# A Residue-type phenomenon and its applications to higher order nonlinear systems of Poisson type

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**Abstract**

In this paper, we establish a Residue-type phenomenon for the fundamental solution of the Laplacian. With the aid of the formula, we derive a higher order derivative formula for the Newtonian potential and its Hölder estimates with a gain of two derivatives. The estimates allow us to obtain the solvability of a type of higher order nonlinear Poisson system.

## 1 Introduction and the main theorems

Denote by $\mathbf{B}_R$ the ball of radius $R$ centered at 0 in $\mathbb{R}^n$, and by $\Gamma(x) := \frac{1}{(2-n)\omega_n|x|^{n-2}}$ the fundamental solution of the Laplacian in $\mathbb{R}^n \setminus \{0\}, n \geq 3$, where $\omega_n$ is the surface area of the unit sphere in $\mathbb{R}^n$. Motivated by the classical Residue theorem for holomorphic functions on the complex plane, we establish for the fundamental solution of the Laplacian an analogous phenomenon which seems to have been overlooked in the literature. In detail, denote by $\mathcal{P}_k$ the space of polynomials of degree $k$ restricted in $\mathbf{B}_R$, and by $\mathbb{Z}^+$ the set of nonnegative integers. We have, making use of zonal spherical harmonics, the following Residue-type theorem for $\Gamma$.

**Theorem 1.1.** *For any $f \in \mathcal{P}_k$ with $k \in \mathbb{Z}^+$,*

$$\int_{\partial \mathbf{B}_R} \Gamma(\cdot - y) f(y) d\sigma_y \in \mathcal{P}_k.$$

*Here $d\sigma_y$ is the surface area element of $\partial \mathbf{B}_R$.*

The Residue-type theorem allows us to obtain the explicit higher order derivative formula for the Newtonian potential on $\mathbf{B}_R$. Recall for any integrable function $f$, the Newtonian potential on $\mathbf{B}_R$ is defined by $\mathcal{N}(f) := \int_{\mathbf{B}_R} \Gamma(\cdot - y) f(y) dy = \Gamma * f$.

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**Theorem 1.2.** *Let $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$ with $k \in \mathbb{Z}^+, 0 < \alpha < 1$, and let $\beta$ be a multi-index with $|\beta| = k+2$. Then $D^\beta \mathcal{N}(f)(x)$ exists for $x \in \mathbf{B}_R$. Moreover, for $x \in \mathbf{B}_R$,*

$$D^\beta \mathcal{N}(f)(x) = \int_{\mathbf{B}_R} D_x^\beta \Gamma(x-y)\big(f(y) - T_k^x(f)(y)\big)dy - \sum_{|\mu|=k} c_\mu D^\mu f(x),$$

*where $T_k^x(f)$ is the $k$-th order power series expansion of $f$ at $x$ and $c_\mu$ is some constant dependent only on $\mu$ and $n$.*

When $k = 0$, the theorem is reduced to the classical second order derivative formula for the Newtonian potential (cf. [Fr]). We note that different from the explicit derivative formula as in [GT], the formula in Theorem 1.2 does not require to shrink the domain, and hence allows us to derive a uniform Hölder estimates of $(k+2)$-th order derivatives for the Newtonian potential on the function space $\mathcal{C}^{k,\alpha}$. Here the Hölder norm $\| \cdot \|_\alpha$ is in the sense of (3) whose definition is postponed till Section 2.

**Theorem 1.3.** *If $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$ with $k \in \mathbb{Z}^+, 0 < \alpha < 1$, then $\mathcal{N}(f) \in \mathcal{C}^{k+2,\alpha}(\mathbf{B}_R)$. Moreover, for any multi-index $\beta$ with $|\beta| = k+2$,*

$$\| D^\beta \mathcal{N}(f) \|_\alpha \leq C \sum_{|\gamma|=k} \| D^\gamma f \|_\alpha,$$

*where $C$ is a constant dependent only on $k, \alpha$ and $n$. In particular, $C$ is independent of $R$.*

Consequently, we investigate the existence of solutions $u = (u_1, \ldots, u_N)$ to the following nonlinear system in $\mathbb{R}^n$, $n \geq 3$:

$$\triangle^m u(x) = a(x, u, \nabla u, \ldots, \nabla^{2m} u). \tag{1}$$

Here $1 \leq m \in \mathbb{Z}^+$, $\nabla^j u$ represents all $j$-th order partial derivatives of the components of $u$, and $a := (a_1, \ldots, a_N)$ is a vector-valued function on $x$ and the derivatives of $u$ up to order $2m$. Label the variables of $a$ by $(p_{-1}, p_0, p_1, \ldots, p_{2m})$, with $p_{-1}$ representing the position of the variable $x$ and $p_j$ representing the position of $\nabla^j u$, $0 \leq j \leq m$.

The solvability for linear equations was widely explored since the counterexample of Hans Lewy [Lw] in 1957. See [Ho1], [NT], [Mo], [Ho2], [BF], [Ln], [De] and the references therein. Unlike linear equations, there is in general no systematic theory about the solvability for nonlinear equations or systems. Recently, Pan investigated the existence problem to (1) through Cauchy-Riemann operator in the case when $n = 2$ in [Pan1] and when $m = 1$ for general $n \geq 3$ in [Pan2].

We call a family of vectors $\{c_j\}_{0 \leq j \leq 2m-1}$ **an appropriate jet** if there exists a vector-valued function such that for each $0 \leq j \leq 2m-1$, $c_j$ is the vector whose components consist of all the $j$-th order derivatives of the function at a point. Namely, $\{c_j\}_{0 \leq j \leq 2m-1}$ satisfies the necessary compatibility conditions being the derivatives up to order $2m-1$ of some vector-valued function at a point. A vector-valued function $u$ of class $\mathcal{C}^k$ is said to be **of vanishing order** $m$ ($m \leq k$) at 0 if $\nabla^j u(0) = 0$ for all $0 \leq j \leq m-1$ and $\nabla^m u(0) \neq 0$. Making use of Theorem 1.3, especially the fact that the estimate is independent of $R$, we obtain the solvability for the following case.

**Theorem 1.4.** *Let* $a \in \mathcal{C}^{1,\alpha}(0 < \alpha < 1)$. *For any given appropriate jet* $\{c_j\}_{0 \le j \le 2m-1}$, *there exist infinitely many solutions of class* $\mathcal{C}^{2m,\alpha}$ *satisfying*

$$\begin{aligned} &\triangle^m u(x) = a(x, u, \nabla u, \ldots, \nabla^{2m-1} u); \\ &u(0) = c_0; \\ &\nabla u(0) = c_1; \\ &\ldots \\ &\nabla^{2m-1} u(0) = c_{2m-1} \end{aligned} \tag{2}$$

*in some small neighborhood of 0. Moreover, all those solutions are of vanishing order at most* $2m$ *and not radially symmetric.*

Since the solutions are not radially symmetric, they are not obtained by, if possible, reducing the system into an ODE system with respect to the radial variable $r = |x|$. This phenomenon can be compared with the case in [GNN], where there are only radial solutions satisfying an additional assumption.

