \phi_1 = \frac{z^2}{1 + ie_1 w + e_2 w^2},$$
$$\phi_2 = \frac{c_1 zw}{1 + ie_1 w + e_2 w^2}, \ \phi_3 = \frac{c_3 w^2}{1 + ie_1 w + e_2 w^2}, \ g = \frac{w + ie_1 w^2}{1 + ie_1 w + e_2 w^2},$$

where $c_1, c_3 > 0, -e_1, -e_2 \geq 0,\ e_1 e_2 = c_3^2,\ -e_1 - e_2 = \frac{1}{4} + c_1^2$, *satisfying one of the following conditions: either*

$$\begin{cases} e_1 = \frac{-(\frac{1}{4}+c_1^2)-\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2}, \ e_2 = \frac{-(\frac{1}{4}+c_1^2)+\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2}, \\ 0 < 4c_3^2 \leq (\frac{1}{4} + c_1^2)^2, \end{cases} \tag{3}$$

or

$$\begin{cases} e_1 = \frac{-(\frac{1}{4}+c_1^2)+\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2}, \ e_2 = \frac{-(\frac{1}{4}+c_1^2)-\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2}, \\ \frac{1}{2}c_1^2 + c_1^4 \leq 4c_3^2 \leq (\frac{1}{4} + c_1^2)^2. \end{cases} \tag{4}$$

(ii) Any two maps in $Rat(\mathbb{B}^2, \mathbb{B}^5)$ *in the form of types (I), (IIA), and (IIC) above are equivalent if and only if they are identical.*

Next, we give a review on the development of this problem and outline the proof for Theorem 1.1 as follows. For some notations to be used, we refer the reader to §2.

• **A result obtained in [HJX06]** A classification result was proved in the last section of [HJX06] under the action of the isotropic automorphism groups of the Heisenberg hypersurfaces, which gives in particular the following: Any map F in $Rat(\mathbb{H}^2, \mathbb{H}^N)$ with $deg(F) = 2$ is equivalent to a map $(G, 0)$ where $G = (f, \phi_1, \phi_2, \phi_3, g) \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ is of the form (see also Lemma 2.3 below)

$$\begin{array}{l} f(z,w) = \frac{z-2ibz^2+(\frac{i}{2}+ie_1)zw}{1+ie_1w+e_2w^2-2ibz}, \\ \phi_1(z,w) = \frac{z^2+bzw}{1+ie_1w+e_2w^2-2ibz}, \ \phi_2(z,w) = \frac{c_2w^2+c_1zw}{1+ie_1w+e_2w^2-2ibz}, \\ \phi_3(z,w) = \frac{c_3w^2}{1+ie_1w+e_2w^2-2ibz}, \ g(z,w) = \frac{w+ie_1w^2-2ibzw}{1+ie_1w+e_2w^2-2ibz}, \end{array} \tag{5}$$

where $b, -e_1, -e_2, c_1, c_2, c_3$ are real non-negative numbers satisfying $e_1e_2 = c_2^2 + c_3^2$, $-e_1 - e_2 = \frac{1}{4} + b^2 + c_1^2$, $-be_2 = c_1c_2$, and $c_3 = 0$ if $c_1 = 0$.

Since b and c_2 are determined by c_1, c_3, e_1 and e_2, a map in the form of (5) is determined by c_1, c_3, e_1 and e_2. We denote a map of the form (5) determined by c_1, c_3, e_1 and e_2 to be

$$F_{(c_1,c_3,e_1,e_2)} \in \mathcal{K}. \tag{6}$$

Sometimes we regard a such map $F_{(c_1,c_3,e_1,e_2)}$ as a point: $(c_1, c_3, e_1, e_2) \in \mathcal{K}$. It was unclear in [HJX06] which of the coefficients e_1, e_2, c_1 and c_3 of F are independent parameters.

• **Review of the result in [CJX06]** In [CJX06], by obtaining an extra equation, we got a clearer picture on the maps in (5).

For any $F \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ with $deg(F) = 2$, if the geometric rank of F at the origin is one: $Rk_F(0) = 1$, then by a normalization procedure (see Lemma 2.2 and 2.3 below, or [Hu 03][HJX06]), F is equivalent to another map $F^{***} \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ of the form (5). Also we can associate a family of maps $F_p \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ for any $p \in \partial\mathbb{H}^2$ (see § 2 below). Let us define $\Xi_F := \{p \in \partial\mathbb{H}^2 \mid Rk_{F_p}(0) = 0\}$ to be the set of p at which the geometric rank of F_p at the origin is zero. If $p \notin \Xi_F$, we obtain a normalized map $(F_p)^{***}$ that is of the form (5), and we define a real analytic function $\mathcal{W}(F_p^{***}) = c_1(p)^2 - e_1(p) - e_2(p)$ where $c_1(p), e_1(p)$ and $e_2(p)$ are the coefficients of F_p^{***} as in (5).

The desired extra equation is obtained by moving up p to the extremal value as follows. We choose a sequence of $p_m \in \partial\mathbb{H}^2 - \Xi_F$ such that $Rk_{F_{pm}}(0) = 1$, $p_m \to p_0 \in \overline{\partial\mathbb{H}^2}$ and $\lim_m \mathcal{W}(F_{p_m}^{***}) = \inf_{p\in\partial\mathbb{H}^2-\Xi_F}\{\mathcal{W}(F_p^{***})\}$.

If $p_0 \in \partial\mathbb{H}^2$, by [CJX06, § 4], we can write

$$F_{p_m}^{***} = (F_{p_0})_{q_m}^{***} \tag{7}$$

where $q_m \in \partial\mathbb{H}^2$ and $q_m \to 0$. Then it implies by [CJX06, Lemma 2.5] that $Rk_{F_{p_0}}(0) = 1$, and that F is equivalent to $F_{p_0}^{***}$ which is of the form (5) and with the minimum property $\mathcal{W}(F_{p_0}^{***}) = \inf_{p\in\partial\mathbb{H}^2-\Xi_F} \mathcal{W}(F_p^{***})$. The minimum property implies the vanishing of derivatives of the function $\mathcal{W}(F_p^{***})$ at p_0, which derives the extra equation.

If $p_0 = \infty$, by [CJX06, § 4] we can similarly write

$$F_{p_m}^{***} = (\tau_\infty \circ F \circ \sigma_\infty)_{q_m}^{***} \tag{8}$$

where $\sigma_\infty \in Aut(\partial\mathbb{B}^2)$, $\tau_\infty \in Aut(\partial\mathbb{B}^5)$, $q_m \in \partial\mathbb{H}^2$ and $q_m \to 0$ so that, by the same argument above, $Rk_{\tau_\infty\circ F\circ\sigma_\infty}(0) = 1$ and that F is equivalent to $(\tau_\infty \circ F \circ \sigma_\infty)^{***}$ which is of the form (5). The minimum property also derives the extra equation.

With the extra equation described above, it was proved in [CJX06] that F is equivalent to another map $F_{c_1,c_3.e_1,e_2} \in \mathcal{K}$ satisfying the property

$$\mathcal{W}\big((F_{c_1,c_3.e_1,e_2})_p^{***}\big) \geq \mathcal{W}\big((F_{c_1,c_3.e_1,e_2})_0^{***}\big),\ \forall p \in \partial\mathbb{H}^2 \textit{ near } 0. \tag{9}$$

and that the new map $F_{c_1,c_3.e_1,e_2}$ is of the form in one of the following types:

(I) $F_{0,0,e_1,e_2} = (f, \phi_1, \phi_2, \phi_3, g)$ is of the form

$$\begin{array}{l} f = \frac{z+(\frac{i}{2}+ie_1)zw}{1+ie_1w+e_2w^2},\ \phi_1 = \frac{z^2}{1+ie_1w+e_2w^2}, \\ \phi_2 = \frac{c_2w^2}{1+ie_1w+e_2w^2},\ \phi_3 = 0,\ g = \frac{w+ie_1w^2}{1+ie_1w+e_2w^2} \end{array} \tag{10}$$

where $e_1e_2 = c_2^2$ and $-e_1 - e_2 = \frac{1}{4}$. Here $e_2 \in [-\frac{1}{4}, 0)$ is a parameter. It then corresponds to the family $\{G_t\}_{0\le t<\pi/2}$ in (1). When $e_2 = -\frac{1}{4}$, $F_{0,0,e_1,e_2}$ corresponds to G_0, i.e. $(z,w) \mapsto (z^2, \sqrt{2}zw, w^2, 0)$; when $e_2 \to 0$, $F_{0,0,e_1,e_2}$ goes to $G_{\pi/2} = F_{\pi/2}$, i.e., $(Z,w) \mapsto (z, zw, w^2)$.

(IIA) $F_{c_1,0,e_1,0} = (f, \phi_1, \phi_2, \phi_3, g)$ is of the form

$$f = \frac{z + (\frac{i}{2} + ie_1)zw}{1 + ie_1w},\ \phi_1 = \frac{z^2}{1+ie_1w},\ \phi_2 = \frac{c_1zw}{1+ie_1w},\ \phi_3 = 0,\ g = w \tag{11}$$

where $-e_1 = \frac{1}{4} + c_1^2$ and $c_1 \in [0,\infty)$ is a parameter. It corresponds to the family $\{F_\theta\}_{0<\theta\le\pi/2}$ in (2). When $c_1 = 0$, $F_{c_1,0,e_1,0}$ corresponds to $F_{\pi/2}$; when $c_1 \to \infty$, $F_{c_1,0,e_1,0}$ goes to the linear map, i.e., $(z,w) \mapsto (z,w,0)$.

(IIB) $F_{c_1,0,0,e_2} = (f, \phi_1, \phi_2, \phi_3, g)$ is of the form:

$$f = \frac{z + \frac{i}{2}zw}{1+e_2w^2},\ \phi_1 = \frac{z^2}{1+e_2w^2},\ \phi_2 = \frac{c_1zw}{1+e_2w^2},\ \phi_3 = 0,\ g = \frac{w}{1+e_2w^2}, \tag{12}$$

where $-e_2 = \frac{1}{4} + c_1^2$ and $c_1 \in (0,\infty)$ is a parameter. Notice that when $c_1 \to 0$, the map $F_{c_1,0,0,e_2}$ goes to the map G_0, i.e. the one in type (I) when $e_2 = -\frac{1}{4}$.

(IIC) $F_{c_1,c_3,e_1,e_2} = (f, \phi_1, \phi_2, \phi_3, g)$ is of the form:

$$\begin{array}{l} f = \frac{z+(\frac{i}{2}+ie_1)zw}{1+ie_1w+e_2w^2},\ \phi_1 = \frac{z^2}{1+ie_1w+e_2w^2}, \\ \phi_2 = \frac{c_1zw}{1+ie_1w+e_2w^2},\ \phi_3 = \frac{c_3w^2}{1+ie_1w+e_2w^2},\ g = \frac{w+ie_1w^2}{1+ie_1w+e_2w^2}, \end{array} \tag{13}$$

where $c_1, c_3 > 0, -e_1, -e_2 \ge 0,\quad e_1e_2 = c_3^2,\quad -e_1 - e_2 = \frac{1}{4} + c_1^2$.

For any map F_{c_1,c_3,e_1,e_2} in one of these four types, we denote F_{c_1,c_3,e_1,e_2}, or (c_1, c_3, e_1, e_2), $\in \mathcal{K}_I$, $\mathcal{K}_{IIA}$, $\mathcal{K}_{IIB}$, and $\mathcal{K}_{IIC}$, respectively.

Recall from (33) [CJX06]

$$F \textit{ can be embedded into } \mathbb{H}^4 \ \Leftrightarrow\ c_3 = 0. \tag{14}$$

Concerning the proof of Theorem 1.1, our main idea to establish following formula (see (33)):

$$\mathcal{W}(F^{***}_{\Gamma(t+\Delta t)}) = \mathcal{W}(F^{***}_{\Gamma(t)}) + [4c_1(bc_1 + 2c_2) - 8b(e_1+e_2)](\Gamma(t))\Im(q_1(t))\Delta t + o(|\Delta t|). \tag{15}$$

One crucial point is that the term $[4c_1(bc_1 + 2c_2) - 8b(e_1+e_2)](\Gamma(t))$ is always non-negative so that it allows us to reduce the study of (9) into the study for the term $\Im(q_1(t))$.

We'll prove in Lemma 3.4 below that indeed

$$\textit{there is no map } F \textit{ satisfying both } (9) \textit{ and } (12), \tag{16}$$

and that a map

$$F\ satisfies\ (9)\ and\ (13) \quad \Leftrightarrow \quad F\ satisfies\ (13),\ (3)\ and\ (4), \tag{17}$$

which proves Theorem 1.1(i). To prove Theorem 1.1(ii), we first prove its local version (see Corollary 4.3). Then we shall find a way to reduce the global problem into the local one.

