

# A remark on optimal data spaces for classical solutions of $\bar{\partial}$

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## Abstract

We study the minimal regularity required on the datum to guarantee the existence of classical  $C^1$  solutions to the inhomogeneous Cauchy–Riemann equations on planar domains.

## 1 Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{C}$  with  $C^{1,\alpha}$  boundary, where  $\alpha > 0$ , and let  $\mathbf{f}$  be a  $(0, 1)$  form on  $\Omega$ . Consider the Cauchy–Riemann equations

$$\bar{\partial}u = \mathbf{f} \quad \text{on } \Omega. \quad (1)$$

The standard singular integral theory (see [5, Chapter 1]) implies the solvability of (1) in several function spaces. For instance, if  $\mathbf{f} \in L^p(\Omega)$ ,  $1 < p \leq 2$ , then there exists a weak solution  $u \in L^q(\Omega)$ ,  $q < \frac{2p}{2-p}$  to (1). Moreover, if  $\mathbf{f} \in L^p(\Omega)$ ,  $p > 2$ , then there exists a weak solution  $u \in C^\gamma(\Omega)$ ,  $\gamma = \frac{p-2}{p}$ . On the other hand, if  $\mathbf{f} \in C^\alpha(\Omega)$  for some  $0 < \alpha < 1$ , then (1) admits a classical (i.e.,  $C^1(\bar{\Omega})$ ) solution  $u \in C^{1,\alpha}(\Omega)$ . The purpose of the note is to study the minimal regularity required on the datum  $\mathbf{f}$  to guarantee the existence of a classical solution to (1). The following example shows that continuity is not sufficient.

**Example 1.1.** Consider the equation  $\bar{\partial}u = \mathbf{f}_\nu = f_\nu d\bar{z}$  on the disk  $D(0, \frac{1}{2}) := \{z \in \mathbb{C} : |z| < \frac{1}{2}\}$ , where  $\nu > 0$  is fixed and

$$f_\nu(z) := \begin{cases} \frac{\bar{z}}{\bar{z} \ln^\nu |z|^2} & z \neq 0 \\ 0 & z = 0. \end{cases} \quad (2)$$

Clearly,  $\mathbf{f}_\nu \in C(\overline{D(0, \frac{1}{2})})$ . However (see the proof at the end of the paper) if  $\nu \leq 1$ , then there exists no solution  $u \in C^1(\overline{D(0, \frac{1}{2})})$ .

We consider the following subspaces of  $C(\bar{\Omega})$  consisting of functions satisfying a logarithmic continuity condition. Similar subspaces of  $C^\alpha(\Omega)$ ,  $0 < \alpha < 1$  with additional logarithmic continuity were discussed in [4]. As will be seen in Corollary 3.5 and Example 3.6, such refined subspaces naturally capture the optimal regularity of the Cauchy singular integral operator in the critical case  $C(\bar{\Omega})$ .

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**Definition 1.2.** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $k \in \mathbb{Z}^+ \cup \{0\}$ ,  $\nu \in \mathbb{R}^+$ . A function  $f \in C^k(\Omega)$  is said to be in  $C^{k, \text{Log}^\nu L}(\Omega)$  if

$$\|f\|_{C^{k, \text{Log}^\nu L}(\Omega)} := \sum_{|\gamma|=0}^k \sup_{w \in \Omega} |D^\gamma f(w)| + \sum_{|\gamma|=k} \sup_{w, w+h \in \Omega} |D^\gamma f(w+h) - D^\gamma f(w)| |\ln |h||^\nu < \infty.$$

A  $(0, 1)$  form  $\mathbf{f}$  is said to be in  $C^{k, \text{Log}^\nu L}(\Omega)$  if all its components are in  $C^{k, \text{Log}^\nu L}(\Omega)$ .