We note that, due to the flexibility of $a$, Theorem 1.4 can be used to construct local $m$-harmonic maps from Euclidean space to any given Riemannian manifold. The resulting image in the target manifold can be either smooth or singular, depending on the given jet.

When $a$ is dependent also on the $p_{2m}$ variable, we obtain the following existence theorem with some additional assumption on $a$.

**Theorem 1.5.** *If* $a \in \mathcal{C}^2$ *and* $a(0) = \nabla_{p_{2m}} a(0) = \nabla^2_{p_{2m}} a(0) = 0$, *then there exist infinitely many solutions in the class of* $\mathcal{C}^{2m,\alpha}$ $(0 < \alpha < 1)$ *to the system*

$$\triangle^m u(x) = a(x, u, \nabla u, \ldots, \nabla^{2m} u)$$

*in some small neighborhood of 0. Moreover, all those solutions are of vanishing order* $2m$ *and not radially symmetric.*

On the other hand, when the system (1) is autonomous, i.e., independent of the variable $x$, then there exist solutions on arbitrarily large domains in the following sense.

**Theorem 1.6.** *If* $a \in \mathcal{C}^2$ *and* $a(0) = \nabla a(0) = 0$, *then for any* $R > 0$, *there exist infinitely many solutions in the class of* $\mathcal{C}^{2m,\alpha}$ *to the autonomous system*

$$\triangle^m u = a(u, \nabla u, \ldots, \nabla^{2m} u)$$

*in* $\{x \in \mathbb{R}^n : |x| < R\}$. *Moreover, all those solutions are of vanishing order* $2m$ *and are not radially symmetric.*

Although the autonomous system in Theorem 1.6 is itself translation invariant, none of the solutions is obtained by trivial translation of the radial solution, from the proof of Theorem 1.6. On the other hand, the regularity of $a$ in Theorem 1.6 can be reduced to $\mathcal{C}^{1,\alpha}$ if $a$ is in addition independent of $\nabla^{2m}u$ variable. This fact will be seen from the proof of Theorem 1.5 and 1.6 in Section 4 and will be used in some of the examples in Section 5.

The rest of the paper is outlined as follows. The notations for the function spaces with the corresponding norms and a preparation lemma are given in Section 2. In Section 3, we prove Theorem 1.1. The higher order derivative formula for the Newtonian potential along the line of [GT] is derived in Section 4. Restricting on $\mathbf{B}_R$ and applying the residue-type phenomenon, we obtain Theorem 1.2. The corresponding Hölder estimates is proved in Section 5. Section 6 is devoted to the construction of the contraction map with the necessary estimates for the application of the fixed point theorem. After a delicate chasing of the parameters, we show the main theorems hold in Section 7, following the idea of [Pan2]. Examples and remarks concerning solvability of the nonlinear system are discussed in the last section. In Appendix, we compute an interesting integral concerning the fundamental solution over the sphere, making use of Gegenbauer polynomials. This approach provides a practical way to compute the explicit values of all the residue-type formulas for the fundamental solution. Our approach throughout the whole paper is purely elementary.

# 2 Notations and an elementary lemma

Denote $\mathbf{B}_R := \{x \in \mathbb{R}^n : |x| < R\}$ and $\partial\mathbf{B}_R := \{x \in \mathbb{R}^n : |x| = R\}$, $n \geq 3$. Here $|\cdot|$ is the standard Euclidean norm. We consider the following function spaces and norms over $\mathbf{B}_R$ following [Pan2].

Let $\mathcal{C}(\mathbf{B}_R)$ be the set of continuous functions in $\mathbf{B}_R$ and $\mathcal{C}^\alpha(\mathbf{B}_R)$ the Hölder space in $\mathbf{B}_R$ with order $\alpha$. For $f \in \mathcal{C}^\alpha(\mathbf{B}_R)$, the norm of $f$ is defined by

$$\| f \|_\alpha := \| f \| + R^\alpha H_\alpha[f], \tag{3}$$

where

$$\| f \| := \sup\{|f(x)| : x \in \mathbf{B}_R\};$$
$$H_\alpha[f] := \sup\Big\{\frac{|f(x) - f(x')|}{|x - x'|^\alpha} : x, x' \in \mathbf{B}_R, x \neq x'\Big\}.$$

We note when $f \in \mathcal{C}^\alpha(\mathbf{B}_R)$, the trivial unique extension of $f$ onto $\bar{\mathbf{B}}_R$ is identified as an element in $\mathcal{C}^\alpha(\bar{\mathbf{B}}_R)$. $\mathcal{C}^\alpha(\mathbf{B}_R)$ is then a Banach space under the norm $\|\cdot\|_\alpha$.

For $0 < k \in \mathbb{Z}^+$, denote by $\mathcal{C}^k(\mathbf{B}_R)$ the collection of all functions in $\mathbf{B}_R$ whose partial derivatives exist and are continuous up to order $k$. Denote by $\|\cdot\|_{\mathcal{C}^k}$ the corresponding norm, where

$$\|f\|_{\mathcal{C}^k} := \sup\{\| D^\beta f \| : |\beta| = k\}$$

if $\|f\|_{\mathcal{C}^k}$ is finite.

$\mathcal{C}^{k,\alpha}(\mathbf{B}_R)$ is the subset of $\mathcal{C}^k(\mathbf{B}_R)$ whose $k$-th order derivatives belong to $\mathcal{C}^\alpha(\mathbf{B}_R)$. For any multi-index $\beta = (\beta_1, \ldots, \beta_n)$ with nonnegative entries, define $|\beta| := \sum_{j=1}^n \beta_j$ and $\beta! :=$

$\beta_1! \cdots \beta_n!$. Given any $f \in \mathcal{C}^k(\mathbf{B}_R)$, we represent $D^\beta f := \partial_1^{\beta_1}\partial_2^{\beta_2}\cdots\partial_n^{\beta_n} f$ with $\partial_j$ the partial derivative with respect to the $x_j$ variables. If $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$, we define the semi-norm

$$\| f \|_\alpha^{(k)} := \sup\{\| D^\beta f \|_\alpha : |\beta| = k\}.$$

The notation and norm $\| \cdot \|_\alpha^{(k)}$ are naturally extended to vector-valued functions by defining $\| (f_1, \ldots, f_N) \|_\alpha^{(k)} := \sup_{1\le j\le N} \| f_j \|_\alpha^{(k)}$.