2 Notation and preliminaries

• **Maps between balls** Write $\mathbb{H}^n := \{(z,w) \in \mathbf{C}^{n-1} \times \mathbf{C} : \ \mathrm{Im}(w) > |z|^2\}$ for the Siegel upper-half space. Similarly, we can define the space $\mathrm{Rat}(\mathbb{H}^n, \mathbb{H}^N)$, $\mathrm{Prop}_k(\mathbb{H}^n, \mathbb{H}^N)$ and $\mathrm{Prop}(\mathbb{H}^n, \mathbb{H}^N)$ respectively. Since the Cayley transformation

$$\rho_n : \mathbb{H}^n \to \mathbb{B}^n, \quad \rho_n(z,w) = \left(\frac{2z}{1-iw}, \frac{1+iw}{1-iw}\right)$$

is a biholomorphic mapping between $\mathbb{H}^n$ and $\mathbb{B}^n$, we can identify a map $F \in \mathrm{Prop}_k(\mathbb{B}^n, \mathbb{B}^N)$ or $\mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ with $\rho_N^{-1} \circ F \circ \rho_n$ in the space $\mathrm{Prop}_k(\mathbb{H}^n, \mathbb{H}^N)$ or $\mathrm{Rat}(\mathbb{H}^n, \mathbb{H}^N)$, respectively.

Parametrize $\partial\mathbb{H}^n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i|z|^2)$. In what follows, we will assign the weight of z and u to be 1 and 2, respectively. For a non-negative integer m, a function $h(z, \overline{z}, u)$ defined over a small ball U of 0 in $\partial\mathbb{H}^n$ is said to be of quantity $o_{wt}(m)$ if $\frac{h(tz, t\overline{z}, t^2 u)}{|t|^m} \to 0$ uniformly for (z,u) on any compact subset of U as $t (\in \mathbb{R}) \to 0$.

• **Partial normalization of** F Let $F = (f, \phi, g) = (\widetilde{f}, g) = (f_1, \cdots, f_{n-1}, \phi_1, \cdots, \phi_{N-n}, g)$ be a non-constant C^2-smooth CR map from $\partial\mathbb{H}^n$ into $\partial\mathbb{H}^N$ with $F(0) = 0$. For each $p \in \partial\mathbb{H}^2$, we write $\sigma_p^0 \in \mathrm{Aut}(\mathbb{H}^n)$ and $\tau_p^F \in \mathrm{Aut}(\mathbb{H}^N)$ for the maps

$$\begin{aligned} &\sigma_p^0(z,w) = (z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle), \\ &\tau_p^F(z^*, w^*) = (z^* - \widetilde{f}(z_0,w_0), w^* - \overline{g(z_0,w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0,w_0)}\rangle). \end{aligned} \tag{18}$$

F is equivalent to $F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p)$. Notice that $F_0 = F$ and $F_p(0) = 0$. The following is basic for the understanding of the geometric properties of F.

Lemma 2.1 *([§2, Lemma 5.3, Hu99], [Lemma 2.0, Hu03]): Let F be a C^2-smooth CR map from $\partial\mathbb{H}^n$ into $\partial\mathbb{H}^N$, $2 \le n \le N$ with $F(0) = 0$. For each $p \in \partial\mathbb{H}^n$, there is an automorphism $\tau_p^{**} \in Aut_0(\mathbb{H}^N)$ such that $F_p^{**} := \tau_p^{**} \circ F_p$ satisfies the following normalization:*

$$f_p^{**} = z + \frac{i}{2} a_p^{**(1)}(z) w + o_{wt}(3), \ \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2), \ g_p^{**} = w + o_{wt}(4), \tag{19}$$

$$\langle \overline{z}, a_p^{**(1)}(z)\rangle |z|^2 = |\phi_p^{**(2)}(z)|^2.$$

Let $\mathcal{A}(p) = -2i(\frac{\partial^2 (f_p)_l^{**}}{\partial z_j \partial w}|_0)_{1\le j,l\le(n-1)}$. We call the rank of $\mathcal{A}(p)$, which we denote by $Rk_F(p)$, the *geometric rank* of F at p. $Rk_F(p)$ depends only on p and F, and is a lower semi-continuous function on p. We define the *geometric rank* of F to be $Rk_F := max_{p\in\partial\mathbb{H}^n} Rk_F(p)$. Notice that we always have $0 \le Rk_F \le n-1$. We define the geometric rank of $F \in \mathrm{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$ to be the one for the map $\rho_N^{-1}\circ F\circ \rho_n \in \mathrm{Prop}_2(\mathbb{H}^n, \mathbb{H}^N)$. It is proved that F is linear fractional if and only if the geometric rank $Rk_F = 0$ ([Theorem 4.3, Hu99]). Hence, in all that follows, we assume that $Rk_F = \kappa_0 \ge 1$.

Denote by $\mathcal{S}_0 = \{(j,l) : 1 \le j \le \kappa_0, 1 \le l \le (n-1), j \le l\}$ and write $\mathcal{S} := \{(j,l) : (j,l) \in \mathcal{S}_0, \text{ or } j = \kappa_0+1, l \in \{\kappa_0+1, \cdots, \kappa_0 + N - n - \frac{(2n-\kappa_0-1)\kappa_0}{2}\}\}$. Then we further have the following normalization for F:

Lemma 2.2 *([Lemma 3.2, Hu03]): Let F be a C^2-smooth CR map from an open piece $M \subset \partial\mathbb{H}^n$ into $\partial\mathbb{H}^N$ with $F(0) = 0$ and $Rk_F(0) = \kappa_0$. Let $P(n,\kappa_0) = \frac{\kappa_0(2n-\kappa_0-1)}{2}$. Then $N \ge n + P(n,\kappa_0)$ and there are $\sigma \in Aut_0(\partial\mathbb{H}^n)$ and $\tau \in Aut_0(\partial\mathbb{H}^N)$ such that $F_p^{***} = \tau\circ F\circ\sigma := (f,\phi,g)$ satisfies the following normalization conditions:*

$$\begin{cases} f_j = z_j + \dfrac{i\mu_j}{2} z_j w + o_{wt}(3), \quad \dfrac{\partial^2 f_j}{\partial w^2}(0) = 0,\ j = 1\cdots,\kappa_0,\ \mu_j > 0, \\ f_j = z_j + o_{wt}(3), \quad j = \kappa_0+1,\cdots,n-1, \\ g = w + o_{wt}(4), \\ \phi_{jl} = \mu_{jl} z_j z_l + o_{wt}(2),\ \textit{where}\ \ (j,l)\in\mathcal{S}\ \textit{with}\ \ \mu_{jl} > 0\ \textit{for}\ \ (j,l)\in\mathcal{S}_0 \\ \qquad \textit{and}\ \mu_{jl} = 0\ \textit{otherwise}. \end{cases} \tag{20}$$

Moreover $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ for $j,l \le \kappa_0$ $j \ne l$, $\mu_{jl} = \sqrt{\mu_j}$ if $j \le \kappa_0$ and $l > \kappa_0$ or if $j = l \le \kappa_0$.

Here we denote $Aut_0(\partial\mathbb{H}^n) = \{\psi \in Aut(\partial\mathbb{H}^n) \mid \psi(0) = 0\}$.

• **Degree of a rational map** For a rational holomorphic map $H = \frac{(P_1,\dots,P_m)}{Q}$ over $\mathbb{C}^n$, where P_j, Q are holomorphic polynomials and $(P_1,\dots,P_m,Q) = 1$, we define

$$deg(H) = max\{deg(P_j),\ 1 \le j \le m,\ deg(Q)\}.$$

For a rational map H and a complex affine subspace S of dimension k, we say that H is linear fractional along S, if S is not contained in the singular set of H and for any linear parametrization $z_j = z_j^0 + \sum_{l=1}^k a_{jl} t_l$ of S with $j = 1,\cdots,n$, $H^*(t_1,\cdots,t_k) := H(z_1^0 + \sum_{l=1}^k a_{1l} t_l, \cdots, z_n^0 + \sum_{l=1}^k a_{jn} t_j)$ has degree 1 in $(t_1,\cdots,t_k)$.

• **Actions of the isotropic groups of the Heisenberg hypersurfaces** Recall from [(2.4.1), Hu03] and [(2.4.2), Hu03], we define $\sigma \in Aut_0(\partial\mathbb{H}^2)$ and $\tau^* \in Aut_0(\partial\mathbb{H}^5)$ by

$$\sigma = \frac{(\lambda(z + aw) \cdot U,\ \lambda^2 w)}{q(z, w)}, \quad \tau^*(z^*, w^*) = \frac{(\lambda^*(z^* + a^* w^*) \cdot U^*, \lambda^{*2} w^*)}{q^*(z^*, w^*)}, \tag{21}$$

with $q(z, w) = 1 - 2i\langle\overline{a}, z\rangle + (r - i|a|^2)w$, $\lambda > 0$, $r \in \mathbb{R}$, $a, U \in \mathbb{C}$, $|U| = 1$, and $q^*(z^*, w^*) = 1 - 2i\langle\overline{a^*}, z^*\rangle + (r^* - i|a^*|^2)w^*$, $\lambda^* > 0$, $r^* \in \mathbb{R}$, $a^* = (a_1^*, a_2^*) \in \mathbb{C}^1 \times \mathbb{C}^3$ and U^* is an 4×4 unitary matrix, such that [((2.5.1), (2.5.2), Hu03] holds:

$$\lambda^* = \lambda^{-1},\ a_1^* = -\lambda^{-1} aU,\ a_2^* = 0,\ r^* = -\lambda^{-2} r,\ U^* = \begin{pmatrix} U^{-1} & 0 \\ 0 & U_{22}^* \end{pmatrix}, \tag{22}$$

where $a^* = (a_1^*, a_2^*)$, U_{22}^* is an 3×3 unitary matrix. Define $F^* = \tau^* \circ F \circ \sigma$. By [Lemma 2.3(A), Hu03], we can write

$$\begin{aligned} &f(z, w) = z + \tfrac{i}{2} zAw + o_{wt}(3),\ f^*(z, w) = z + \tfrac{i}{2} zA^* w + o_{wt}(3), \\ &\phi(z, w) = \tfrac{1}{2} z(B^1, B^2, B^3)z + z\mathcal{B}w + \tfrac{1}{2}\tfrac{\partial^2\phi}{\partial w^2}(0)w^2 + o(|(z, w)|^2), \\ &\phi^*(z, w) = \tfrac{1}{2} z(B^{*1}, B^{*2}, B^{*3})z + z\mathcal{B}^* w + \tfrac{1}{2}\tfrac{\partial^2\phi^*}{\partial w^2}(0)w^2 + o(|(z, w)|^2), \end{aligned} \tag{23}$$

where $B^i = \frac{\partial^2\phi_i}{\partial z^2}(0)$, $B^{*i} = \frac{\partial^2\phi_i^*}{\partial z^2}(0)$ for $i = 1, 2, 3$ and $\mathcal{B} = (\frac{\partial^2\phi_1}{\partial z\partial w}, \frac{\partial^2\phi_2}{\partial z\partial w}, \frac{\partial^2\phi_3}{\partial z\partial w})$, $\mathcal{B}^* = (\frac{\partial^2\phi_1^*}{\partial z\partial w}, \frac{\partial^2\phi_2^*}{\partial z\partial w}, \frac{\partial^2\phi_3^*}{\partial z\partial w})$. Also, the same computation in [Hu03, Lemma 2.3 (A)] gives the following:

$$\begin{aligned} &\tfrac{\partial^2 g^*}{\partial z^2}(0) = 0,\ \tfrac{\partial^2 g^*}{\partial z\partial w}(0) = 0,\ \tfrac{\partial^2 g^*}{\partial w^2}(0) = 0,\ \tfrac{\partial^2 f^*}{\partial z^2}(0) = 0,\ \mathcal{A}^* = \lambda^2 U\mathcal{A}U^{-1}, \\ &\tfrac{\partial^2 f^*}{\partial w^2}(0) = i\lambda^2 aU\mathcal{A}U^{-1} + \lambda^3\tfrac{\partial^2 f}{\partial w^2}(0)U^{-1}, \\ &[B^{*1},\ B^{*2}, B^{*3}] = \lambda U[B^1,\ B^2,\ B^3]U^t U_{22}^*, \\ &\mathcal{B}^* = \lambda U[B^1,\ B^2,\ B^3]U^t a^t U_{22}^* + \lambda^2 U\mathcal{B}U_{22}^*, \\ &\tfrac{\partial^2\phi^*}{\partial w^2}(0) = \lambda aU[B^1,\ B^2,\ B^3]U^t a^t U_{22}^* + 2\lambda^2 aU\mathcal{B}U_{22}^* + \lambda^3\tfrac{\partial^2\phi}{\partial w^2}(0)U_{22}^*. \end{aligned} \tag{24}$$