When  $k = 0$ , the space  $C^{0, \text{Log}^\nu L}(\Omega)$  is abbreviated as  $C^{\text{Log}^\nu L}(\Omega)$ . For any  $k \in \mathbb{Z}^+ \cup \{0\}$ ,  $0 < \nu < \mu$ , and  $0 < \alpha < 1$ , we have

$$C^{k, \alpha}(\Omega) \hookrightarrow C^{k, \text{Log}^\mu L}(\Omega) \hookrightarrow C^{k, \text{Log}^\nu L}(\Omega) \hookrightarrow C^k(\bar{\Omega}),$$

where every inclusion map is a continuous embedding. In our main result we prove, in particular, that a classical solution to (1) exists whenever  $\mathbf{f}$  is in  $C^{\text{Log}^\nu L}(\Omega)$  for some  $\nu > 1$ . Here is the precise statement.

**Theorem 1.3.** *Let  $\Omega \subset \mathbb{C}$  be a bounded domain with  $C^{1, \alpha}$  boundary, where  $\alpha > 0$ . Assume that  $f \in C^{\text{Log}^\nu L}(\Omega)$  for some  $\nu > 1$ . Then there exists a solution  $u \in C^{1, \text{Log}^{\nu-1} L}(\Omega)$  to  $\bar{\partial}u = \mathbf{f} := fd\bar{z}$  such that  $\|u\|_{C^{1, \text{Log}^{\nu-1} L}(\Omega)} \leq C\|\mathbf{f}\|_{C^{\text{Log}^\nu L}(\Omega)}$ , where  $C$  depends only on  $\Omega$  and  $\nu$ . In particular,  $u \in C^1(\bar{\Omega})$ , with  $\|u\|_{C^1(\bar{\Omega})} \leq C\|\mathbf{f}\|_{C^{\text{Log}^\nu L}(\Omega)}$ .*

Example 1.1 shows that the assumption  $\nu > 1$  in Theorem 1.3 cannot be relaxed. In this sense, Theorem 1.3 identifies the largest possible data set guaranteeing the existence of classical solutions to (1).

The following example shows another way in which Theorem 1.3 is sharp: the loss of 1 in the order of Log-continuity of the solution is optimal.

**Example 1.4.** Let  $\nu > 0$  be fixed, and consider the equation  $\bar{\partial}u = \mathbf{f}_\nu = f_\nu d\bar{z}$  on the disk  $D(0, \frac{1}{2})$ , where  $f_\nu$  is defined by (2). Then  $\mathbf{f}_\nu \in C^{\text{Log}^\nu L}(D(0, \frac{1}{2}))$ . However (see the proof at the end of the paper) if  $\nu > 1$ , there does not exist a weak solution in  $C^{1, \text{Log}^\mu L}(D(0, \frac{1}{2}))$  for any  $\mu > \nu - 1$ .

Finally, we mention that Coffman, together with the second author and the third author of this paper, constructed in [2, Example 3.3] a  $(0, 1)$  form  $\mathbf{f} \in C(\mathbb{C})$  such that  $\bar{\partial}u = \mathbf{f}$  has a solution that fails to be in  $C^1$  near 0, yet is *differentiable* (i.e., it has a real linear approximation everywhere) on  $\mathbb{C}$ .

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## 2 Preliminaries on the integral operators $T$ and ${}^2T$

Let  $\Omega$  be a bounded domain in  $\mathbb{C}$  with  $C^{1, \alpha}$  boundary, where  $\alpha > 0$ . Given a function  $f \in C(\bar{\Omega})$ , define

$$Tf(z) := \frac{-1}{2\pi i} \int_{\Omega} \frac{f(\zeta)}{\zeta - z} d\bar{\zeta} \wedge d\zeta, \quad z \in \Omega.$$

It is well known that  $T$  is a solution operator to  $\bar{\partial}$  on planar domains in several function spaces (see for instance [1] and [5]). For  $f \in C(\bar{\Omega})$ , we have

$$\frac{\partial}{\partial \bar{z}} T f = f, \quad \frac{\partial}{\partial z} T f = p.v. \frac{-1}{2\pi i} \int_{\Omega} \frac{f(\zeta)}{(\zeta - \cdot)^2} d\bar{\zeta} \wedge d\zeta =: \Pi f \quad (3)$$

in  $\Omega$  in the sense of distributions [5, Theorem 1.32]. Here *p.v.* represents the principal value.