Of special interest, we introduce the subset of $\mathcal{C}^{k,\alpha}(\mathbf{B}_R)$ as follows.

$$\mathcal{C}_0^{k,\alpha}(\mathbf{B}_R) := \{f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R) : D^\beta f(0) = 0, |\beta| \le k-1\}.$$

The following two preparation lemmas are elementary but essential in deriving the desired estimates in the rest of the paper. The original proof of the lemmas can be found in [Pan2]. For the convenience of the reader, we enclose the details as follows.

**Lemma 2.1.** *If $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$, then for any $x, x' \in \mathbf{B}_R$ and $0 < \alpha < 1$,*

$$|f(x') - T_k^x(f)(x')| \le |x - x'|^{k+\alpha}\Big(\sum_{|\mu|=k} H_\alpha[D^\mu f]\Big).$$

*Here $T_k^x(f)(\cdot)$ is the $k$-th order power series expansion of $f$ at $x$.*

**Proof of Lemma 2.1**: The proof is based on the following identity in calculus. Indeed, for $x, x' \in \mathbf{B}_R$,

$$\begin{aligned} & f(x') - T_k^x(f)(x') \\ &= \int_0^1\int_0^{t_{k-1}}\cdots\int_0^{t_1} \frac{d^k}{dt^k}\big(f(tx' + (1-t)x)\big)dtdt_1\ldots dt_{k-1} - \sum_{|\mu|=k}\frac{1}{\mu!}D^\mu f(x)(x'-x)^\mu \\ &= \int_0^1\int_0^{t_{k-1}}\cdots\int_0^{t_1} \sum_{|\mu|=k}\big((D^\mu f)(tx' + (1-t)x) - (D^\mu f)(x)\big)(x'-x)^\mu dtdt_1\ldots dt_{k-1}. \end{aligned}$$

Hence

$$\begin{aligned} |f(x') - T_k^x(f)(x')| \le & \int_0^1\int_0^{t_{k-1}}\cdots\int_0^{t_1}\sum_{|\mu|=k} H_\alpha[D^\mu f]|x'-x|^\alpha|x'-x|^\mu dtdt_1\ldots dt_{k-1} \\ \le & |x - x'|^{k+\alpha}\Big(\sum_{|\mu|=k} H_\alpha[D^\mu f]\Big). \blacksquare \end{aligned}$$

**Lemma 2.2.** *If $f \in \mathcal{C}_0^{k,\alpha}(\mathbf{B}_R)$, then for any $l < k$ and $0 \le \alpha \le 1$,*

$$\| f \|_\alpha^{(l)} \le CR^{k-l} \| f \|_\alpha^{(k)},$$

*where $C$ depends only on $k$ and $n$.*

**Proof of Lemma 2.2**: If $f \in \mathcal{C}_0^{k,\alpha}(\mathbf{B}_R)$ and $\beta$ with $|\beta| = l < k$, $D^\beta f \in \mathcal{C}_0^{k-l,\alpha}(\mathbf{B}_R)$. For any $x \in \mathbf{B}_R$,

$$\begin{aligned}
D^\beta f(x) =& D^\beta f(x) - T^0_{k-l-1} D^\beta f(x) \\
=& \sum_{|\mu|=k-l} \int_0^1 \int_0^{t_{k-l-1}} \cdots \int_0^{t_1} (D^\mu D^\beta f)(tx) x^\mu dt dt_1 \ldots dt_{k-l-1} \\
=& \sum_{|\mu|=k-l} F_\mu(x) x^\mu,
\end{aligned}$$

where $F_\mu := \int_0^1 \int_0^{t_{k-l-1}} \cdots \int_0^{t_1} (D^\mu D^\beta f)(t\cdot) dt dt_1 \ldots dt_{k-l-1}$. Note that $\| F_\mu \|_\alpha \leq \| f \|_\alpha^{(k)}$.

Hence

$$\| D^\beta f \|_\alpha \leq \sum_{|\mu|=k-l} \| F_\mu \|_\alpha \cdot \| x^\mu \|_\alpha \leq C R^{k-l} \| f \|_\alpha^{(k)} . \blacksquare$$

$\mathcal{C}_0^{k,\alpha}(\mathbf{B}_R)$ $(0 < \alpha < 1)$ thus becomes a Banach space under the norm $\| \cdot \|_\alpha^{(k)}$, as a consequence of Lemma 2.2.

From now on and throughout the rest of the paper, we use $C$ to represent any positive constant number dependent only on $n, \alpha, m, k$ and $N$, where $0 < \alpha < 1$, $n \geq 3$ and $N \geq 1$. Especially, $C$ is independent of $R$.

# 3 The Residue-type theorem for the fundamental solution of the Laplacian in $\mathbb{R}^n$

In one complex variable, the Residue theorem implies, given any holomorphic function $f$ defined in $\Omega \subset \mathbb{C}$ with $\bar{\mathbf{B}}_R \subset \Omega$, and for any $z \in \mathbf{B}_R$,

$$\int_{|\xi|=R} \frac{f(\xi)}{\xi - z} d\xi = 2\pi i f(z),$$

Here $\frac{1}{z}$ is the Cauchy kernel for the $\bar{\partial}$ operator in $\mathbb{C}$. It is also related to the first derivative of the fundamental solution of the Laplacian in $\mathbb{R}^2$. As a special case, if $f$ is a holomorphic polynomial of degree $k$ in $\Omega$, then for $z \in \mathbf{B}_R$, $\int_{|\xi|=R} \frac{f(\xi)}{\xi - z} d\xi$ is a polynomial of the same degree. Theorem 1.1 in this sense serves as a suitable substitute of the Residue theorem in $\mathbb{R}^n$.

The proof of Theorem 1.1 makes use of zonal spherical harmonics $Z_x^{(l)}$ and their reproducing properties for spherical harmonics. In particular, let $H_l$ be the set of all spherical harmonics of degree $l$, then for any $f \in H_l$ and $x \in \mathbf{B}_1$,

$$f(x) = \int_{\partial \mathbf{B}_1} Z_x^{(l)}(y) f(y) d\sigma_y.$$

Moreover, if $f \in H_k$ with $l \neq k$, then for $x \in \mathbf{B}_1$,

$$0 = \int_{\partial \mathbf{B}_1} Z_x^{(l)}(y) f(y) d\sigma_y.$$

Moreover, denote by $\mathcal{P}_k^h$ the space of all homogeneous polynomials of degree $k$ restricted in $\mathbf{B}_R$. For any $f \in \mathcal{P}_k^h$, there exist $P_j$'s, some homogenous harmonic polynomials of degree $j$, such that for $x \in \mathbf{B}_R$,

$$f(x) = P_k(x) + |x|^2 P_{k-2}(x) + \cdots + |x|^k P_0(x), \text{ when } k \text{ is even}, \tag{4}$$

and

$$f(x) = P_k(x) + |x|^2 P_{k-2}(x) + \cdots + |x|^{k-1} P_1(x), \text{ when } k \text{ is odd}. \tag{5}$$

Note $P_j|_{\partial \mathbf{B}_1} \in H_j$. See [SW] for more details.