Lemma 2.3 *([HXJ06, theorem 4.1]) Let $F \in Rat(\partial\mathbb{H}^2, \partial\mathbb{H}^N)$ have degree 2 with $F(0) = 0$ and $Rk_F(0) = 1$ ($N \geq 4$). Then*

(1) F is equivalent to $(F^{***}, 0)$ where $F^{***} = (f, \phi_1, \phi_2, \phi_3, g) \in Rat(\partial\mathbb{H}^2, \partial\mathbb{H}^5)$ defined by

$$\begin{aligned} f(z, w) &= \tfrac{z - 2ibz^2 + (\frac{i}{2} + ie_1)zw}{1 + ie_1 w + e_2 w^2 - 2ibz}, \\ \phi_1(z, w) &= \tfrac{z^2 + bzw}{1 + ie_1 w + e_2 w^2 - 2ibz}, \\ \phi_2(z, w) &= \tfrac{c_2 w^2 + c_1 zw}{1 + ie_1 w + e_2 w^2 - 2ibz}, \\ \phi_3(z, w) &= \tfrac{c_3 w^2}{1 + ie_1 w + e_2 w^2 - 2ibz}, \\ g(z, w) &= \tfrac{w + ie_1 w^2 - 2ibzw}{1 + ie_1 w + e_2 w^2 - 2ibz}. \end{aligned} \tag{25}$$

Here $b, -e_1, -e_2, c_1, c_2, c_3$ are real non-negative numbers satisfying

$$e_1e_2 = c_2^2 + c_3^2,\ -e_1 - e_2 = \frac{1}{4} + b^2 + c_1^2,\ -be_2 = c_1c_2,\ c_3 = 0\ if\ c_1 = 0. \tag{26}$$

(2) $c_1, c_2, c_3, e_1, e_2, b$ are uniquely determined by F. Conversely, for any non-negative real numbers $c_1, c_2, c_3, e_1, e_2, b$ satisfying the relations in (26), the map F defined in (25) is an element in $Rat(\partial\mathbb{H}^2, \partial\mathbb{H}^5)$ of degree 2 with $F(0) = 0$ and $Rk_F(0) = 1$.

(3) If $e_2 = 0$, then F is equivalent to $(F_\theta, 0)$ with F_θ as in (1).

Remarks (i) The new normalized map in Lemma 2.3(1) can be obtained by $F^{***} = \tau^* \circ F^{**} \circ \sigma$ where F^{**} is as in Lemma 2.2 and σ and τ^* are as in (21).
(ii) For any map F in Lemma 2.3(1), $b = \sqrt{-e_1 - e_2 - \frac{1}{4} - c_1^2}$ and $c_2 = \sqrt{e_1e_2 - c_3^2}$ are determined by c_1, c_3, e_1 and e_2. Then c_1, c_3, e_1 and e_2 can be regarded as parameters, and we denote $F = F_{c_1,c_3,e_1,e_2}$.
(iii) We denote by $\mathcal{K}$ a subset of $\mathbb{R}^4$ such that (c_1, c_3, e_1, e_2), or $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}$ if and only if F_{c_1,c_3,e_1,e_2} is a map as above.

Lemma 2.4 *([CJX06, Lemma 2.5]) Let $F \in Rat(\partial\mathbb{H}^2, \partial\mathbb{H}^5)$ with $F(0) = 0$ and $deg(F) = 2$. Suppose that $p_m \in \partial\mathbb{H}^2$ is a sequence converging to $0 \in \partial\mathbb{H}^2$ and F_{p_m} is of rank 1 at 0 for any m and $F^{***}_{p_m}$ converges such that $\frac{\partial^2\phi^{***}_{1,m}}{\partial z\partial w}|_0$, $\frac{\partial^2\phi^{***}_{2,m}}{\partial w^2}|_0$, $\frac{\partial^2\phi^{***}_{2,m}}{\partial z\partial w}|_0$ and $\frac{\partial^2\phi^{***}_{3,m}}{\partial w^2}|_0$ are bounded for all m. Then*

(i) F is of rank 1 at 0.

(ii) $F^{***}_{p_m} \to F^{***}$.

(iii) If we write $F^{***}_{p_m} = G_{2,m} \circ \tau_{p_m} \circ F \circ \sigma_{p_m} \circ G_{1,m}$ where σ_{p_m} and $\tau_{p_m} := \tau^F_{p_m}$ are as in (18), $G_{1,m}$ and $G_{2,m}$ are as in (21), then $G_{1,m}$ and $G_{2,m}$ are convergent to some $G_1 \in Aut_0(\partial\mathbb{H}^2)$ and $G_2 \in Aut_0(\partial\mathbb{H}^5)$ respectively.

Let F be as in Lemma 2.3 (1). By Lemma 2.3, F_p is equivalent to a map of the following form $F^{***}_p = (f^{***}_p, \phi^{***}_{1,p}, \phi^{***}_{2,p}, \phi^{***}_{3,p}, g^{***}_p)$ for any $p \in \partial\mathbb{H}^2$ where $Rk_F(p) = 1$:

$$f^{***}_p(z, w) = \frac{z - 2ib(p)z^2 + (\frac{i}{2} + ie_1(p))zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},$$

$$\phi^{***}_{1,p}(z, w) = \frac{z^2 + b(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},$$

$$\phi^{***}_{2,p}(z, w) = \frac{c_2(p)w^2 + c_1(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},$$

$$\phi^{***}_{3,p}(z,w) = \frac{c_3(p)w^2}{1+ie_1(p)w+e_2(p)w^2-2ib(p)z},$$
$$g^{***}_p(z,w) = \frac{w+ie_1(p)w^2-2ib(p)zw}{1+ie_1(p)w+e_2(p)w^2-2ib(p)z}.$$

Here $b(p), e_1(p), e_2(p), c_1(p), c_2(p), c_3(p)$ satisfy $e_2(p)e_1(p) = c_2^2(p)+c_3^2(p)$, $-e_2(p) = \frac{1}{4}+e_1(p)+b^2(p)+c_1^2(p)$, and $-b(p)e_2(p) = c_1(p)c_2(p)$, $c_3(p) = 0$ if $c_1(p) = 0$, with $c_1(p), c_2(p), b(p) \geq 0$, $e_2(p), e_1(p) \leq 0$.

Lemma 2.5 *Let F and F^{***}_p be as above. Let $p = (z_0, w_0) = (z_0, u_0 + i|z_0|^2) \in \partial\mathbb{H}^2$ near 0. Then the followings hold.*

(i) The real analytic functions have the formulas

$$\begin{aligned}
&b^2(p) = b^2 - 4b(2e_1 + c_1^2)\Im(z_0) + o(1),\\
&c_1^2(p) = c_1^2 + 4c_1(bc_1 + 2c_2)\Im(z_0) + o(1),\\
&e_2(p) + e_1(p) = e_2 + e_1 + 8b(e_1 + e_2)\Im(z_0) + o(1),\\
&c_1^2(p) - e_1(p) - e_2(p) = c_1^2 - e_1 - e_2 + \Big(4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)\Big)\Im(z_0) + o(1)
\end{aligned}$$

where we denote $o(k) = o(|(z_0, u_0)|^k)$.

(ii) If $c_1 > 0$, the real analytic function has the formula

$$c_3^2(p) = c_3^2 + 4(c_3)^2(5b - \frac{2c_2}{c_1})\Im(z_0) + o(1),$$

(iii) If $c_1 = 0$, then $c_3(p) \equiv 0$.

Proof: (1) All these formulas were proved in [CJX06, lemma 3.1].

(ii) We use the formulas in [CJX06, Step 3 and 4, § 5] and the notation to obtain

$$c_3^2 = \left|\frac{1}{2}\frac{\partial^2 \phi^{***}_{p3}}{\partial w^2}(0)\right|^2 = \left|\frac{1}{2}\frac{\partial^2 \phi^{**}_{pe3}}{\partial w^2}(0)\right|^2 = c_3^2 + 4(c_3)^2(5b - \frac{2c_2}{c_1})\Im(z_0) + o(1).$$

(iii) If $c_1 = 0$, by Lemma 2.3, $c_3 = 0$ and $F \in Rat(\mathbb{H}^2, \mathbb{H}^4)$. We modify slightly on the normalization F^{***} so that $\phi^{***}_{p3} \equiv 0$ and hence $c_3(p) \equiv 0$. □

3 A Monotone Lemma

Recall that for any $(c_1, c_3, e_1, e_2) \in \mathcal{K}$, we denote

- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_I$ (i.e. F_{c_1,c_3,e_1,e_2} is of the form of type (I)) if $c_1 = 0$ and $b = 0$;
- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{II}$ (i.e. F_{c_1,c_3,e_1,e_2} is of the form of type (II)) if $c_1 > 0$ and $b = c_2 = 0$.

Also recall that for any map $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{II}$, we denote

- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{IIA}$ (i.e. F_{c_1,c_3,e_1,e_2} is of the form of type (IIA)) if $c_1 > 0$, $b = c_2 = 0$ and $c_3 = e_2 = 0$;
- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{IIB}$ (i.e. F_{c_1,c_3,e_1,e_2} is of the form of type (IIB)) if $c_1 > 0$, $b = c_2 = 0$ and $c_3 = e_1 = 0$;
- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{IIC}$ (i.e. F_{c_1,c_3,e_1,e_2} is of the form of type (IIC)) if $c_1 > 0$, $b = c_2 = 0$ and $c_3 > 0$.

For any $(c_1, c_3, e_1, e_2) \in \mathcal{K}_I \cup \mathcal{K}_{II}$, we denote

- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2>0}$, if $1 + 4e_2 + 2c_1^2 > 0$;
- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2=0}$, if $1 + 4e_2 + 2c_1^2 = 0$;
- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$, if $1 + 4e_2 + 2c_1^2 < 0$.

For any $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}$, we define $\mathcal{W}(F_{c_1,c_3,e_1,e_2}) := \mathcal{W}(c_1, c_3, e_1, e_2) := c_1^2 - e_1 - e_2$. We also consider curves

$$\Gamma(t) = (\alpha t, \beta_1 t + i|\alpha|^2 t^2) \in \partial\mathbb{H}^2, \quad \forall t \in [0, 1], \quad |\alpha| \le 1 \textit{ and } |\beta_1| \le 1 \tag{27}$$

where $\alpha = \alpha_1 + i\alpha_2, \alpha_j, \beta_1$ are real numbers.

Lemma 3.1 *Let Γ be any curve as in (27).*

(a) If $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2>0}$, then there exists $\delta = \delta(\Gamma) > 0$ such that

$$\mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_{\Gamma(t_1)}) \le \mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_{\Gamma(t_2)}), \ \forall\ 0 \le t_1 < t_2 \le \delta. \tag{28}$$

(b) If $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2=0}$, then there exists $\delta = \delta(\Gamma) > 0$ such that

$$\mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_{\Gamma(t)}) \equiv constant, \ \forall t. \tag{29}$$

(c) If $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$, then there exists $\delta = \delta(\Gamma) > 0$ such that

$$\mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_{\Gamma(t_1)}) \ge \mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_{\Gamma(t_2)}), \ \forall\ 0 \le t_1 < t_2 \le \delta. \tag{30}$$

Proof of Lemma 3.1: **Step a. The basic setup** The monotonicity (28) in (a) means

$$\frac{d\mathcal{W}(F^{***}_{\Gamma(t)}))}{dt} = \lim_{\Delta t\to 0}\frac{\mathcal{W}(F^{***}_{\Gamma(t+\Delta t)}) - \mathcal{W}(F^{***}_{\Gamma(t)})}{\Delta t} \geq 0,\ \forall t \in [0,\delta]. \tag{31}$$

For any $0 < t < \delta$ and sufficiently small $\Delta t > 0$, if we can write

$$F^{***}_{\Gamma(t+\Delta t)} = \left(F^{***}_{\Gamma(t)}\right)^{***}_{q(t,\Delta t)} \tag{32}$$

for some differentiable map $q(t, \Delta t) \in \partial\mathbb{H}^2$, then from Lemma 2.5 we should have

$$\mathcal{W}(F^{***}_{\Gamma(t+\Delta t)}) = \mathcal{W}(F^{***}_{\Gamma(t)}) + \left[4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)\right](\Gamma(t))\Im(q_1(t))\Delta t + o(|\Delta t|), \tag{33}$$

where we write $q(t,\Delta t) := (q_1(t), q_2(t))\Delta t + o(|\Delta t|)$. Notice that $[4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)](\Gamma(t)) \geq 0$ always holds because $c_1, c_2, -e_1 - e_2 \geq 0$. Then (31) follows if $\Im(q_1(t)) \geq 0$ holds. In particular, if $[4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)](\Gamma(t)) \neq 0$ for any fixed $t \in [0,\delta)$, and if the following condition is satisfied:

$$\Im(q_1(t)) > 0, \qquad \forall t \in [0,\delta], \tag{34}$$

then the strict inequality (31) holds. To prove (31), it suffices to prove (34).