In their inspiring paper [3], Nijenhuis and Woolf introduced the related integral operator  ${}^2T$ . For functions  $f \in C^\alpha(\Omega)$ , where  $0 < \alpha < 1$ , define

$${}^2T f(z) := \frac{-1}{2\pi i} \int_{\Omega} \frac{f(\zeta) - f(z)}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta, \quad z \in \Omega.$$

${}^2T$  is a bounded operator from the space  $C^\alpha(\Omega)$  to itself whenever  $0 < \alpha < 1$  (see [3, Appendix 6.1.e] for a proof in the case of  $\Omega$  being a disk, and [5, Theorem 1.32] for the general case). The next proposition shows that  ${}^2T f$  is well-defined also for functions  $f$  in Log-continuous spaces, and describes the connection between the integral operators  $\Pi$  and  ${}^2T$ .

**Proposition 2.1.** *Let  $\Omega \subset \mathbb{C}$  be a bounded domain with  $C^{1,\alpha}$  boundary, where  $\alpha > 0$ . For every  $f \in C^{\text{Log}^\nu L}(\Omega)$ , with  $\nu > 1$ , the function  ${}^2T f$  is well-defined in  $\Omega$ . Moreover, letting*

$$\Psi(z) := \frac{1}{2\pi i} \int_{b\Omega} \frac{1}{\zeta - z} d\bar{\zeta}, \quad z \in \Omega. \quad (4)$$

we have

$$\Pi f(z) = {}^2T f(z) - f(z)\Psi(z), \quad z \in \Omega. \quad (5)$$

In the special case of  $\Omega$  being a disk centered at 0, then

$$\Pi f(z) = {}^2T f(z), \quad z \in \Omega. \quad (6)$$

To prove Proposition 2.1 we need the two elementary lemmas below. Throughout the paper, unless otherwise specified, we use  $C$  to represent a positive constant which depends only on  $\Omega$  or  $\nu$ , and which may be different at each occurrence.

**Lemma 2.2.** *Let  $\Omega \subset \mathbb{C}$  be a bounded domain with  $C^{1,\alpha}$  boundary, where  $0 < \alpha < 1$ . Then  $\Psi$  defined in (4) is in  $C^\alpha(\Omega)$ . Moreover, if  $\Omega$  is a disk centered at 0, then  $\Psi \equiv 0$  on  $\Omega$ .*

*Proof.* Write  $\partial\Omega = \cup_{j=1}^N \Gamma_j$ , where each Jordan curve  $\Gamma_j$  is connected, positively oriented with respect to  $\Omega$ , and of arclength  $s_j$ . Let  $\zeta(s)$  be a parameterization of  $b\Omega$  in terms of the arclength  $s$  such that, for every  $j$ ,  $\zeta|_{s \in [\sum_{m=1}^{j-1} s_m, \sum_{m=1}^j s_m]}$  is a  $C^{1,\alpha}$  parametrization of  $\Gamma_j$ . In particular,  $\bar{\zeta}' = \frac{1}{\zeta'} \neq 0$  on  $b\Omega$ .

For any  $z \in \Omega$ ,

$$\Psi(z) = \frac{1}{2\pi i} \int_0^{s_0} \frac{1}{\zeta(s) - z} \bar{\zeta}'(s) ds = \frac{1}{2\pi i} \int_0^{s_0} \frac{(\bar{\zeta}'(s))^2}{\zeta(s) - z} \zeta'(s) ds =: S f_0(z).$$

Here  $f_0$  is a function on  $b\Omega$  satisfying  $f_0(\zeta(s)) = (\bar{\zeta}'(s))^2$  for  $s \in [0, s_0)$ , and  $S$  is the Cauchy singular integral defined by

$$S f(z) := \frac{1}{2\pi i} \int_{b\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \Omega,$$

for any  $f \in C(b\Omega)$ . Consequently,  $f_0 \in C^\alpha(b\Omega)$ .