We are now in a position to prove the residue-type Theorem 1.1 for the fundamental solution of the Laplacian in $\mathbb{R}^n$.

**Proof of Theorem 1.1**: Without loss of generality, we assume $f$ is a monomial of degree $k$. We also assume that $R = 1$. This is due to the following simple fact that for any $f \in \mathcal{P}_k^h$ and $x \in \mathbf{B}_R$,

$$\int_{\partial \mathbf{B}_R} \Gamma(x-y) f(y) d\sigma_y = R^{k+1} \int_{\partial \mathbf{B}_1} \Gamma(\frac{x}{R} - y) f(y) d\sigma_y.$$

Under the zonal spherical harmonics, we have when $x \in \mathbf{B}_1 \setminus \{0\}$,

$$\Gamma(x-y) = \sum_{l=0}^{\infty} C_{n,l} \frac{|x|^l}{|y|^{n+k-2}} Z^{(l)}_{\frac{x}{|x|}}(\frac{y}{|y|}),$$

where $C_{n,l} = \frac{2l+n-2}{(n-2)\omega_n}$. Letting $y \in \partial \mathbf{B}_1$, the above expression for $x \in \mathbf{B}_1 \setminus \{0\}$ simplifies as

$$\Gamma(x-y) = \sum_{l=0}^{\infty} C_{n,l} |x|^l Z^{(l)}_{\frac{x}{|x|}}(y). \tag{6}$$

On the other hand, since $f \in \mathcal{P}_k^h$, letting $y \in \partial \mathbf{B}_1$ and making use of (5), one has when $k$ is odd,

$$f(y) = P_k(y) + P_{k-2}(y) + \cdots + P_1(y) \tag{7}$$

for some harmonic spherics $P_j \in H_j$. Therefore, combining (6) and (7) together with the reproducing property of the zonal spherical harmonics, we have for $x \in \mathbf{B}_1 \setminus \{0\}$,

$$\begin{aligned}\int_{\partial \mathbf{B}_1} \Gamma(x-y) f(y) d\sigma_y &= \int_{\partial \mathbf{B}_1} \left( \sum_{l=0}^{\infty} C_{n,l} |x|^l Z^{(l)}_{\frac{x}{|x|}}(y) \right) \Big( P_k(y) + P_{k-2}(y) + \cdots + P_1(y) \Big) d\sigma_y \\ &= C_{n,k} |x|^k P_k(\frac{x}{|x|}) + C_{n,k-2} |x|^{k-2} P_{k-2}(\frac{x}{|x|}) + \cdots + C_{n,1} |x| P_1(\frac{x}{|x|}) \\ &= C_{n,k} P_k(x) + C_{n,k-2} P_{k-2}(x) + \cdots + C_{n,1} P_1(x) \in \mathcal{P}_k.\end{aligned}$$

The formula extends automatically to $x = 0$ for the sake of continuity on both sides at 0. The case when $k$ is even can be treated similarly and is omitted here. ■

It is not clear to us whether the Residue-type phenomenon is true on general domains due to a lack of symmetry of their boundaries. On the other hand, despite of the constructive proof of the Residue-type formula in Theorem 1.1 for the fundamental solution of the Laplacian, the integral can actually be computed directly. See Appendix for a computation of the formula when $k = 1$. The same method could practically be used for general $k > 1$.

As immediate consequences of Theorem 1.1, we obtain the following two corollaries.

**Corollary 3.1.** *For any $f \in \mathcal{P}_k$ and any multi-index $\beta$ and $x \in \mathbf{B}_R$,*

$$\int_{\partial \mathbf{B}_R} D_x^\beta \Gamma(x-y) f(y) d\sigma_y = \begin{cases} 0 & \text{when } |\beta| \geq k+1; \\ C(f)R & \text{when } |\beta| = k. \end{cases}$$

*Here $C(f)$ is some constant dependent on $f$. In particular, $C(f)$ is independent of $R$.*

**Proof of Corollary 3.2**: We only need to show $C(f)$ is independent of $R$ for any monomial $f$ of degree $k$ when $|\beta| = k$. Indeed, for $x \in \mathbf{B}_R$,

$$\begin{aligned}\int_{\partial \mathbf{B}_R} D_x^\beta \Gamma(x-y) f(y) d\sigma_y =& D_x^\beta \int_{\partial \mathbf{B}_R} \Gamma(x-y) f(y) d\sigma_y \\ =& R^{1+k} D^\beta \int_{\partial \mathbf{B}_1} \Gamma(\frac{x}{R} - y) f(y) d\sigma_y.\end{aligned}$$

According to Theorem 1.1, $\int_{\partial \mathbf{B}_1} \Gamma(\frac{x}{R} - y) f(y) d\sigma_y$ is a polynomial of degree $k$ for $x \in \mathbf{B}_R$. Write $\int_{\partial \mathbf{B}_1} \Gamma(\frac{x}{R} - y) f(y) d\sigma_y = P_k(\frac{x}{R}) + R_{k-1}(\frac{x}{R})$ for some homogeneous polynomial $P_k$ of degree $k$ and another polynomial $R_{k-1}$ of degree $k-1$. Hence for $x \in \mathbf{B}_R$,

$$\begin{aligned}\int_{\partial \mathbf{B}_R} D_x^\beta \Gamma(x-y) f(y) d\sigma_y =& R^{1+k} D^\beta \big(P_k(\frac{x}{R}) + R_{k-1}(\frac{x}{R})\big) \\ =& R^{1+k} D^\beta P_k(\frac{x}{R}) \\ =& R D^\beta P_k(x) = C(f) R\end{aligned}$$

with $C(f) = D^\beta P_k$. ■

**Corollary 3.2.** *For any $f \in \mathcal{P}_k$ and any multi-index $\beta$ with $|\beta| \geq k+2$,*

$$\int_{\mathbf{B}_R \setminus \mathbf{B}_\epsilon(z)} D_x^\beta \Gamma(x-y) f(y) dy = 0, \tag{8}$$

*when $x \in \mathbf{B}_\epsilon(z) \subset \mathbf{B}_R$. Here $\mathbf{B}_\epsilon(z)$ is the ball centered at $z$ with radius $\epsilon$.*