Step b. $\Gamma(t)$ **determines** $q(t,\Delta t)$ To prove (32), we define $q(t,\Delta t)$ by

$$\Gamma(t+\Delta t) = \sigma_{\Gamma(t)} \circ G_1(q(t,\Delta t)) \tag{35}$$

where $G_1 = G_1(t) \in Aut_0(\partial\mathbb{H}^2)$ and $G_2 \in Aut_0(\partial\mathbb{H}^5)$ are defined such that

$$(F_{\Gamma(t)})^{***} = G_2 \circ \tau^F_{\Gamma(t)} \circ F \circ \sigma_{\Gamma(t)} \circ G_1. \tag{36}$$

By (35), $q(t,\Delta t)$ is a function uniquely determined by $\Gamma(t)$ given by

$$q(t,\Delta t) = G_1^{-1} \circ \sigma^{-1}_{\Gamma(t)} \circ \Gamma(t+\Delta t). \tag{37}$$

The definition (37) will be justified in Step c. Here we derive a formula (39).

By the definition of σ (see (18)),

$$\sigma^{-1}_{\Gamma(t)}(z,w) = (z - z(t), w - w(t) - 2i\langle z, \overline{z(t)}\rangle + 2i|z(t)|^2),$$

and

$$\Gamma(t+\Delta t) = \Big(\alpha(t+\Delta t), \beta_1(t+\Delta t) + i|\alpha|^2(t^2+2t\Delta t+\Delta t^2)\Big)$$
$$= \Gamma(t) + (\alpha,\ \beta_1 + i|\alpha|^2(2t+\Delta t))\Delta t = \Gamma(t) + (\alpha\Delta t, (\beta_1+2i|\alpha|^2 t)\Delta t) + o(|\Delta t|). \tag{38}$$

Then

$$\sigma_{\Gamma(t)}^{-1} \circ \Gamma(t+\Delta t) = (\alpha\Delta t,\ \beta_1\Delta t) + o(|\Delta t|).$$

We denote $G_1 \in Aut_0(\partial\mathbb{H}^2)$ as in (21), and we have

$$G_1(z,w) = \left(\frac{\lambda(z+\vec{a}w)U}{1-2i\langle\vec{a},z\rangle-(r+i|\vec{a}|^2)w},\ \frac{\lambda^2 w}{1-2i\langle\vec{a},z\rangle-(r+i|\vec{a}|^2)w}\right)$$

where $U = U(t) = e^{i\theta}$, $\theta = \theta(t) \in \mathbb{R}$, $\lambda = \lambda(t) > 0$ and $\vec{a} = \vec{a}(t) \in \mathbb{C}$, and $r = r(t) \in \mathbb{R}$, and

$$G_1^{-1}(z^*,w^*) = \left(\frac{\frac{1}{\lambda}(z-\frac{\vec{a}}{\lambda}Uw)U^{-1}}{1+2i\langle\frac{\vec{a}}{\lambda}U,z\rangle+(\frac{1}{\lambda^2}r-i|\frac{\vec{a}}{\lambda}|^2)w},\ \frac{\frac{1}{\lambda^2}w}{1+2i\langle\frac{\vec{a}}{\lambda}U,z\rangle+(\frac{1}{\lambda^2}r-i|\frac{\vec{a}}{\lambda}|^2)w}\right).$$

Therefore

$$q(t,\Delta t) = G_1^{-1}\circ\sigma_{\Gamma(t)}^{-1}\circ\Gamma(t+\Delta t) = G_1^{-1}(\alpha\Delta t,\ \beta_1\Delta t) + o(|\Delta t|)$$
$$= \left(\frac{1}{\lambda^2}(\lambda\alpha U^{-1}-\vec{a}\beta_1),\ \frac{1}{\lambda^2}\beta_1\right)\Delta t + o(|\Delta t|).$$

By using the notation in (34), we have

$$\Im(q_1(t)) = \frac{1}{\lambda(t)^2}\Im\Big(\lambda(t)\alpha U(t)^{-1} - \vec{a}(t)\beta_1\Big). \tag{39}$$

Step c. The identity We want to prove that the identity (32) holds:

$$(F_{\Gamma(t+\Delta t)})^{***} = \left(\left((F_{\Gamma(t)})^{***}\right)_{q(t,\Delta t)}\right)^{***}, \tag{40}$$

for sufficiently small t and Δt, i.e., to prove the following identity

$$G_4\circ\tau^F_{\Gamma(t+\Delta t)}\circ F\circ\sigma_{\Gamma(t+\Delta t)}\circ G_3 = G_6\circ\tau^F_q\circ\Big(G_2\circ\tau^F_{\Gamma(t)}\circ F\circ\sigma_{\Gamma(t)}\circ G_1\Big)\circ\sigma_{q(t,\Delta t)}\circ G_5. \tag{41}$$

Here by abusing of notion, we still use τ_q^F to denote τ_q^H where $H = (F_{\Gamma(t)})^{***}$. Notice that $G_1, G_5, G_3 \in Aut_0(\partial\mathbb{H}_2)$, $\sigma_{\Gamma(t)}$, σ_q, $\sigma_{\Gamma(t+\Delta t)} \in Aut(\partial\mathbb{H}_2)$, and $G_2, G_6, G_4 \in Aut_0(\partial\mathbb{H}_5)$, $\tau^F_{\Gamma(t)}$, τ^F_q, $\tau^F_{\Gamma(t+\Delta t)} \in Aut(\partial\mathbb{H}_5)$ are uniquely determined by F, $\Gamma(t), q$ and $\Gamma(t+\Delta t)$ in the normalization process, respectively.

If we can write

$$\left(\left((F_{\Gamma(t)})^{***}\right)_{q(t,\Delta t)}\right)^{***} = B \circ (F_{\Gamma(t+\Delta t)})^{***} \circ A \tag{42}$$

for some $A \in Aut_0(\partial\mathbb{H}^2)$ and $B \in Aut_0(\partial\mathbb{H}^5)$, then (40) holds by Lemma 2.3(2).

In fact, we write

$$\begin{aligned}
&\left(\left((F_{\Gamma(t)})^{***}\right)_{q(t,\Delta t)}\right)^{***}\\
&= G_6 \circ \tau_q^F \circ \left(G_2 \circ \tau^F_{\Gamma(t)} \circ F \circ \sigma_{\Gamma(t)} \circ G_1\right) \circ \sigma_{q(t,\Delta t)} \circ G_5\\
&= \left(G_6 \circ \tau_q^F \circ G_2 \circ \tau^F_{\Gamma(t)} \circ (\tau^F_{\Gamma(t+\Delta t)})^{-1} \circ G_4^{-1}\right) \circ \left(G_4 \circ \tau^F_{\Gamma(t+\Delta t)} \circ F \circ \sigma_{\Gamma(t+\Delta t)} \circ G_3\right) \circ\\
&\quad \circ \left(G_3^{-1} \circ \sigma^{-1}_{\Gamma(t+\Delta t)} \circ \sigma_{\Gamma(t)} \circ G_1 \circ \sigma_{q(t,\Delta t)} \circ G_5\right)\\
&= B \circ (F_{\Gamma(t+\Delta t)})^{***} \circ A
\end{aligned}$$

where $B = G_6 \circ \tau_q^F \circ G_2 \circ \tau^F_{\Gamma(t)} \circ (\tau^F_{\Gamma(t+\Delta t)})^{-1} \circ G_4^{-1}$ and $A = G_3^{-1} \circ \sigma^{-1}_{\Gamma(t+\Delta t)} \circ \sigma_{\Gamma(t)} \circ G_1 \circ \sigma_{q(t,\Delta t)} \circ G_5$. Writing $A = G_3^{-1} \circ \left(\sigma^{-1}_{\Gamma(t+\Delta t)} \circ \sigma_{\Gamma(t)} \circ G_1 \circ \sigma_{q(t,\Delta t)}\right) \circ G_5$. Notice G_3^{-1}, $G_5 \in Aut_0(\partial\mathbb{H}^2)$. By (35), we know $\sigma^{-1}_{\Gamma(t+\Delta t)} \circ \sigma_{\Gamma(t)} \circ G_1 \circ \sigma_{q(t,\Delta t)} \in Aut_0(\partial\mathbb{H}^2)$. Then $A \in Aut_0(\partial\mathbb{H}^2)$. Similarly, we can show $B \in Aut_0(\partial\mathbb{H}^5)$.

Step d. Proof of (a) - the case $\alpha \neq 0$ Let α be as in (39). Suppose $\alpha \neq 0$. By our construction (see [CJX06, Step 3 in § 5]), the vector $\vec{a}$ and the matrix U in (39) are given by

$$\vec{a} = \vec{a}(t) = i\frac{\partial^2 f^{**}_{pb}}{\partial w^2}(0) = i(e_1 - 2e_2)z_0 + 2ic_1c_2u_0 + (|p|) = i(e_1 - 2e_2)\alpha t + o(t), \tag{43}$$

$$U = U(t) = \begin{cases} e^{i\theta} = \frac{\partial^2 \phi^{**}_{pe1}}{\partial z \partial w}(0) / \left|\frac{\partial^2 \phi^{**}_{pe1}}{\partial z \partial w}(0)\right|, & if\ \frac{\partial^2 \phi^{**}_{pe1}}{\partial z \partial w}(0) \neq 0,\\ 1, & if\ \frac{\partial^2 \phi^{**}_{pe1}}{\partial z \partial w}(0) = 0, \end{cases} \tag{44}$$

and (see [CJX06, Step 3 in § 5])

$$\frac{\partial^2\phi^{**}_{pe1}}{\partial z\partial w}(0)=\frac{\partial^2\phi^{**}_{pd1}}{\partial z\partial w}(0)=b-2ib^3u_0-ibe_1u_0-4ib^2z_0-\frac{i}{2}bu_0$$
$$-iz_0-4ie_2z_0+4ic_1c_2u_0-2ibc_1^2u_0-2ic_1^2z_0=-i(1+4e_2+2c_1^2)z_0+o(|p|),$$

where $p=(z_0,w_0)=\Gamma(t)=(\alpha t,\beta_1t+i|\alpha|^2t^2)\in\partial\mathbb{H}^2$. Here we used the fact that $b=c_2c_1=0$ because $(c_1,c_3,e_1,e_2)\in\mathcal{K}_I\cup\mathcal{K}_{II}$. Then we obtain

$$\frac{\partial^2\phi^{**}_{pe1}}{\partial z\partial w}(0)=-i(1+4e_2+2c_1^2)\alpha t+o(t)\tag{45}$$

Now $1+4e_2+2c_1^2>0$. Since $\alpha\neq0$, we have $\frac{\partial^2\phi^{**}_{pe1}}{\partial z\partial w}(0)\neq0$ by (45) so that $\vec{a}$, U^{-1} and q_1 are real analytic neat 0 from their construction (cf. [CJX06]). Then

$$U(t)^{-1}=e^{-i\theta}=\frac{\overline{\frac{\partial^2\phi^{**}_{pe1}}{\partial z\partial w}(0)}}{|\frac{\partial^2\phi^{**}_{pe1}}{\partial z\partial w}(0)|}=\frac{i(1+4e_2+2c_1^2)\overline{\alpha}t+o(|t|)}{|\frac{\partial^2\phi^{**}_{pe1}}{\partial z\partial w}(0)|}=\frac{i(1+4e_2+2c_1^2)\overline{\alpha}}{|(1+4e_2+2c_1^2)\overline{\alpha}|}+O(|t|).$$

and there exists a constant $\delta>0$ such that

$$\begin{aligned}\Im(q_1(t))&=\tfrac{1}{\lambda(t)^2}\Im\Big(\lambda(t)\alpha U(t)^{-1}-\vec{a}(t)\beta_1\Big)=\tfrac{1}{\lambda(t)}\Im\Big(\alpha U(t)^{-1}\Big)+O(t)\\&=\tfrac{1}{\lambda}\Im\left(\tfrac{i(1+4e_2+2c_1^2)|\alpha|^2}{|(1+4e_2+2c_1^2)\alpha|}\right)+O(|t|)=|\alpha|+O(|t|),\quad\forall t\in[0,\delta]\end{aligned}\tag{46}$$

because $\lambda=\lambda(t)=1+O(|t|)$. This proves (34) as well as (28).