Recall that  $S$  sends  $C^\alpha(b\Omega)$  into  $C^\alpha(\Omega)$ , with  $\|Sf\|_{C^\alpha(\Omega)} \leq C\|f\|_{C^\alpha(b\Omega)}$  for some constant  $C$  dependent only on  $\Omega$  (see [5, Theorem 1.10]). We then have  $Sf_0 \in C^\alpha(\Omega)$  with

$$\|\Psi\|_{C^\alpha(\Omega)} = \|Sf_0\|_{C^\alpha(\Omega)} \leq C\|f_0\|_{C^\alpha(b\Omega)} \leq C.$$

If  $\Omega \subset \mathbb{C}$  is a disk centered at 0 with radius  $R$ , then using the relation  $\bar{\zeta} = R^2/\zeta$  on  $b\Omega$  and the Residue Theorem, we get

$$\Psi(z) = -\frac{1}{2\pi i} \int_{b\Omega} \frac{R^2 d\zeta}{\zeta^2(\zeta - z)} = \begin{cases} R^2(\frac{1}{z^2} - \frac{1}{z^2}) & \text{if } z \in \Omega, z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

Hence  $\Psi \equiv 0$  in  $\Omega$ . ■

*Remark 2.3.* In [5, Theorem 1.32], Vekua proved  $\Psi \in C^\alpha(\Omega)$  by writing

$$\Psi(z) = \frac{1}{2\pi i} \int_{b\Omega} \frac{d\bar{\zeta}}{\zeta - z} = \frac{1}{2\pi i} \int_{b\Omega} \frac{\bar{\zeta} d\zeta}{(\zeta - z)^2} = \frac{1}{2\pi i} \left( - \int_{b\Omega} \frac{\bar{\zeta} d\zeta}{\zeta - z} \right)' = -(S\bar{z})'$$

for  $z \in \Omega$ . Here the second identity makes use of Stokes' Theorem. However, in order that  $\Psi \in C^\alpha(\Omega)$ , or equivalently  $S\bar{z} \in C^{1,\alpha}(\Omega)$ , Vekua's approach seems to necessarily require  $b\Omega \in C^{2,\alpha}$ , instead of the claimed  $C^{1,\alpha}$  boundary regularity in his theorem. By using the parameterization method as in the proof Lemma 2.2, we were able to successfully lower that unnecessary boundary regularity assumption to the desired  $C^{1,\alpha}$ .

**Lemma 2.4.** *Let  $\nu \in \mathbb{R}^+$ . There exists a constant  $C$  depending only on  $\nu$  such that, for every choice of  $h, h_0$  with  $0 < h \leq h_0 < 1$ , the following hold:*

1.  $\int_h^{h_0} s^{-2} |\ln s|^{-\nu} ds \leq h^{-1} |\ln h|^{-\nu}$ .
2. If  $\nu > 1$ , then  $\int_0^h s^{-1} |\ln s|^{-\nu} ds \leq C |\ln h|^{1-\nu}$ .

*Proof.* 1. Integration by parts yields, when  $\nu > 0$  and  $0 < h \leq h_0 < 1$ ,

$$\int_h^{h_0} s^{-2} |\ln s|^{-\nu} ds = h^{-1} |\ln h|^{-\nu} - h_0^{-1} |\ln h_0|^{-\nu} - \nu \int_h^{h_0} s^{-2} |\ln s|^{-\nu-1} ds.$$