**Proof of Corollary 3.2**: Write $\beta = (\beta_1, \dots, \beta_n)$. Without loss of generality, assume $R = 1$, $\beta_1 > 0$ and $f$ is a monomial of degree $k$. Moreover we write $\beta' = (\beta_1 - 1, \dots, \beta_n)$. Hence applying Stokes' Theorem on $D^{\beta'}\Gamma(x-y)f(y)$ over the domain $\mathbf{B}_R \setminus \mathbf{B}_\epsilon(z)$, one has

$$\begin{aligned}
&\int_{\mathbf{B}_1\setminus\mathbf{B}_\epsilon(z)} D_y^{\beta}\Gamma(y-x)f(y)dy \\
=& -\int_{\mathbf{B}_1\setminus\mathbf{B}_\epsilon(z)} D_y^{\beta'}\Gamma(y-x)\partial_1 f(y)dy + \int_{\partial\mathbf{B}_1} D_y^{\beta'}\Gamma(y-x)f(y)y_1 d\sigma_y \\
&- \int_{\partial\mathbf{B}_\epsilon(z)} D_y^{\beta'}\Gamma(y-x)f(y)\frac{y_1-z_1}{|y-z|}d\sigma_y.
\end{aligned} \tag{9}$$

Write $I := \int_{\partial\mathbf{B}_1} D_y^{\beta'}\Gamma(y-x)f(y)y_1 d\sigma_y$ and $II := \int_{\partial\mathbf{B}_\epsilon(z)} D_y^{\beta'}\Gamma(y-x)f(y)\frac{y_1-z_1}{|y-z|}d\sigma_y$. We show next that $I = II$ in $\mathbf{B}_1$ and therefore

$$\int_{\mathbf{B}_1\setminus\mathbf{B}_\epsilon(z)} D_y^{\beta}\Gamma(y-x)f(y)dy = -\int_{\mathbf{B}_1\setminus\mathbf{B}_\epsilon(z)} D_y^{\beta'}\Gamma(y-x)\partial_1 f(y)dy. \tag{10}$$

First, after a change of coordinates by letting $y = z + \epsilon\tau$, $II$ is computed as follows.

$$\begin{aligned}
II =& \epsilon^{2-|\beta|}\int_{\partial\mathbf{B}_1} D_y^{\beta'}\Gamma(\frac{z-x}{\epsilon}+\tau)f(z+\epsilon\tau)\tau_1 d\sigma_\tau \\
=& \epsilon^{2-|\beta|}\int_{\partial\mathbf{B}_1} D_y^{\beta'}\Gamma(\frac{z-x}{\epsilon}+\tau)\big(f(\epsilon\tau)+P_{k-1}(\tau)\big)\tau_1 d\sigma_\tau \\
=& \epsilon^{2-|\beta|+k}\int_{\partial\mathbf{B}_1} D_y^{\beta'}\Gamma(\frac{z-x}{\epsilon}+\tau)f(\tau)\tau_1 d\sigma_\tau.
\end{aligned} \tag{11}$$

Here $P_{k-1}(\cdot)$ is some polynomial of degree $k-1$ such that $f(z+\epsilon\tau) = f(\epsilon\tau) + P_{k-1}(\tau)$. The last identity is due to the fact that $f(\epsilon\tau) = \epsilon^k f(\tau)$, together with an application of Corollary 3.1 onto $P_{k-1}(\tau)\tau_1$.

We divide the proof of (10) into two cases. When $|\beta| \geq k+3$ and hence $|\beta'| \geq k+2$, $I$ and $II$ are both zero because of Corollary 3.1 and we are done. When $|\beta| = k+2$ so $|\beta'| = k+1$, $I$ is a constant in $x \in \mathbf{B}_1$ dependent only on the degree $k+1$ polynomial $f(y)y_1$ by Corollary 3.1. On the other hand, for $II$, (11) is further simplified as

$$II = \int_{\partial\mathbf{B}_1} D_y^{\beta'}\Gamma(\tau + \frac{z-x}{\epsilon})f(\tau)\tau_1 d\sigma_\tau,$$

which by Corollary 3.1 again is the same constant. Therefore $I = II$ when $|\beta| \geq k+2$ and hence (10) holds.

Now applying the induction process on (10), we get for $x \in \mathbf{B}_1$,

$$\begin{aligned}\int_{\mathbf{B}_1\setminus\mathbf{B}_\epsilon(z)} D_y^\beta\Gamma(y-x)f(y)dy =& -\int_{\mathbf{B}_1\setminus\mathbf{B}_\epsilon(z)} D_y^{\beta'}\Gamma(y-x)\partial_1 f(y)dy\\ =&\cdots\\ =&C(f)\int_{\mathbf{B}_1\setminus\mathbf{B}_\epsilon(z)} D_y^{\mu}\Gamma(y-x)dy\\ =&0.\end{aligned}$$

Here $\mu$ is some multi-index with $|\mu| \geq 2$ and $C(f)$ is some constant dependent only on $f$ and $\beta$. ∎

We briefly note that by Corollary 3.2, the principal value of $D^\beta\Gamma * f$ $(:= \lim_{\epsilon\to 0}\int_{\mathbb{R}^n\setminus\mathbf{B}_\epsilon(x)} D^\beta\Gamma(x-y)f(y)dy)$ is well defined in $\mathbf{B}_R$ whenever $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$. Moreover, for $x \in \mathbf{B}_R$,

$$p.v.(D^\beta\Gamma * f)(x) = \int_{\mathbf{B}_R} D_x^\beta\Gamma(x-y)\big(f(y) - T_k^x(f)(y)\big)dy.$$

# 4 The higher order derivative formula of the Newtonian potential

Due to the nonintegrability of the fundamental solution of the Laplacian after differentiation more than once, the second order derivatives of the Newtonian potential becomes a distribution. However, when the function space is regular enough, such as $\mathcal{C}^\alpha$, the Newtonian potential is twice differentiable. See Lemma 4.4 in [GT] and references therein.

We first compute the higher order derivatives of the Newtonian potential on general domains following the idea of [GT]. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded open set. The Newtonian potential over $\Omega$ is denoted accordingly by

$$\mathcal{N}_\Omega(f) := \int_\Omega \Gamma(\cdot - y)f(y)dy$$

for any given integrable function $f$ in $\Omega$.

**Definition 4.1.** *Given two multi-indices $\beta$ and $\mu$, and $j$ with $1 \leq j \leq n$, we define for $x \in \Omega$,*

$$\mathcal{I}_\Omega(\beta,\mu,j)(x) := \int_{\partial\Omega} D_x^\beta\Gamma(x-y)(y-x)^\mu \nu_j d\sigma_y,$$

*where $d\sigma_y$ is the surface area element of $\partial\Omega$ with the unit outer normal vector $(\nu_1,\ldots,\nu_n)$.*

It is clear to see that $\mathcal{I}_\Omega(\beta,\mu,j) \in \mathcal{C}^\infty(\Omega)$.