Step e. Proof of (a) - the case $\alpha=0$ Next we will prove (a) for the case $\alpha=0$. In this case $\Gamma(t)=(0,\beta_1t)$, and $\Im(q_1(t))=-\frac{\beta_1}{\lambda(t)^2}\Im(\vec{a}(t))$ and $\vec{a}(t)=i\frac{\partial^2f^{**}_{pb}}{\partial w^2}(0)$. From [CJX06, § 5, step 3 and step 2], we have $\frac{\partial^2f^{**}_{pb}}{\partial w^2}(0)=\frac{\partial^2f^{**}_{p}}{\partial w^2}(0)=$

$$=\frac{1}{\lambda(p)}T^2\widetilde{f}(p)\cdot\overline{L\widetilde{f}(p)}^t-\frac{1}{\lambda(p)^2}(T\widetilde{f}\cdot\overline{L\widetilde{f}}^t)(T^2g-2iT^2\widetilde{f}\cdot\overline{\widetilde{f}}^t-2i\|T\widetilde{f}\|^2)(p)\tag{47}$$

We want to prove $\vec{a}(t)\equiv0$ which implies (28). This will be done by direct computation. Write F as in the following form:

$$f=zh+(\frac{i}{2}+ie_1)zwh,\phi_1=z^2h,\phi_2=c_1zwh,\phi_3=c_3w^2h,g=wh+ie_1w^2h,$$

where $h = h(w) = \frac{1}{1+ie_1w+e_2w^2}$. Then

$$h' = (-ie_1 - 2e_2w)h^2, \;\; h'' = (-2e_2 - 2e_1^2 + 6ie_1e_2w + 6e_2^2w^2)h^3.$$

From the definition of F_p where $p = (z, w)$, we have [CJH06, § 5]

$$\begin{aligned} &f(p) = zh + (\frac{i}{2} + ie_1)zwh, \\ &Lf(p) = h + (\frac{i}{2} + ie_1)wh + 2i\overline{z}\bigg(zh' + (\frac{i}{2} + ie_1)z(h + wh')\bigg), \end{aligned}$$

$$Tf(p) = zh' + (\frac{i}{2} + ie_1)z(h + wh'),$$

$$T^2f(p) = zh'' + (\frac{i}{2} + ie_1)z(2h' + wh''),$$

$$\phi_1(p) = z^2h, \quad L\phi_1(p) = 2zh + 2i\overline{z}z^2h', \quad T\phi_1(p) = z^2h',$$

$$\phi_2(p) = c_1zwh, \quad L\phi_2(p) = c_1wh + 2ic_1\overline{z}z(h + wh'), \quad T\phi_2(p) = c_1z(h + wh'),$$

$$T^2\phi_1(p) = z^2h'',$$

$$\begin{aligned} &L^2\phi_2(p) = 2ic_1\overline{z}(h + wh') + 2i\overline{z}\Big[c_1(h + wh') + 2ic_1\overline{z}z(2h' + wh'')\Big] \\ &= 4ic_1\overline{z}(h + wh') - 4c_1\overline{z}^2z(2h' + wh''), \end{aligned}$$

$$T^2\phi_2(p) = c_1z(2h' + wh''),$$

$$\phi_3(p) = c_3w^2h, \quad L\phi_3(p) = 2ic_3\overline{z}(2wh + w^2h'), \quad T\phi_3(p) = c_3(2wh + w^2h'),$$

$$T^2\phi_3(p) = c_3(2h + 2wh' + 2wh' + w^2h'') = c_3(2h + 4wh' + w^2h''),$$

When $p = (0, t)$, we have

$$\lambda(p) = |Lf(p)|^2 + |L\phi_1(p)|^2 + |L\phi_2(p)|^2 + |L\phi_3(p)|^2 = |h(t)|^2 + |c_1th(t)|^2 = 1 + o(t)$$

and $Tf(p) = T\phi_1(p) = T\phi_2(p) = L\phi_3(p) = T^2f(p) = T^2\phi_1(p) = T^2\phi_2(p) = 0$ so that $(T\widetilde{f} \cdot \overline{L\widetilde{f}}^t)(p) = 0$ and that $(T^2\widetilde{f} \cdot \overline{L\widetilde{f}}^t)(p) = 0$. Hence by (47) we obtain $\Im(q_1(t)) = -\frac{\beta_1}{\lambda(t)^2}\Im(\vec{a}(t)) \equiv 0$. The proof of (a) is complete.

Step f. Proof of (b) and (c) Similarly we can prove (c). To prove (b), we first consider the case when $\alpha \neq 0$. In this case, we can take a sequence of points $(c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \in \mathcal{K}_{IIC,1+4e_2+2c_1^2>0}$ such that $(c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \to (c_1, c_3, e_1, e_2)$. Then (46) holds for such maps $F_{c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}}$:

$$\Im(q_1^{(k)}(t))) = |\alpha| + O(|t|), \qquad \forall t \in [0, \delta] \tag{48}$$

Also, we can take another sequence of points $(\widetilde{c}_1^{(k)}, \widetilde{c}_3^{(k)}, \widetilde{e}_1^{(k)}, \widetilde{e}_2^{(k)}) \in \mathcal{K}_{IIC,1+4e_2+2c_1^2<0}$ such that $(\widetilde{c}_1^{(k)}, \widetilde{c}_3^{(k)}, \widetilde{e}_1^{(k)}, \widetilde{e}_2^{(k)}) \to (c_1, c_3, e_1, e_2)$. Then by letting $k \to \infty$ and the same argument in the proof for (c), we get

$$\Im(\widetilde{q}_1^{(k)}(t))) = -|\alpha| + O(|t|), \qquad \forall t \in [0, \delta] \tag{49}$$

for maps $F_{\widetilde{c}_1^{(k)}, \widetilde{c}_3^{(k)}, \widetilde{e}_1^{(k)}, \widetilde{e}_2^{(k)}}$. Such estimate is uniform for all k. Notice that the function $[4c_1(bc_1+2c_2)-8b(e_1+e_2)](\Gamma(t))\Im(q_1(t))$ in (33) is real analytic but $4c_1(bc_1+2c_2)-8b(e_1+e_2)$ and $\Im(q_1)$ may be not (see Remark (a) following the proof of Lemma 3.1 below). Then by (48) and (49) and by letting $k \to \infty$, we must have

$$[4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)](\Gamma(t))\Im(q_1(t)) \equiv 0, \quad \forall t \in [0, \delta]$$

for the map F_{c_1,c_3,e_1,e_2} so that $\Im(q_1(t)) \equiv 0$ is proved.

Next we consider the case when $\alpha = 0$, by Step e, we have $\Im(q_1(t)) \equiv 0$ so that (c) is proved $\square$.

Remark (a) We notice that if $1 + 4e_2 + 2c_1^2 = 0$, $\frac{\partial^2\phi_{pe1}^{**}}{\partial z\partial w}(0)$ may be zero so that U and hence U^{-1} may not be differentiable. By the way, $\mathcal{W}(F_p^{***}) = c_1^2(p) - e_1(p) - e_2(p) = \frac{1}{4} + 2c_1^2(p) + b^2(p)$ is real analytic but $c_1(p)$ and $b(p)$ may not be differentiable; this is because of some definitions such as (44) (cf. [CJX06, p.1521-1522]). Then the function

$[4c_1(bc_1+2c_2)-8b(e_1+e_2)](\Gamma(t))\Im(q_1(t))$ in (33) is real analytic but $4c_1(bc_1+2c_2)-8b(e_1+e_2)$ and $\Im(q_1)$ may be not.

(b) If we replace the curve $\Gamma(t)=(\alpha t,\beta_1 t+i|\alpha|^2t^2)$ by another curve

$$\Gamma(t)=(\alpha t,\beta_0+\beta_1 t+i|\alpha|^2t^2), \tag{50}$$

then (38) and hence (46) holds.

Recall $(c_1,c_3,e_1,e_2)\in\mathcal{K}_{II}\iff$ (5) holds with $c_1>0$ and $b=c_2=0\iff c_1>0$ and either

$$e_1=\frac{-(\frac{1}{4}+c_1^2)-\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2},\ e_2=\frac{-(\frac{1}{4}+c_1^2)+\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2}, \tag{51}$$

where $4c_3^2\le(\frac{1}{4}+c_1^2)^2$, or

$$e_1=\frac{-(\frac{1}{4}+c_1^2)+\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2},\ e_2=\frac{-(\frac{1}{4}+c_1^2)-\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2}, \tag{52}$$

where $4c_3^2\le(\frac{1}{4}+c_1^2)^2$. Here c_1 and c_3 are parameters.

We can write a disjoint union $\mathcal{K}_{II}=\mathcal{K}_{II,e_1<e_2}\cup\mathcal{K}_{II,e_1=e_2}\cup\mathcal{K}_{II,e_1>e_2}$, where

$$\mathcal{K}_{II,e_1<e_2}=\{(c_1,c_3,e_1,e_2)\in\mathcal{K}_{II}\mid e_1<e_2\}$$

$$\mathcal{K}_{II,e_1=e_2}=\{(c_1,c_3,e_1,e_2)\in\mathcal{K}_{II}\mid e_1=e_2\},$$

and

$$\mathcal{K}_{II,e_1>e_2}=\{(c_1,c_3,e_1,e_2)\in\mathcal{K}_{II}\mid e_1>e_2\}.$$

Then $\mathcal{K}_{II,e_1<e_2}=\{(c_1,c_3,e_1,e_2)\in\mathcal{K}_{II}\mid (51)\ \textit{and}\ 4c_3^2<(\frac{1}{4}+c_1^2)^2\ \textit{hold}\}$, $\mathcal{K}_{II,e_1=e_2}=\{(c_1,c_3,e_1,e_2)\in\mathcal{K}_{II}\mid (51)\textit{or}\ (52)\ \textit{and}\ 4c_3^2=(\frac{1}{4}+c_1^2)^2\ \textit{hold}\}$, and $\mathcal{K}_{II,e_1<e_2}=\{(c_1,c_3,e_1,e_2)\in\mathcal{K}_{II}\mid (52)\ \textit{and}\ 4c_3^2<(\frac{1}{4}+c_1^2)^2\ \textit{hold}\}$.

Lemma 3.2 *(i)* $\mathcal{K}_{II,e_1<e_2}\subseteq\mathcal{K}_{I,II,1+4e_2+2c_1^2>0}$, *and* $\mathcal{K}_{II,e_1=e_2}\subseteq\mathcal{K}_{I,II,1+4e_2+2c_1^2>0}$.

(ii) Let $(c_1,c_3,e_1,e_2)\in\mathcal{K}_{II,e_1>e_2}$. *Then*
(a) $(c_1,c_3,e_1,e_2)\in\mathcal{K}_{I,II,1+4e_2+2c_1^2>0}$ *if and only if* $\frac{1}{2}c_1^2+c_1^4<4c_3^2<(\frac{1}{4}+c_1^2)^2$ *holds.*
(b) $(c_1,c_3,e_1,e_2)\in\mathcal{K}_{I,II,1+4e_2+2c_1^2=0}$ *if and only if* $\frac{1}{2}c_1^2+c_1^4=4c_3^2$ *holds.*
(c) $(c_1,c_3,e_1,e_2)\in\mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$ *if and only if* $0\le 4c_3^2<\frac{1}{2}c_1^2+c_1^4$ *holds.*

Proof of Lemma 3.2: (i) For any $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{II,e_1<e_2} \cup \mathcal{K}_{II,e_1=e_2}$, by $-e_1 - e_2 = \frac{1}{4} + c_1^2$ and (51), we have

$$1 + 4e_2 + 2c_1^2 = \frac{1}{2} + 2e_2 - 2e_1 = \frac{1}{2} + 2\sqrt{\left(\frac{1}{4} + c_1^2\right)^2 - 4c_3^2} \geq \frac{1}{2} > 0.$$

(ii) For any $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{II,e_1>e_2}$, we know that $1 + 4e_2 + 2c_1^2 > 0$ is equivalent to $\frac{1}{2} + 2e_2 - 2e_1 = \frac{1}{2} - 2\sqrt{(\frac{1}{4} + c_1^2)^2 - 4c_3^2} > 0$, i.e., $\frac{1}{2}c_1^2 + c_1^4 < 4c_3^2$, so that (a) is proved. (b) and (c) are proved similarly. $\square$.

Lemma 3.3 *Let* $\mathcal{E} := \{(c_1, c_3, e_1, e_2) \in \mathcal{K}_I \cup \mathcal{K}_{II} \mid (F_{c_1,c_3,e_1,e_2})^{***}_p \equiv F_{c_1,c_3,e_1,e_2}, \quad \forall p \in \partial\mathbb{H}^2$ *near* $0\}$*. Then* $F_{c_1,c_3,e_1,e_2} \in \mathcal{E}$ *if and only if for any curve* Γ *as in (27),*

$$(4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2))(\Gamma(t)) \equiv 0, \ \forall t \in [0, 1]. \tag{53}$$

Proof: It is clear

$$F_{c_1,c_3,e_1,e_2} \in \mathcal{E} \Longleftrightarrow c_1(p), c_3(p) \textit{ are constant}, \ \forall p \in \partial\mathbb{H}^2 \textit{ near } 0. \tag{54}$$

If $F_{c_1,c_3,e_1,e_2} \in \mathcal{E}$, then either $c_1(p) = b(p) = 0$ or $c_1(p) > 0, b(p) = c_2(p) = 0$, $\forall p \in \partial\mathbb{H}^2$ near 0 (i.e., the case (I) or (IIA), (IIB) and (IIC)). Then the equality in (53) holds.