In particular,

$$\int_h^{h_0} s^{-2} |\ln s|^{-\nu} ds \leq h^{-1} |\ln h|^{-\nu}.$$

2. Direct integration gives, for  $\nu > 1$  and  $0 < h < 1$ ,

$$\int_0^h s^{-1} |\ln s|^{-\nu} ds = \frac{1}{\nu - 1} |\ln h|^{-\nu+1},$$

which proves the second part of the lemma. ■

*Proof of Proposition 2.1:* Fix  $z \in \Omega$ , and let  $h_0$  be such that  $0 < h_0 < 1$ . By Lemma 2.4 part 2,

$$\begin{aligned}
|2\pi ({}^2Tf(z))| &= \left| \int_{\Omega} \frac{f(\zeta) - f(z)}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta \right| \\
&\leq \left| \int_{D(z, h_0) \cap \Omega} \frac{f(\zeta) - f(z)}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta \right| + \left| \int_{\Omega \setminus D(z, h_0)} \frac{f(\zeta) - f(z)}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta \right| \\
&\leq C \|f\|_{C^{Log^\nu L}(\Omega)} \int_0^{h_0} |\ln s|^{-\nu} s^{-1} ds + Ch_0^{-2} \|f\|_{C(\Omega)} \\
&\leq C \|f\|_{C^{Log^\nu L}(\Omega)} |\ln h_0|^{-\nu+1} \\
&\leq C \|f\|_{C^{Log^\nu L}(\Omega)}.
\end{aligned} \tag{7}$$

In particular,  ${}^2Tf$  is well-defined pointwise in  $\Omega$ .

A direct computation gives

$$\begin{aligned}
p.v. \int_{\Omega} \frac{f(\zeta)}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta &= \lim_{\epsilon \rightarrow 0} \int_{\Omega \setminus D(z, \epsilon)} \frac{f(\zeta)}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta \\
&= \lim_{\epsilon \rightarrow 0} \left( \int_{\Omega \setminus D(z, \epsilon)} \frac{f(\zeta) - f(z)}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta + f(z) \int_{\Omega \setminus D(z, \epsilon)} \frac{1}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta \right).
\end{aligned} \tag{8}$$

Note that

$$\left| \frac{f(\zeta) - f(z)}{(\zeta - z)^2} \right| \leq C \|f\|_{C^{Log^\nu L}(\Omega)} |\zeta - z|^{-2} |\ln |\zeta - z||^{-\nu}. \tag{9}$$

By Lemma 2.4 part 1, the function on the right side of (9) belongs to  $L^1(\Omega)$ . Hence the dominated convergence theorem implies

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega \setminus D(z, \epsilon)} \frac{f(\zeta) - f(z)}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta = -2\pi i ({}^2Tf(z)). \tag{10}$$

On the other hand,

$$\int_{\Omega \setminus D(z, \epsilon)} \frac{1}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta = \int_{b\Omega} \frac{1}{\zeta - z} d\bar{\zeta} - \int_{bD(z, \epsilon)} \frac{1}{\zeta - z} d\bar{\zeta} = \int_{b\Omega} \frac{1}{\zeta - z} d\bar{\zeta}. \tag{11}$$

Here the first equality makes use of Stokes' Theorem, and the second equality follows from Lemma 2.2. Combining (8), (10), and (11), we conclude

$$\Pi f(z) = {}^2Tf(z) - f(z)\Psi(z), \quad z \in \Omega,$$

which proves (5).

In the case of  $\Omega$  being a disk, (6) follows from (5) together with Lemma 2.2. ■

### 3 Optimal bounds for ${}^2T$ and $T$ in Log-continuous spaces

In this section we study the boundedness of the operator  ${}^2T$  in the space  $C^{Log^\nu L}(\Omega)$ . We will show in Theorem 3.4 that  ${}^2T$  is a bounded linear operator from  $C^{Log^\nu L}(\Omega)$  into  $C^{Log^{\nu-1} L}(\Omega)$  when  $\nu > 1$ . As a consequence, we derive our Main Theorem 1.3.

We begin by pointing out that  ${}^2T$  does not send  $C(\bar{\Omega})$  into itself, as shown by the following example.