The following notations will be adopted throughout the rest of the paper. Unless otherwise indicated, we always regard derivatives inside the integration as derivatives with respect to $y$ variables. For instance, inside an integral, $\partial_1\Gamma(x-y) := \frac{\partial(\Gamma(x-y))}{\partial y_1}$ while $\partial_{x_1}\Gamma(x-y) := \frac{\partial\Gamma(x-y)}{\partial x_1}$. Given any two multi-indices $\beta = (\beta_1, \ldots, \beta_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$, we say $\beta < \mu$ if $\beta_j \le \mu_j$ for each $1 \le j \le n$ and $|\beta| < |\mu|$. Moreover, we define $\mu - \beta := (\mu_1 - \beta_1, \cdots, \mu_n - \beta_n)$ if $\beta < \mu$. If in addition $|\mu| = |\beta| + 1$ with $\partial_j D^\beta = D^\mu$ for some $1 \le j \le n$, we write $\mu - \beta = j$.

**Definition 4.2.** *Given a multi-index $\beta$, we call $\{\beta^{(j)}\}_{j=1}^k$ a* **continuously increasing nesting of length** *$k$ for $\beta$ if $|\beta^{(j)}| = j$ for $1 \le j \le k$ and $\beta^{(j)} < \beta^{(j+1)} \le \beta$ for $1 \le j \le k-1$. Given two multi-indices $\gamma$ and $\gamma'$, we say $\gamma'$ is* **the dual** *of $\gamma$ with respect to $\beta$ if $D^\beta = D^\gamma D^{\gamma'}$.*

**Proposition 4.3.** *Let $\beta$ be a multi-index with $|\beta| = k+2$. Let $\{\beta^{(j)}\}$ be a continuously increasing nesting of length $k+2$ for $\beta$ and let $\beta^{(j)'}$ be the dual of $\beta^{(j)}$ with respect to $\beta$ for $1 \le j \le k+2$. Then given a bounded and locally $\mathcal{C}^{k,\alpha}$ function $f$ in $\Omega$ and for any $x \in \Omega$,*

$$
\begin{aligned}
D^\beta \mathcal{N}_\Omega(f)(x) = &\int_\Omega D_x^\beta \Gamma(x-y)\big(f(y) - T_k^x(f)(y)\big)dy \\
&- \sum_{j=2}^{k+2} D^{\beta^{(j)'}}\Big(\sum_{|\mu|=j-2} \frac{D^\mu f(x)}{\mu!}\mathcal{I}_\Omega(\beta^{(j-1)}, \mu, \beta^{(j)} - \beta^{(j-1)})(x)\Big).
\end{aligned}
\tag{12}
$$

*Here $T_k^x(f)$ is the $k$-th order power series expansion of $f$ at $x$.*

**Proof of Proposition 4.3**: The proposition is proved by induction on $k$. When $k = 0$, the theorem reduces to the case in [GT]. Fix $x \in \Omega$ and assume (12) is true for $k = k_0 \ge 0$, i.e., for any $f \in \mathcal{C}^{k_0,\alpha}$ and $\beta$ with $|\beta| = k_0 + 2$,

$$
\begin{aligned}
D^\beta \mathcal{N}_\Omega(f)(x) = &\int_\Omega D_x^\beta \Gamma(x-y)\big(f(y) - T_{k_0}^x(f)(y)\big)dy \\
&- \sum_{j=2}^{k_0+2} D^{\beta^{(j)'}}\Big(\sum_{|\mu|=j-2} \frac{D^\mu f(x)}{\mu!}\mathcal{I}_\Omega(\beta^{(j-1)}, \mu, \beta^{(j)} - \beta^{(j-1)})(x)\Big).
\end{aligned}
\tag{13}
$$

We want to show for any $\beta$ with $|\beta| = k_0 + 3$ and $f \in \mathcal{C}^{k_0+1,\alpha}$,

$$
\begin{aligned}
D^\beta \mathcal{N}_\Omega(f)(x) = &\int_\Omega D_x^\beta \Gamma(x-y)\big(f(y) - T_{k_0+1}^x(f)(y)\big)dy \\
&- \sum_{j=2}^{k_0+3} D^{\beta^{(j)'}}\Big(\sum_{|\mu|=j-2} \frac{D^\mu f(x)}{\mu!}\mathcal{I}_\Omega(\beta^{(j-1)}, \mu, \beta^{(j)} - \beta^{(j-1)})(x)\Big).
\end{aligned}
\tag{14}
$$

Without loss of generality, assume $D^\beta = \partial_1 D^{\beta^{(k_0+2)}}$ with $|\beta^{(k_0+2)}| = k_0 + 2$. Choose positive

$\epsilon \leq \frac{dist\{x,\partial\Omega\}}{2}$ and let

$$v_\epsilon(x) := \int_\Omega D_x^{\beta^{(k_0+2)}}\Gamma(x-y)\eta_\epsilon(x-y)\big(f(y)-T^x_{k_0}(f)(y)\big)dy \\ - \sum_{j=2}^{k_0+2} D^{\beta^{(k_0+2)}-\beta^{(j)}}\big(\sum_{|\mu|=j-2}\frac{D^\mu f(x)}{\mu!}\mathcal{I}_\Omega(\beta^{(j-1)},\mu,\beta^{(j)}-\beta^{(j-1)})(x)\big),$$

where $\eta_\epsilon(x-y) = \eta(\frac{|x-y|}{\epsilon})$ with $\eta$ some smooth increasing function such that $\eta(t) = 0$ when $t \leq 1$ and $\eta(t) = 1$ when $t \geq 2$. When $\epsilon \to 0$, $v_\epsilon(x) \to D^{\beta^{(k_0+2)}}\mathcal{N}_\Omega(f)(x)$ for all $x \in \Omega$ by induction.

Now compute

$$\begin{aligned}\partial_1 v_\epsilon(x) =& -\int_\Omega \partial_1\big(D_x^{\beta^{(k_0+2)}}\Gamma(x-y)\eta_\epsilon(x-y)\big)\big(f(y)-T^x_{k_0}(f)(y)\big)dy \\ &+\int_\Omega D_x^{\beta^{(k_0+2)}}\Gamma(x-y)\eta_\epsilon(x-y)\partial_{x_1}\big(f(y)-T^x_{k_0}(f)(y)\big)dy \\ &-\partial_1\big[\sum_{j=2}^{k_0+2} D^{\beta^{(k_0+2)}-\beta^{(j)}}\big(\sum_{|\mu|=j-2}\frac{D^\mu f(x)}{\mu!}\mathcal{I}_\Omega(\beta^{(j-1)},\mu,\beta^{(j)}-\beta^{(j-1)})(x)\big)\big] \\ =& A+B-\sum_{j=2}^{k_0+2} D^{\beta^{(j)'}}\big(\sum_{|\mu|=j-2}\frac{D^\mu f(x)}{\mu!}\mathcal{I}_\Omega(\beta^{(j-1)},\mu,\beta^{(j)}-\beta^{(j-1)})(x)\big).\end{aligned} \tag{15}$$