Conversely, suppose that $(4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2))(\Gamma(t)) \equiv 0$ for any choice of curve $\Gamma(t)$ and for any (c_1, c_3) in some open subset of $\mathbb{R}^2$. Then $b_1(p) = 0$ and $c_1(p)c_2(p) = 0$, $\forall p \in \partial\mathbb{H}^2$ near 0. If $c_1 \equiv 0$, then by Lemma 2.5(iii), $c_3(p) = 0, \forall p$ so that $F_{c_1,c_3,e_1,e_2} \in \mathcal{E}$. If $c_1(p) > 0$ for any p in some open subset of $\partial\mathbb{H}^2$, then $c_2(p) = 0$, $\forall p$. Then we apply Lemma 2.5(ii) to know

$$c_3^2(p) = c_3^2 + 4(c_3)^2(5b - \frac{2c_2}{c_1})\Im(z_0) + o(|p|) = c_3^2 + o(|p|), \quad \textit{where } p = (z_0, w_0) \in \partial\mathbb{H}^2 \tag{55}$$

which implies as in (33) that $c_3(p) = \textit{constant}$, $\forall p$. Also, by (33), from $(4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2))(\Gamma(t)) \equiv 0$ it implies $\mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_{\Gamma(t)}) = \textit{constant}$, $\forall \Gamma$ and $\forall t$. Then

$$\mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_{\Gamma(t)}) = (c_1^2 - e_1 - e_2)(\Gamma(t)) = (\frac{1}{4} + 2c_1^2)(\Gamma(t)) = \textit{constant},$$

which implies that $c_1(\Gamma(t)) = \textit{constant}$ for any $t \in [0, t_0]$, i.e., $c_1 \equiv \textit{constant}$. By (54), we obtain $F_{c_1,c_3,e_1,e_2} \in \mathcal{E}$. Claim (53) is proved. $\square$

Theorem 1.1(i) will follow by Lemma 3.2 and the following lemma.

Lemma 3.4 *Let* $(c_1, c_3, e_1, e_2) \in \mathcal{K}_I \cup \mathcal{K}_{II}$. *Then* F_{c_1,c_3,e_1,e_2} *satisfies (9) if and only if* $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}^* := \mathcal{K}_I \cup \mathcal{K}_{II} - \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$.

Proof: $(\Longleftarrow)$ It follows from Lemma 3.1.

$(\Longrightarrow)$ Take any map $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$ satisfying the minimum property (9). We first show that $F_{c_1,c_3,e_1,e_2} \in \mathcal{E}$ where $\mathcal{E}$ was defined in above lemma.

By Step d in the proof of Lemma 3.1, we know that for any curve Γ as in Lemma 3.1, there is $\delta > 0$ such that

$$\Im(q_1(t)) = -|\alpha| + O(|t|),\ \forall t \in [0, \delta].$$

Suppose that F_{c_1,c_3,e_1,e_2} satisfies (9). By (33), it implies $(4c_1(bc_1+2c_2)-8b(e_1+e_2))(\Gamma(t)) \equiv 0$ for any such curves $\Gamma(t)$ and for any (c_1, c_3) with $0 \le 4c_3^2 \le (\frac{1}{4}+c_1^2)^2$. Then by above lemma, $F_{c_1,c_3,e_1,e_2} \in \mathcal{E}$.

$\mathcal{E} \cap \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$ is a real analytic set in $\mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$. We claim:

$$\mathcal{E} \cap \mathcal{K}_{I,II,1+4e_2+2c_1^2<0} = \emptyset. \tag{56}$$

Suppose (56) is not true. Then we can take

$$(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2<0} \cap \mathcal{E}. \tag{57}$$

We can take a sequence of points $(c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2<0} - \mathcal{E}$ such that

$$(c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \to (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}).$$

By our choice of $(c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)})$, the corresponding maps $F_{c_1^{(k)},c_3^{(k)},e_1^{(k)},e_2^{(k)}}$ has the property that the associated function $\mathcal{W}\big((F_{c_1^{(k)},c_3^{(k)},e_1^{(k)},e_2^{(k)}})^{***}_{\Gamma(t)}\big)$ is strictly decreasing as t goes from 0 to 1. Then $F_{c_1^{(k)},c_3^{(k)},e_1^{(k)},e_2^{(k)}}$ is equivalent to some map $F_{\widetilde{c}_1^{(k)},\widetilde{c}_3^{(k)},\widetilde{e}_1^{(k)},\widetilde{e}_2^{(k)}} \in \mathcal{K}^* = \mathcal{K}_I \cup \mathcal{K}_{II} - \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$ with the minimum $\mathcal{W}$ value. Since the function value $\mathcal{W}((F_{c_1^{(k)},c_3^{(k)},e_1^{(k)},e_2^{(k)}})^{***}_{\Gamma})$ is decreasing, the sequence of points $(\widetilde{c}_1^{(k)}, \widetilde{c}_3^{(k)}, \widetilde{e}_1^{(k)}, \widetilde{e}_2^{(k)})$ is also bounded in $\mathcal{K}$. By taking subsequence, we may assume that $(\widetilde{c}_1^{(k)}, \widetilde{c}_3^{(k)}, \widetilde{e}_1^{(k)}, \widetilde{e}_2^{(k)}) \to (\widetilde{c}_1^{(0)}, \widetilde{c}_3^{(0)}, \widetilde{e}_1^{(0)}, \widetilde{e}_2^{(0)}) \in \mathcal{K}^*$. Then $F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}}$ is equivalent to $F_{\widetilde{c}_1^{(0)},\widetilde{c}_3^{(0)},\widetilde{e}_1^{(0)},\widetilde{e}_2^{(0)}} \in \mathcal{K}^*$, i.e.,

$$F_{\widetilde{c}_1^{(0)},\widetilde{c}_3^{(0)},\widetilde{e}_1^{(0)},\widetilde{e}_2^{(0)}} = \left(F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}}\right)^{***}_q \tag{58}$$

for some non zero $q \in \partial\mathbb{H}^2$, by the same argument as in (7) and (8) (or [CJX06, Step 1, § 4]). On the other hand, since $F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}} \in \mathcal{E}$, by the definition of $\mathcal{E}$, (58) cannot occur. This contradiction shows that (57) cannot occur. Thus Claim (56) is proved. □

4 Local version of Theorem 1.1(ii)

For each point $p = (a, b + i|a|^2) \in \partial\mathbb{H}^2$ where $b \in \mathbb{R}$ and $a \in \mathbb{C}$, we denote $\pi(p) = \pi(a, b + i|a|^2) := (|a|, |b|) \in \mathbb{R}^2$. We denote by $\square_c := [0, c] \times [0, c]$ a square and $\triangle_c := \{(x, y) \mid 0 \le x \le c,\ 0 \le y \le x\}$ a triangle inside $\square_c$. Let $\Gamma(t) = (\alpha t, \beta_1 t + i|\alpha|^2 t^2)$ with $t \in [0, 1]$ be line segments, The set $\{\pi(\Gamma(t)) = \pi(\alpha t, \beta_1 t + i|\alpha|^2 t^2) \mid |\alpha| = 1,\ |\beta_1| \le 1,\ \ 0 \le t \le t_0\}$ is equal to $\triangle_{t_0}$. Notice that $\pi(a, b + i|a|^2) \in \triangle_{t_0}$ if and only if there exists such a line segment $\Gamma(t)$ so that $(a, b + i|a|^2) = \Gamma(t^*)$ for some $t^* \in [0, t_0]$.

Lemma 4.1 *For any $P^{(0)} = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \in \mathcal{K}^*$, there is a neighborhood U of $P^{(0)}$ in $\mathcal{K}^*$ and a constant $c > 0$ such that for any point $(c_1', c_3', e_1', e_2'), (c_1'', c_3'', e_1'', e_2'') \in U$ with $F_{c_1'', c_3'', e_1'', e_2''} = (F_{c_1', c_3', e_1', e_2'})^{***}_p$ where $p = (a, b + i|a|^2) \in \partial\mathbb{H}^2$, $a \in \mathbb{C}$, $b \in \mathbb{R}$, $|p| := \max\{|a|, |b|\}$ $\le c$, we have*

$$(c_1'', c_3'', e_1'', e_2'') = (c_1', c_3', e_1', e_2'). \tag{59}$$

Proof of Lemma 4.1: **Step 1. Choose U and c** For the point $P^{(0)} \in \mathcal{K}^*$, by Lemma 3.1 and the uniform estimate (46), there exists a neighborhood U of this point and a constant $0 < t_0 < 1$ such that for any point $(c_1', c_3', e_1', e_2') \in U$ and for any curve $\Gamma(t) = \{(\alpha t, \beta_1 t + i|\alpha|^2 t^2)\}$ with $\alpha \in \mathbb{C}$, $\beta_1 \in \mathbb{R}$ with $|\beta_1| \le 1$, $|\alpha| = 1$, $0 \le t \le t_0$, we have the property

$$\mathcal{W}((F_{c_1', c_3', e_1', e_2'})^{***}_{\Gamma(t)}) \ \textit{is nondecreasing}, \quad \forall t \in [0, t_0]. \tag{60}$$

Since $F_{c_1'', c_3'', e_1'', e_2''} = (F_{c_1', c_3', e_1', e_2'})^{***}_p = H \circ \tau \circ F_{c_1', c_3', e_1', e_2'} \circ \sigma_p \circ G$ where $G \in Aut_0(\partial\mathbb{H}^2)$, $H \in Aut_0(\partial\mathbb{H}^5)$, τ and and σ_p are as in (18), we can write

$$F_{c_1', c_3', e_1', e_2'} = (F_{c_1'', c_3'', e_1'', e_2''})^{***}_q,$$

where $q = G^{-1}(-z_0, -\overline{w_0})$. Since $G(0) = 0$ and $G^{-1}(0) = 0$, by continuity, $q \to 0$ as $p \to 0$. Then we can choose a number $0 < c < t_0$ such that $\forall p = (a, b + i|a|^2) \in \partial\mathbb{H}^2$ with $|p| \le c$, the point $q = (A, B + i|A|^2)$ satisfies $|q| \le t_0$. Let us verify that c is the desired number.

Step 2. There exists a curve from 0 to p with monotone property We have to put the condition $|\alpha| = 1$ in (60); otherwise we may not be able to find the t_0 for all curves. We want to remove this condition by adding one more piece of the line segment, namely, we claim that for any p and (c_1', c_3', e_1', e_2') as above, there is a curve $\Gamma(t)$, $t \in [0, t^*]$, consisting of one or two pieces of line segments, such that (60) is still true: $\mathcal{W}((F_{c_1', c_3', e_1', e_2'})^{***}_{\Gamma(t)})$ is nondecreasing along Γ.

Write $p = (a, b + i|a|^2) \in \partial\mathbb{H}^2$. We distinguish two cases: (i) $\pi(a, b + i|a|^2) \in \triangle_c$; and (ii) $\pi(a, b + i|a|^2) \in \square_c - \triangle_c$.

In the first case (i): for any $p = (a, b + i|a|^2)$ with $|a| \le c$ and $|b| \le |a|c$, assuming $p \ne 0$, we have $p = \Gamma(t^*)$ for some curve $\Gamma(t) = (\alpha t, \beta_1 t + i|\alpha|^2 t^2)$ with $0 \le \beta_1 \le 1$ and $|\alpha| = 1$ as above with some $t^* \in [0, c]$. In fact, we have $\alpha = \frac{a}{|a|}$, $\beta_1 = \frac{b}{|a|}$ and $t^* = |a|$. By (60) the function $\mathcal{W}((F_{c'_1,c'_3,e'_1,e'_2})^{***}_{\Gamma(t)})$ is increasing as t varies from 0 to t^*.

In the second case (ii): $p = (a, b + i|a|^2)$ with $|a| \le c$ and $|a| < |b| \le c$. Let us assume $b > 0$; otherwise it can be proved by the same argument. In this case, we cannot find Γ such that it connects 0 and p as in the case (i). However, we can define two pieces of curves:

$$\Gamma(t) = \begin{cases} \Gamma_1(t), & 0 \le t \le b - |a|, \\ \Gamma_2(t), & b - |a| \le t \le b. \end{cases}$$

$$:= \begin{cases} (0, t), & 0 \le t \le b - |a|, \\ \left(\frac{a}{|a|}(t - b + |a|), t + i \Big| t - b + |a| \Big|^2 \right), & b - |a| \le t \le t^* := b. \end{cases}$$

Here $\pi(\Gamma_1) = \{0\} \times [0, b - |a|]$ is a vertical line segment; and $\pi(\Gamma_2)$ is another line segment connecting $\Gamma_1(b - |a|)$ and the point p.