**Example 3.1.** Let  $f$  be the function defined on the disk  $D(0, \frac{1}{2})$  by

$$f(z) = \begin{cases} \frac{z}{\bar{z} \ln |z|} & z \neq 0 \\ 0 & z = 0. \end{cases}$$

Then  $f \in C^{Log^1 L}(D(0, \frac{1}{2})) \subset C(\overline{D(0, \frac{1}{2})})$ . However,

$${}^2Tf(0) = \int_{D(0, \frac{1}{2})} \frac{1}{|\zeta|^2 \ln |\zeta|} d\zeta \wedge d\bar{\zeta} = C \int_0^{\frac{1}{2}} \frac{1}{r \ln r} dr = -\infty,$$

and therefore  ${}^2Tf(0)$  is not defined. In particular,  ${}^2Tf \notin C(\overline{D(0, \frac{1}{2})})$ .

The following important inequality is proved in [3].

**Lemma 3.2.** [3, Appendix 6.1d] *Let  $z \in D(0, R)$  and  $r > 0$ . Then*

$$\left| \int_{D(0, R) \setminus D(z, r)} \frac{1}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta \right| \leq 8\pi. \quad (12)$$

Note that the disk  $D(z, r)$  may or may not be completely contained in the ambient disk  $D(0, R)$ . Moreover, the bound in (12) is independent of both  $R$  and  $r$ . The proof of Lemma 3.2 in [3] relies on the symmetries of the disk, and unfortunately does not carry through when the disk  $D(0, R)$  is replaced by more general domains. In the next result, using Lemma 2.2, we achieve a generalization of Lemma 3.2 to arbitrary smoothly bounded planar domains.

**Lemma 3.3.** *Let  $\Omega \subset \mathbb{C}$  be a bounded domain with  $C^{1, \alpha}$  boundary, where  $\alpha > 0$ . Let  $z \in \Omega$  and  $r > 0$ . There exists a positive constant  $C$  depending only on  $\Omega$  such that*

$$\left| \int_{\Omega \setminus D(z, r)} \frac{1}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta \right| \leq C.$$

*Proof.* Let  $R > 0$  be such that  $\Omega \subset\subset D(0, R)$ . Since  $D(0, R) \setminus D(z, r) = (D(0, R) \setminus \Omega) \sqcup (\Omega \setminus D(z, r))$ , Lemma 3.2 yields

$$\begin{aligned} \left| \int_{\Omega \setminus D(z, r)} \frac{1}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta \right| &= \left| \int_{D(0, R) \setminus D(z, r)} \frac{1}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta - \int_{D(0, R) \setminus \Omega} \frac{1}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta \right| \\ &\leq 8\pi + \left| \int_{D(0, R) \setminus \Omega} \frac{1}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta \right|. \end{aligned} \quad (13)$$

By Stokes' Theorem and Lemma 2.2,

$$\int_{D(0, R) \setminus \Omega} \frac{1}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta = \int_{bD(0, R)} \frac{1}{\zeta - z} d\bar{\zeta} - \int_{b\Omega} \frac{1}{\zeta - z} d\bar{\zeta} = - \int_{b\Omega} \frac{1}{\zeta - z} d\bar{\zeta} \in C^\alpha(\Omega). \quad (14)$$

In particular,

$$\left| \int_{D(0, R) \setminus \Omega} \frac{1}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta \right| \leq C$$

for some constant  $C$  independent of  $r$ . ■

We are now ready to prove the main theorem of this section, following the ideas in [3, Appendix, 6.1e].

**Theorem 3.4.** *Let  $\Omega \subset \mathbb{C}$  be a bounded domain with  $C^{1,\alpha}$  boundary, where  $\alpha > 0$ . Then  ${}^2T$  is a bounded linear operator from  $C^{\text{Log}^\nu L}(\Omega)$  into  $C^{\text{Log}^{\nu-1} L}(\Omega)$ ,  $\nu > 1$ .*