Here $A := -\int_\Omega \partial_1\big(D_x^{\beta^{(k_0+2)}}\Gamma(x-y)\eta_\epsilon(x-y)\big)\big(f(y)-T^x_{k_0}(f)(y)\big)dy$ and $B := \int_\Omega D_x^{\beta^{(k_0+2)}}\Gamma(x-y)\eta_\epsilon(x-y)\partial_{x_1}\big(f(y)-T^x_{k_0}(f)(y)\big)dy$. We will show as $\epsilon \to 0$,

$$\begin{aligned}A+B \to& \int_{\Omega'} D_x^\beta\Gamma(x-y)\big(f(y)-T^x_{k_0+1}(f)(y)\big)dy \\ &-\sum_{|\mu|=k_0+1}\frac{D^\mu f(x)}{\mu!}\mathcal{I}_{\Omega'}(\beta^{(k_0+2)},\mu,\beta^{(k_0+3)}-\beta^{(k_0+2)})(x).\end{aligned} \tag{16}$$

Then (15) gives

$$\partial_1 v_\epsilon(x) \to \int_\Omega D_x^\beta\Gamma(x-y)\big(f(y)-T^x_{k_0+1}(f)(y)\big)dy \\ -\sum_{j=2}^{k_0+3} D^{\beta^{(j)'}}\big(\sum_{|\mu|=j-2}\frac{D^\mu f(x)}{\mu!}\mathcal{I}_\Omega(\beta^{(j-1)},\mu,\beta^{(j)}-\beta^{(j-1)})(x)\big).$$

and (14) is concluded.

To prove (16), firstly

$$A = -\int_\Omega \partial_1\big(D_x^{\beta^{(k_0+2)}}\Gamma(x-y)\eta_\epsilon(x-y)\big)\big(f(y)-T^x_{k_0+1}(f)(y)\big)dy$$
$$-\sum_{|\mu|=k_0+1}\frac{D^\mu f(x)}{\mu!}\int_\Omega \partial_1\big(D_x^{\beta^{(k_0+2)}}\Gamma(x-y)\eta_\epsilon(x-y)\big)(y-x)^\mu dy.$$

Applying Stokes' Theorem to the second term of the above expression, we then have

$$A = -\int_\Omega \partial_1\big(D_x^{\beta^{(k_0+2)}}\Gamma(x-y)\eta_\epsilon(x-y)\big)\big(f(y)-T^x_{k_0+1}(f)(y)\big)dy$$
$$-\sum_{|\mu|=k_0+1}\frac{D^\mu f(x)}{\mu!}\int_{\partial\Omega} D_x^{\beta^{(k_0+2)}}\Gamma(x-y)\eta_\epsilon(x-y)(y-x)^\mu \nu_1 d\sigma_y$$
$$+\sum_{|\mu|=k_0+1}\frac{D^\mu f(x)}{\mu!}\int_\Omega D_x^{\beta^{(k_0+2)}}\Gamma(x-y)\eta_\epsilon(x-y)\partial_1(y-x)^\mu dy.$$

On the other hand,

$$B = -\int_\Omega D_x^{\beta^{(k_0+2)}}\Gamma(x-y)\eta_\epsilon(x-y)\partial_{x_1}\big(T^x_{k_0}(f)(y)\big)dy.$$

Therefore,

$$\begin{aligned}A+B =& -\int_\Omega \partial_1\big(D_x^{\beta^{(k_0+2)}}\Gamma(x-y)\eta_\epsilon(x-y)\big)\big(f(y)-T^x_{k_0+1}(f)(y)\big)dy\\
&-\sum_{|\mu|=k_0+1}\frac{D^\mu f(x)}{\mu!}\int_{\partial\Omega} D_x^{\beta^{(k_0+2)}}\Gamma(x-y)\eta_\epsilon(x-y)(y-x)^\mu \nu_1 d\sigma_y\\
&+\int_\Omega D_x^{\beta^{(k_0+2)}}\Gamma(x-y)\eta_\epsilon(x-y)\big[\sum_{|\mu|=k_0+1}\frac{D^\mu f(x)}{\mu!}\partial_1(y-x)^\mu - \partial_{x_1}\big(T^x_{k_0}(f)(y)\big)\big]dy\\
=&I+II+III.\end{aligned}$$

As $\epsilon \to 0$,

$$\begin{aligned}I &\to \int_\Omega D_x^\beta\Gamma(x-y)\big(f(y)-T^x_{k_0+1}(f)(y)\big)dy\\
II &\to -\sum_{|\mu|=k_0+1}\frac{D^\mu f(x)}{\mu!}\int_{\partial\Omega} D_x^{\beta^{(k_0+2)}}\Gamma(x-y)(y-x)^\mu\nu_1 d\sigma_y\\
&= -\sum_{|\mu|=k_0+1}\frac{D^\mu f(x)}{\mu!}\mathcal{I}_\Omega(\beta^{(k_0+2)},\mu,1)(x).\end{aligned} \tag{17}$$

For $III$, notice $T^x_{k_0}(f)(y) = \sum_{|\mu|\le k_0} \frac{D^\mu f(x)(y-x)^\mu}{\mu!}$, so

$$\partial_{x_1}\big(T^x_{k_0}(f)(y)\big) = \sum_{|\mu|\le k_0} \frac{\partial_1 D^\mu f(x)(y-x)^\mu}{\mu!} + \sum_{|\mu|\le k_0} \frac{D^\mu f(x)\partial_{x_1}(y-x)^\mu}{\mu!}.$$

One also observes the following identities:

$$\sum_{|\mu|\le k_0} \frac{\partial_1 D^\mu f(x)(y-x)^\mu}{\mu!} = \sum_{|\mu|=k_0+1} \frac{D^\mu f(x)}{\mu!}\partial_1(y-x)^\mu + \sum_{|\mu|\le k_0-1} \frac{\partial_1 D^\mu f(x)(y-x)^\mu}{\mu!},$$

$$\sum_{|\mu|\le k_0} \frac{D^\mu f(x)\partial_{x_1}(y-x)^\mu}{\mu!} = -\sum_{|\mu|\le k_0-1} \frac{\partial_1 D^\mu f(x)(y-x)^\mu}{\mu!}.$$

Hence

$$\sum_{|\mu|=k_0+1} \frac{D^\mu f(x)}{\mu!}\partial_1(y-x)^\mu - \partial_{x_1}\big(T^x_{k_0}(f)(y)\big) = 0,$$

and

$$III = 0. \tag{18}$$

Combining (17) and (18), (16) is thus proved. ∎

Restricting on $\mathbf{B}_R$, the higher order derivative formula of the Newtonian potential can be simplified in a fashion that the global Hölder estimates can be achieved as in the next section. The approach is motivated by the following global formula and estimate for the second order derivative of the Newtonian potential in [Fr].