By Step e in § 3, the function $\mathcal{W}((F_{c'_1,c'_3,e'_1,e'_2})^{***}_{\Gamma_1(t)})$ is constant for $0 \le t \le b - |a|$. Next we consider $\mathcal{W}((F_{c'_1,c'_3,e'_1,e'_2})^{***}_{\Gamma_2(t)})$. If we use a new variable $u = t - b + |a|$, then $\Gamma_2(t)$ can be written as

$$\Gamma_2(u) = \left(\frac{a}{|a|} u, \quad (b - |a|) + u + iu^2 \right), \quad 0 \le u \le |a|.$$

By the remark (b) in (50), (46) is still valid for $\Gamma_2(u)$ so that $\mathcal{W}((F_{c'_1,c'_3,e'_1,e'_2})^{***}_{\Gamma_2(t)})$ is nondecreasing for any $b - |a| \le t \le t^*$. Our claim is proved.

Step 3. The $\mathcal{W}$ function is constant We claim:

$$\mathcal{W}((F_{c'_1,c'_3,e'_1,e'_2})^{***}_{\Gamma}) = constant. \tag{61}$$

In fact, since $F_{c''_1,c''_3,e''_1,e''_2} = (F_{c'_1,c'_3,e'_1,e'_2})^{***}_p$ and $F_{c'_1,c'_3,e'_1,e'_2} = (F_{c''_1,c''_3,e''_1,e''_2})^{***}_q$. We have $F_{c'_1,c'_3,e'_1,e'_2} = ((F_{c'_1,c'_3,e'_1,e'_2})^{***}_p)^{***}_q$.

Since $\pi(p) \in \square_c$, by our choice of c, $q = (A, B + i|A|^2)$ satisfies $\pi(q) \in \square_{t_0}$, i.e., $|A| \le t_0$ and $|B| \le t_0$. Then by Step 2, there exists a curve $\widetilde{\Gamma}(\widetilde{t})$, $0 \le \widetilde{t} \le \widetilde{t}^*$, connecting 0 and q such that the function $\mathcal{W}((F_{c'_1,c'_3,e'_1,e'_2})^{***}_{\widetilde{\Gamma}(\widetilde{t})})$ is nondecreasing along $\widetilde{\Gamma}$. Then we obtain

$$\mathcal{W}(F_{c'_1,c'_3,e'_1,e'_2}) = \mathcal{W}((F_{c'_1,c'_3,e'_1,e'_2})^{***}_{\Gamma(0)}) \le \mathcal{W}((F_{c'_1,c'_3,e'_1,e'_2})^{***}_{\Gamma(t^*)}) = \mathcal{W}(F_{c''_1,c''_3,e''_1,e''_2}), \tag{62}$$

and

$$\mathcal{W}(F_{c''_1,c''_3,e''_1,e''_2}) = \mathcal{W}((F_{c''_1,c''_3,e''_1,e''_2})^{***}_{\widetilde{\Gamma}(0)}) \le \mathcal{W}((F_{c''_1,c''_3,e''_1,e''_2})^{***}_{\widetilde{\Gamma}(\widetilde{t}^*)}) = \mathcal{W}((F_{c'_1,c'_3,e'_1,e'_2}). \tag{63}$$

By (62) and (63), Claim (61) is proved.

Step 4. Proof of the uniqueness We next claim that $(F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma(t)}$ is constant:

$$(F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma(t)} \equiv F_{c_1',c_3',e_1',e_2'}, \quad \forall t \in [0, t_0]. \tag{64}$$

Let us consider the case (i) in Step 2. From (31) and Lemma 2.5, it implies that $\big(4c_1'(b'c_1' + 2c_2') - 8b'(e_1' + e_2')\big)\Gamma(t) = 0$ for any $t \in [0, t^*]$. Thus by the argument in (55), we proved $c_1'(\Gamma(t)) = c_3'(\Gamma(t)) = 0$ for any $t \in [0, t^*]$. This implies that $(F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma(t)}$ is the same map for any $t \in [0, t_0]$. Claim (64) is proved. The case (ii) will be proved by similar argument as the case (i) and by the remark (b) in (50). $\square$

Lemma 4.2 *For any point $P^{(0)} = (c_1^{(0)}, c_3^{(0)},\ e_1^{(0)}, e_2^{(0)}) \in \mathcal{K}^* - \mathcal{E}$ where $\mathcal{E}$ is defined in Lemma 3.3, there is a neighborhood V of $P^{(0)}$ in $\mathcal{K}$, a neighborhood U of $P^{(0)}$ in $\mathcal{K}^* - \mathcal{E}$ and a neighborhood E of 0 in $\partial\mathbb{H}^2$ such that the map $\Psi : U \times E \to V$, $(F, p) \mapsto F_p^{***}$ is surjective.*

Proof: We first claim that for any $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}^* - \mathcal{E}$, the set $N := \{(F_{c_1,c_3,e_1,e_2})_p^{***} \mid p \in \partial\mathbb{H}^2\}$ is of real dimension ≥ 2. In fact, consider a function $\mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_\Gamma)$ on N where $\Gamma(t) = (\alpha t, \beta_1 t + |\alpha|^2 t^2)$ is a curve in $\partial\mathbb{H}^2$ as (27). By (46), we have $\Im(q_1(t)) = |\alpha| + O(|t|)$ for $t > 0$ sufficiently small. Since $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}^* - \mathcal{E}$, by Lemma 3.3, we have $(4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2))(\Gamma(t)) \not\equiv 0$ holds for some curve Γ. Then from (33),

$$\mathcal{W}(F^{***}_{\Gamma(t+\Delta t)}) = \mathcal{W}(F^{***}_{\Gamma(t)}) + \Bigg[4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)\Bigg](\Gamma(t))|\alpha|\Delta t + o(|\Delta t|), \tag{65}$$

Since $\alpha \in \mathbb{C} \cong \mathbb{R}^2$, our claim is proved.

It remains to prove $\dim_{\mathbb{R}} \Psi(U \times E) = 4$. Notice that $\dim_{\mathbb{R}} \mathcal{K} = 4$, $\dim_{\mathbb{R}}(\mathcal{K}^*) \geq 2$, and that the map defined by $(\mathcal{K}^* - \mathcal{E}) \times \partial\mathbb{H}^2 \to \mathcal{K}$, $(F, p) \mapsto F_p^{***}$ is (Nash) algebraic. Then it suffices to show that this map is injective, i.e., for any two distinct points (c_1, c_3, e_1, e_2), $(\widetilde{c}_1, \widetilde{c}_3, \widetilde{e}_1, \widetilde{e}_2) \in \mathcal{K}^*$, which are sufficiently close to $(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)})$, and for any two points $p, \widetilde{p} \in \partial\mathbb{H}^2$, which are sufficiently close to $0 \in \partial\mathbb{H}^2$,

$$(F_{c_1,c_3,e_1,e_2})_p^{***} \neq (F_{\widetilde{c}_1,\widetilde{c}_3,\widetilde{e}_1,\widetilde{e}_2})_{\widetilde{p}}^{***}. \tag{66}$$

If this can be proved, it follows $\dim_{\mathbb{R}} \Psi(U \times E) = 4$.

Recall that for a fixed F, we write

$$F_p^{***} = H_p \circ \tau_p \circ F \circ \sigma_p \circ G_p, \tag{67}$$

where $\sigma_p \in Aut(\mathbb{H}^2)$ and $\tau_p \in Aut(\mathbb{H}^5)$ are defined in (18), $G_p \in Aut_0(\mathbb{H}^2)$ and $H_p \in Aut_0(\partial\mathbb{H}^5)$.

In case (66) does not hold, i.e., we have $(F_{c_1,c_3,e_1,e_2})^{***}_p = (F_{\widetilde{c}_1,\widetilde{c}_3,\widetilde{e}_1,\widetilde{e}_2})^{***}_{\widetilde{p}}$. By (67), we write

$$H_p \circ \tau_p \circ F_{c_1,c_3,e_1,e_2} \circ \sigma_p \circ G_p = \widetilde{H}_p \circ \widetilde{\tau}_p \circ F_{\widetilde{c}_1,\widetilde{c}_3,\widetilde{e}_1,\widetilde{e}_2} \circ \widetilde{\sigma}_p \circ \widetilde{G}_p,$$

i.e.,

$$F_{c_1,c_3,e_1,e_2} = \tau_p^{-1} \circ H_p^{-1} \circ \widetilde{H}_p \circ \widetilde{\tau}_p \circ F_{\widetilde{c}_1,\widetilde{c}_3,\widetilde{e}_1,\widetilde{e}_2} \circ \widetilde{\sigma}_p \circ \widetilde{G}_p \circ G_p^{-1} \circ \sigma_p^{-1} = (F_{\widetilde{c}_1,\widetilde{c}_3,\widetilde{e}_1,\widetilde{e}_2})^{***}_{p_0}, \quad (68)$$

where $p_0 = \widetilde{\sigma}_p \circ \widetilde{G}_p \circ G_p^{-1} \circ \sigma_p^{-1}(0)$.

Notice from (67) that there is $\delta > 0$ such that as $p \to 0$, $\sigma_p, G_p, \tau_p, H_p$ all converge to the identity maps in $Aut(\mathbb{H}^2)$ and $Aut(\mathbb{H}^5)$ respectively. We apply this fact to (68) to conclude that for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $(c_1, c_3, e_1, e_2), (\widetilde{c}_1, \widetilde{c}_3, \widetilde{e}_1, \widetilde{e}_2) \in \mathcal{K}^*$ with

$$dist\big((c_1, c_3, e_1, e_2), (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)})\big) < \delta, \quad dist\big((\widetilde{c}_1, \widetilde{c}_3, \widetilde{e}_1, \widetilde{e}_2), (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)})\big) < \delta,$$

we must have $|p_0| < \epsilon$. We can choose ϵ to be the c as in Lemma 4.1. By applying Lemma 4.1 to (68) to conclude $F_{c_1,c_3,e_1,e_2} = F_{\widetilde{c}_1,\widetilde{c}_3,\widetilde{e}_1,\widetilde{e}_2}$. This contracts with the fact that (c_1, c_3, e_1, e_2) and $(\widetilde{c}_1, \widetilde{c}_3, \widetilde{e}_1, \widetilde{e}_2)$ are distinct. Hence (66) is proved. $\square$

Corollary 4.3 *(Local version of Theorem 1.1(ii)) For any $P^{(0)} = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \in \mathcal{K}^* - \mathcal{E}$ where $\mathcal{E}$ is defined in Lemma 3.3, there is a neighborhood U of $P^{(0)}$ in $\mathcal{K}^* - \mathcal{E}$ such that $\forall (c_1', c_3', e_1', e_2'), (c_1'', c_3'', e_1'', e_2'') \in U$ such that $F_{c_1'',c_3'',e_1'',e_2''}$ and $F_{c_1',c_3',e_1',e_2'}$ are equivalent, we have $(c_1'', c_3'', e_1'', e_2'') = (c_1', c_3', e_1', e_2')$.*

Proof: Let U_1 be a neighborhood of $P^{(0)}$ in $\mathcal{K}^* - \mathcal{E}$, E a neighborhood of 0 in $\partial\mathbb{H}^2$ and V a neighborhood of $P^{(0)}$ in $\mathcal{K}$ as in Lemma 4.2. Let U be a neighborhood of $P^{(0)}$ in $\mathcal{K}^* - \mathcal{E}$ and $c > 0$ be a constant as in Lemma 4.1. We choose $U_1, E = \{(z, u + i|z|^2) \in \partial\mathbb{H}^2 \mid |z| < c, |u| < c\}$, V such that $U_1 \subset U$ and $V \cap (\mathcal{K}^* - \mathcal{E}) \subset U$. Then by Lemma 4.2, we have $F_{c_1'',c_3'',e_1'',e_2''} = (F_{c_1',c_3',e_1',e_2'})^{***}_p$ with $|p| < c$, and by Lemma 4.1, $(c_1'', c_3'', e_1'', e_2'') = (c_1', c_3', e_1', e_2')$. $\square$

5 The proof of Theorem 1.1

Before proving Theorem 1.1, we mention a fact. Let σ_a and $\sigma_b \in Aut(\partial\mathbb{H}^2)$ defined as in (18) and $F \in Rat(\mathbb{H}^2, \mathbb{H}^5)$, then we can define a family of automorphism $\Theta_s = \sigma_{sb+(1-s)a}$, $0 \leq$

$s \leq 1$, and $\Psi_s = \tau^F_{sb+(1-s)a} \in Aut(\partial \mathbb{H}^5)$ defined as in (18) so that $\Psi_s \circ F \circ \Theta_s \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ satisfies $\Theta_0 = \sigma_a$, $\Theta_1 = \sigma_b$ and

$$\Psi_s \circ F \circ \Theta_s(0) = 0, \quad \forall s \in [0,1]. \tag{69}$$

Proof of Theorem 1.1: For any $F \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ with degree 2, by [CJX06] and Lemma 3.3, F is equivalent to another map $F_{\widetilde{c}_1,\widetilde{c}_3,\widetilde{e}_1,\widetilde{e}_2} \in \mathcal{K}^*$ with the minimum property (9). By Lemma 3.2 and 3.4, Theorem 1.1(i) is proved.