*Proof.* Let  $f \in C^{\text{Log}^\nu L}(\Omega)$  and  $z \in \Omega$ . Let  $h_0$  be fixed with  $0 < h_0 < \frac{1}{2}$ . By (7),

$$|{}^2Tf(z)| \leq C\|f\|_{C^{\text{Log}^\nu L}(\Omega)}.$$

Next, given  $z, z+h \in \Omega$ , where  $|h| \leq h_0$ , set  $D_0 := \Omega \cap D(z, 2|h|)$ . Then

$$\begin{aligned} & |{}^2Tf(z) - {}^2Tf(z+h)| \\ &= \left| \int_{\Omega} \frac{f(\zeta) - f(z)}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta - \int_{\Omega} \frac{f(\zeta) - f(z+h)}{(\zeta - z - h)^2} d\bar{\zeta} \wedge d\zeta \right| \\ &= \left| \int_{\Omega \setminus D_0} (f(\zeta) - f(z+h)) \left[ \frac{1}{(\zeta - z)^2} - \frac{1}{(\zeta - z - h)^2} \right] d\bar{\zeta} \wedge d\zeta - \int_{\Omega \setminus D_0} \frac{f(z) - f(z+h)}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta \right. \\ &\quad \left. + \int_{D_0} \frac{f(\zeta) - f(z)}{(\zeta - z)^2} d\bar{\zeta} \wedge d\zeta - \int_{D_0} \frac{f(\zeta) - f(z+h)}{(\zeta - z - h)^2} d\bar{\zeta} \wedge d\zeta \right| =: |I_1 + I_2 + I_3 + I_4|. \end{aligned}$$

Let  $\gamma$  be the segment connecting  $z$  and  $z+h$ . We can rewrite  $|I_1|$  as follows:

$$\begin{aligned} |I_1| &= 2 \left| \int_{\Omega \setminus D_0} (f(\zeta) - f(z+h)) \left[ \int_{\gamma} \frac{dw}{(\zeta - w)^3} \right] d\bar{\zeta} \wedge d\zeta \right| \\ &= 2 \left| \int_{\gamma} dw \int_{\Omega \setminus D_0} \frac{f(\zeta) - f(z+h)}{(\zeta - w)^3} d\bar{\zeta} \wedge d\zeta \right|. \end{aligned}$$

Note, if  $\zeta \in \Omega \setminus D_0$  and  $w \in \gamma$ , then  $|\zeta - w| \geq |\zeta - z| - |z - w| \geq |h|$ . Thus

$$|\zeta - z - h| \leq |\zeta - w| + |w - z - h| \leq |\zeta - w| + |h| \leq 2|\zeta - w|.$$

Writing  $\zeta = w + se^{i\theta}$  we see, in particular, that  $\Omega \setminus D_0 \subset \{\zeta \in \mathbb{C} : |h| < s < 2R\}$ , where  $R$  is the diameter of  $\Omega$ . By Lemma 2.4 part 1 and the fact that  $f \in C^{\text{Log}^\nu L}(\Omega)$ ,

$$\begin{aligned} |I_1| &\leq C\|f\|_{C^{\text{Log}^\nu L}(\Omega)}|h| \left| \int_{|h|}^{2R} |\ln s|^{-\nu} s^{-2} ds \right| \\ &\leq C\|f\|_{C^{\text{Log}^\nu L}(\Omega)}|h| \left( \left| \int_{|h|}^{h_0} |\ln s|^{-\nu} s^{-2} ds \right| + \left| \int_{h_0}^{2R} |\ln s|^{-\nu} s^{-2} ds \right| \right) \\ &\leq C\|f\|_{C^{\text{Log}^\nu L}(\Omega)}(|\ln |h||^{-\nu} + |h|) \\ &\leq C\|f\|_{C^{\text{Log}^\nu L}(\Omega)} |\ln |h||^{-\nu+1}. \end{aligned}$$

The estimate for  $|I_2|$  follows directly from Lemma 3.3, and the estimate of  $|I_3|$  is a straightforward consequence of Lemma 2.4 part 2, as shown below:

$$|I_3| \leq C\|f\|_{C^{\text{Log}^\nu L}(\Omega)} \left| \int_0^{|h|} |\ln s|^{-\nu} s^{-1} ds \right| \leq C\|f\|_{C^{\text{Log}^\nu L}(\Omega)} |\ln |h||^{-\nu+1}.$$

Finally,  $I_4$  is estimated in the same way as  $I_3$ . ■

Combining Proposition 2.1 with Theorem 3.4, we obtain the next corollary on the solution operator  $T$ , from which our Main Theorem 1.3 follows immediately.