It remains to prove Theorem 1.1(ii). We need to show: if two distinct maps $F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}}$ and $F_{\widetilde{c}_1^{(0)},\widetilde{c}_3^{(0)},\widetilde{e}_1^{(0)},\widetilde{e}_2^{(0)}}$ in $\mathcal{K}^*$ are equivalent, then

$$(\widetilde{c}_1^{(0)}, \widetilde{c}_3^{(0)}, \widetilde{e}_1^{(0)}, \widetilde{e}_2^{(0)}) = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}). \tag{70}$$

We assume that $(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \notin \mathcal{E}$ where $\mathcal{E}$ is defined in Lemma 3.3; otherwise these two maps $F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}}$ and $F_{\widetilde{c}_1^{(0)},\widetilde{c}_3^{(0)},\widetilde{e}_1^{(0)},\widetilde{e}_2^{(0)}}$ cannot be equivalent.

Step 1. Construct a curve $\hat{L}_0$ Since $F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}}$ and $F_{\widetilde{c}_1^{(0)},\widetilde{c}_3^{(0)},\widetilde{e}_1^{(0)},\widetilde{e}_2^{(0)}}$ are equivalent,

$$F_{\widetilde{c}_1^{(0)},\widetilde{c}_3^{(0)},\widetilde{e}_1^{(0)},\widetilde{e}_2^{(0)}} = \Psi \circ F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}} \circ \Theta \tag{71}$$

where $\Theta \in Aut(\mathbb{H}^2)$ and $\Psi \in Aut(\mathbb{H}^5)$. Notice $\Psi \circ F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}} \circ \Theta(0) = 0$ holds.

We take a real analytic curve $L = L(s) \in \mathcal{K}^* - \mathcal{E}$, $0 \leq s < 1$, such that $L(0) = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)})$. In fact, since $(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \notin \mathcal{E}$ and $\mathcal{E}$ is closed, L could be taken in a neighborhood of $(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)})$.

By using automorphisms of balls, Cayley transformations and (69), we can take a real analytic family of automorphisms $\Theta_s \in Aut(\partial\mathbb{H}^2)$, $\Psi_s \in Aut(\partial\mathbb{H}^5)$, $s \in [0,1]$, such that when $s = 0$, $\Theta_0 = \Theta$, $\Psi_0 = \Psi$; when $s \in (0,1)$, $\Theta_s(0) \neq \infty$, $\Psi_s \circ F_{L(s)} \circ \Theta_s(0) = 0$; when $s = 1$, $\Theta_1 = Id$, $\Psi_1 = Id$. Then we define

$$\hat{L}_0(s) := \Psi_s \circ F_{L(s)} \circ \Theta_s \in Rat(\mathbb{H}^2, \mathbb{H}^5), \quad 0 \leq s \leq 1,$$

such that $\hat{L}_0(s)(0) = 0$ for all s, $F_{\hat{L}_0(0)} = \Psi \circ F_{L(0)} \circ \Theta$ and $\hat{L}_0(1) = L(1)$. Our goal is to show: $\hat{L}_0(s) = L(s)$, $\forall s \in [0,1]$, so that $\hat{L}_0(0) = L(0)$, i.e., (70) holds.

Step 2. Define a curve $\hat{L}(s)$ Notice that $\hat{L}_0$ must be in $\mathcal{K}$, namely, $F_{\hat{L}_0(s)}$ may geometric rank one at the origin for all $s \in [0,1]$, so that $(F_{\hat{L}_0(s)})^{***}$ is well defined for all $s \in [0,1]$.

Recall $\Theta_s(0) \neq \infty$ for any $s \in (0,1]$ and $\Theta_1 = Id$. Then for any $s \in (0,1]$, we denote $\psi(s) := \Theta_s(0) \in \partial\mathbb{H}^2$ with $\psi(1) = 0$, so that $\Theta_s = \sigma_{\psi(s)} \circ G_s$ where $\sigma_{\psi(s)}$ is defined as in (18) and $G_s \in Aut_0(\partial\mathbb{H}^2)$, i.e., we have a continuous map $\psi(s) \in \partial\mathbb{H}^2$ such that $\psi(1) = 0$ and

$$(F_{\hat{L}_0(s)})^{***} = \left(F_{L(s)}\right)^{***}_{\psi(s)}, \quad \forall s \in (0,1], \quad and \quad (F_{\hat{L}_0(1)})^{***} = F_{L(1)}. \tag{72}$$

Even though $(F_{\hat{L}_0(s)})^{***}$ is in $\mathcal{K}$ for any $s \in (0,1]$, it may not be in $\mathcal{K}^*$ because the minimum property (9) may not be satisfied. We claim that $(F_{\hat{L}_0(s)})^{***}$ is equivalent to another map $F_{\hat{L}(s)} \in \mathcal{K}^*$. More precisely, we want to find $q(s) \in \partial\mathbb{H}^2$ so that

$$F_{\hat{L}(s)} := (F_{\hat{L}_0(s)})^{***}_{q(s)} \in \mathcal{K}^*, \quad \forall s \in (0,1]. \tag{73}$$

To define such $q(s)$, we consider several cases below.

If $s = 1$, since $F_{L(1)} \in \mathcal{K}^*$ and $\psi(1) = 0$, we define $q(1) = 0$.

If $s \in (0,1]$ at which the minimum property (9) holds, we define $q(s) = 0$.

If $s \in (0,1]$ at which (9) does not hold, we consider a continuous curve $\Gamma^{(s)}(t) \in \partial\mathbb{H}^2 - \Xi_F$, $0 \le t \le 1$, with $\Gamma^{(s)}(0) = 0$ such that the function value $\mathcal{W}((F_{\hat{L}_0(s)})^{***}_{\Gamma^{(s)}(t)})$ is decreasing along $\Gamma^{(s)}$. We denote by ℓ_s the infimum of $\mathcal{W}((F_{\hat{L}_0(s)})^{***}_{\Gamma^{(s)}})$ over all such curves. Then there exists a sequence of curves $\Gamma^{(s)}_m$ in $\partial\mathbb{H}^2$ such that

$$\ell_s = \lim_{m\to\infty} \mathcal{W}\left((F_{L(s)})^{***}_{\Gamma^{(s)}_m(1)}\right). \tag{74}$$

Since $\mathcal{W}((F_{\hat{L}_0(s)})^{***}_p) = c_1(p)^2 - e_1(p) - e_2(p)$, the decreasing property implies $c_1(p), -e_1(p)$ and $-e_2(p)$ are bounded (cf. [CJX06, Step 1, §4]), so that $(F_{\hat{L}_0(s)})^{***}_{\Gamma^{(s)}_m(t)}$, regarded as a point, is inside $\mathcal{K}$ and is contained a compact subset of $\mathcal{K}$ that is independent of $\Gamma^{(s)}_m$. Therefore, by taking subsequences, we may assume that the limit $\lim_{m\to\infty}(F_{\hat{L}_0(s)})^{***}_{\Gamma^{(s)}_m(1)}$ exists as a point in $\mathcal{K}^*$ and that $\lim_{m\to\infty}\Gamma^{(s)}_m(1) \in \overline{\partial\mathbb{H}^2}$ exists. Let us define

$$F_{\hat{L}(s)} := \lim_{m\to\infty}(F_{\hat{L}_0(s)})^{***}_{\Gamma^{(s)}_m(1)} \in \mathcal{K}^*. \tag{75}$$

It remains to show that $q(s) \in \partial\mathbb{H}^2$ can be defined such that $F_{\hat{L}(s)} = (F_{\hat{L}_0(s)})^{***}_{q(s)}$.

By the choice of $L(1)$ and Corollary 4.3, there exists a neighborhood U of $L(1)$ in $\mathcal{K}^*$, such that if a point $(c_1, c_3, e_1, e_2) \in U$ such that F_{c_1,c_3,e_1,e_2} and $F_{L(1)}$ are equivalent, then $(c_1, c_3, e_1, e_2) = L(1)$.

Let us consider $\mathcal{K} \cap \mathbb{B}^4(\hat{L}_0(s), r)$, the intersection of $\mathcal{K}$ with the sphere in $\mathbb{C}^4$ which is centered at $\hat{L}_0(s)$ with radius r. We also consider $\mathcal{K}^* \cap \mathbb{B}^2(\hat{L}_0(s), r)$, the intersection of $\mathcal{K}^*$

with the sphere in $\mathbb{C}^2$ which is centered at $\hat{L}_0(s)$ with radius r. We take r so small that $\mathcal{K}^* \cap \mathbb{B}^2(\hat{L}_0(s), r) \subset U$.

Step 3. Claim on $F_{\hat{L}(s)} \to F_{\hat{L}_0(s)}$ Regarding $F_{\hat{L}(s)}$ as points in $\mathcal{K}$, we claim:

$$dist\left(F_{\hat{L}(s)},\quad F_{\hat{L}_0(s)}\right) \to 0, \quad as\ s \to 1. \tag{76}$$

Suppose (76) is not true. Then there exists a sequence $s_k \to 1$ such that

$$dist\left(F_{\hat{L}(s_k)},\quad F_{\hat{L}_0(s_k)}\right) \geq \delta_0, \quad as\ k \to \infty. \tag{77}$$

for a certain $\delta_0 > 0$. By (75), we can take integer m_{s_k} for each s_k such that

$$0 \leq \mathcal{W}((F_{\hat{L}_0(s_k)})^{***}_{\Gamma^{(s_k)}_{ms_k}(1)}) - \ell_{s_k} < \frac{1}{k},\ and\ dist\left((F_{\hat{L}_0(s_k)})^{***}_{\Gamma^{(s_k)}_{ms_k}(1)},\quad F_{\hat{L}(s_k)}\right) < \frac{1}{k}. \tag{78}$$

By (77) we have

$$dist\left((F_{\hat{L}_0(s_k)})^{***}_{\Gamma^{(s_k)}_{ms_k}(1)},\quad F_{\hat{L}_0(s_k)}\right) \geq \frac{\delta_0}{2}. \tag{79}$$

Then we can choose $r < \frac{\delta_0}{2}$. Then $\{(F_{\hat{L}_0(s_k)})^{***}_{\Gamma^{(s_k)}_{ms_k}}\}_{t\in[0,1]}$, regarded as a curve in $\mathcal{K}$ initiated from the point $F_{\hat{L}_0(s_k)}$, must be across through the sphere $(\mathcal{K} \cap \partial B^4(\hat{L}_0(s_k), r))$, i.e.,

$$\{(F_{\hat{L}_0(s_k)})^{***}_{\Gamma^{(s_k)}_{ms_k}}\}_{t\in[0,1]} \cap (\mathcal{K} \cap \partial B^4(\hat{L}_0(s_k), r)) \neq \emptyset. \tag{80}$$

Let $Q^{(s_k)}$ be a point in the intersection (80) and then $Q^{(s_k)} = (F_{\hat{L}_0(s_k)})^{***}_{\Gamma^{s_k}_{ms_k}(t_k)}$ for some $t_k \in [0,1]$. By taking subsequences, we assume $Q := \lim_{k\to\infty} Q^{(s_k)}$ exists. By the construction, we see that the F_Q is equivalent to $F_{L(1)}$ and

$$Q \in \mathcal{K}^*, \quad and \quad dist(Q, L(1)) = r.$$

Since $Q \in \mathcal{K}^* \cap \partial B^2(\hat{L}_0(1), r) \subset U$, by Corollary 4.3, $Q = L(1)$, i.e., $dist(Q, L(1)) = 0$, but this is a contradiction. Claim (76) is proved.

Step 4. Proof of $\hat{L}(s) \equiv L(s)$ From (76), we have

$$dist\left(F_{\hat{L}(s)},\quad F_{L(s)}\right) \to 0, \quad as\ s \to 1.$$

Since both $F_{\hat{L}(s)} \in \mathcal{K}^*$ and $F_{L(s)} \in \mathcal{K}^* - \mathcal{E}$ where $s \in (s_0, 1]$ for some $s_0 > 0$ such that $0 \leq 1 - s_0$ is sufficiently small, by Corollary 4.3 and the choice of $L(1)$, we conclude

$$F_{\hat{L}(s)} = F_{L(s)}, \quad \forall s \in (s_0, 1].$$

Repeating this process. Finally by continuity $F_{\hat{L}(s)} = F_{L(s)}$, $\forall s \in [0, 1]$. When restricted at 0, $F_{\hat{L}_0(0)} = F_{\hat{L}(0)} = F_{L(0)}$, so that (70) is proved. $\square$

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