**Corollary 3.5.** *Let  $\Omega$  be a bounded domain in  $\mathbb{C}$  with  $C^{1,\alpha}$  boundary for some  $\alpha > 0$ . Then  $T$  is a bounded linear operator from  $C^{\text{Log}^\nu L}(\Omega)$  into  $C^{1,\text{Log}^{\nu-1}L}(\Omega)$ ,  $\nu > 1$ .*

*Proof.* Let  $f \in C^{\text{Log}^\nu L}(\Omega)$ . Then  $\Pi f \in C^{\text{Log}^{\nu-1}L}(\Omega)$  by Proposition 2.1 and Theorem 3.4. Hence, by (3),  $Tf$  is a continuous function whose weak derivatives are in  $C^{\text{Log}^{\nu-1}L}(\Omega)$ . Using a standard mollifier argument, we further know that (3) holds pointwise in  $\Omega$ , and in particular  $Tf \in C^1(\Omega)$ . The  $C^{1,\text{Log}^{\nu-1}L}(\Omega)$  estimate for  $Tf$  is again a consequence of Theorem 3.4. ■

Example 3.1 implies that the assumption  $\nu > 1$  in Theorem 3.4 cannot be dropped. Furthermore, the following example shows that the loss by 1 in the order of Log-continuity in Theorem 3.4 and Corollary 3.5 is optimal in a very precise sense.

**Example 3.6.** Let  $f_\nu$  be defined by (2) for some  $\nu > 1$ . Then  $f_\nu \in C^{\text{Log}^\nu L}(D(0, \frac{1}{2}))$ . However,

1.  $Tf_\nu \notin C^{1,\text{Log}^\mu L}(D(0, \frac{1}{2}))$  for any  $\mu > \nu - 1$ .
2.  ${}^2Tf_\nu \notin C^{\text{Log}^\mu L}(D(0, \frac{1}{2}))$  for any  $\mu > \nu - 1$ .

*Proof.* For the first statement, first check that  $u_\nu = \frac{z}{(1-\nu)\ln^{\nu-1}|z|^2}$  satisfies  $\frac{\partial}{\partial \bar{z}}u = f_\nu$  on  $D(0, \frac{1}{2})$ . Hence there exists a holomorphic function  $h$  on  $D(0, \frac{1}{2})$  such that  $Tf_\nu = u_\nu + h$ . In particular,  $Tf_\nu$  has the same regularity at  $z = 0$  as  $u_\nu$ , which is not in  $C^{1,\text{Log}^\mu L}$  near 0 for any  $\mu > \nu - 1$ . The second statement is a direct consequence of the first, in view of the identity  ${}^2Tf_\nu = \frac{\partial}{\partial \bar{z}}Tf_\nu$  on  $D(0, \frac{1}{2})$ . ■

*Proof of Example 1.1 and Example 1.4:* One can check that

$$u_\nu(z) := \begin{cases} \frac{z}{(1-\nu)\ln^{\nu-1}|z|^2} & \nu \neq 1 \\ z \ln(|\ln|z|^2|) & \nu = 1 \end{cases}$$

is a solution to  $\bar{\partial}u = \mathbf{f}_\nu$  on  $D(0, \frac{1}{2})$ . Let  $u$  be any weak solution to  $\bar{\partial}u = \mathbf{f}_\nu$  on  $D(0, \frac{1}{2})$ . Then  $u$  differs from  $u_\nu$  by a holomorphic function, which is always smooth near 0. However, when  $\nu \leq 1$ ,  $u_\nu$  fails to be  $C^1$  near 0. Moreover, for any  $\mu > \nu - 1 > 0$ ,  $u_\nu$  (and therefore  $u$ ) is not in  $C^{1,\text{Log}^\mu L}$  near 0. ■

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