**Theorem 4.4.** *[Fr] Let $f \in \mathcal{C}^\alpha(\mathbf{B}_R)$. Then for any $x \in \mathbf{B}_R$,*

$$\partial_i\partial_j\mathcal{N}(f)(x) = \int_{\mathbf{B}_R} \partial_{x_i}\partial_{x_j}\Gamma(x-y)(f(y)-f(x))dy - \frac{\delta_{ij}}{n}f(x).$$

*Moreover, for all $f \in \mathcal{C}^\alpha(\mathbf{B}_R)$,*

$$\| \mathcal{N}(f) \|^{(2)}_\alpha \le C \| f \|_\alpha .$$

When $n = 2$, [Pan1] derived the derivative formula making use of complex analysis for the higher order derivatives of the Newtonian potential. In light of Theorem 4.4, we shall state the following higher order derivative formula of the Newtonian potential on $\mathbf{B}_R$ when $n \ge 3$.

**Theorem 4.5.** *Let $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$. Let $\beta$ be a multi-index with $|\beta| = k+2$ and $\{\beta^{(j)}\}$ a continuously increasing nesting of length $k+2$ for $\beta$. Then $D^\beta\mathcal{N}(f)(x)$ exists for $x \in \mathbf{B}_R$. Moreover,*

$$D^\beta\mathcal{N}(f) = \int_{\mathbf{B}_R} D^\beta_\cdot\Gamma(\cdot-y)\big(f(y)-T^\cdot_k(f)(y)\big)dy - \sum_{j=2}^{k+2}\sum_{|\mu|=j-2} \frac{C(\beta^{(j-1)},\mu,\beta^{(j)}-\beta^{(j-1)})}{\mu!} D^{\mu+\beta^{(j)\prime}} f. \tag{19}$$

*Here $\beta^{(j)'}$ is the dual of $\beta^{(j)}$ with respect to $\beta$, and $C(\beta^{(j-1)}, \mu, \beta^{(j)} - \beta^{(j-1)})$ is some constant dependent only on $(\beta^{(j-1)}, \mu, \beta^{(j)} - \beta^{(j-1)})$.*

Since on the right hand side of (19), the order $|\mu + \beta^{(j)'}|$ for each term in the summation is equal to $k$ by definition, Theorem 1.2 follows from Theorem 4.5.

**Proof of Theorem 4.5**: $k = 0$ is given by Theorem 4.4. When $k > 0$, for any multi-indices $\beta$ and $\mu$ with $|\beta| = |\mu| + 1$, one has for $x \in \mathbf{B}_R$ by Corollary 3.1,

$$\begin{aligned}\mathcal{I}_{\mathbf{B}_R}(\beta, \mu, j)(x) =& \int_{\partial \mathbf{B}_R} D_x^\beta \Gamma(x-y)(y-x)^\mu \nu_j d\sigma_y \\ =& \frac{1}{R} \int_{\partial \mathbf{B}_R} D_x^\beta \Gamma(x-y)(y-x)^\mu y_j d\sigma_y \\ \equiv& C(\beta, \mu, j)\end{aligned}$$

with $C(\beta, \mu, j)$ some constant dependent only on $(\beta, \mu, j)$. In particular, $C(\beta, \mu, j)$ is independent of $R$. Letting $\Omega = \mathbf{B}_R$ in Proposition 4.3, one obtains (19). ■

# 5 $\mathcal{C}^{k,\alpha}$ estimate of the Newtonian potential on $\mathbf{B}_R$

To simplify the notations, we first define the following operators.

**Definition 5.1.** *Given a multi-index $\beta$ with $|\beta| = k+2$, $k \geq 0$, the operator $\mathcal{N}_\beta : \mathcal{C}^{k,\alpha}(\mathbf{B}_R) \to \mathcal{C}(\mathbf{B}_R)$ is defined by*

$$\mathcal{N}_\beta(f)(x) := \int_{\mathbf{B}_R} D_x^\beta \Gamma(x-y)\big(f(y) - T_k^x(f)(y)\big) dy,$$

*for $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$ and $x \in \mathbf{B}_R$, where $T_k^x(f)(\cdot)$ is the $k$-th order power series expansion of $f$ at $x$.*

**Definition 5.2.** *Given multi-indices $\beta$ and $\beta'$ with $|\beta| = k+2$, $k \geq 0$ and $D^\beta = \partial_j D^{\beta'}$, the operator $\mathcal{S}_\beta : \mathcal{C}^{k,\alpha}(\mathbf{B}_R) \to \mathcal{C}(\mathbf{B}_R)$ is defined by*

$$\mathcal{S}_\beta(f)(x) := \int_{\partial \mathbf{B}_R} D_x^{\beta'} \Gamma(x-y)\big(f(y) - T_k^x(f)(y)\big) \nu_j d\sigma_y$$

*for $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$ and $x \in \mathbf{B}_R$. Here $T_k^x(f)(y)$ is the $k$-th order power series expansion of $f$ at $x$, $d\sigma_y$ is the surface area element of $\partial \mathbf{B}_R$ with the unit outer normal $(\nu_1, \ldots, \nu_n)$.*

Since $|\beta'| = k+1$, $\mathcal{S}_\beta(f)$ differs from $D^{\beta'}(\int_{\partial \mathbf{B}_R} \Gamma(\cdot - y) f(y) \nu_j d\sigma_y)$ by a constant independent of $R$ due to Corollary 3.1.

**Definition 5.3.** *Given a multi-indices* $\beta$ *with* $|\beta| = k+2, k \geq 0$ *and* $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$*, the operator* $\mathcal{T}_\beta : \mathcal{C}^{k,\alpha}(\mathbf{B}_R) \to \mathcal{C}^\alpha(\mathbf{B}_R)$ *is defined by*

$$\mathcal{T}_\beta(f) := \sum_{j=2}^{k+2} \sum_{|\mu|=j-2} \frac{C(\beta^{(j-1)}, \mu, \beta^{(j)} - \beta^{(j-1)})}{\mu!} D^{\mu+\beta^{(j)'}} f.$$

Theorem 1.2 can thus be rewritten as

$$D^\beta \mathcal{N}(f) = \mathcal{N}_\beta(f) - \mathcal{T}_\beta(f) \tag{20}$$

for any $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$.

## 5.1 An induction formula for $D^\beta \mathcal{N}$

We shall first prove an induction formula for the higher order derivative formula of the Newtonian potential.

**Lemma 5.4.** *Let* $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$*. let* $\beta, \beta'$ *be two multi-indices with* $|\beta| = k+2$ *and* $D^\beta = \partial_j D^{\beta'}$*. We have in* $\mathbf{B}_R$*,*

$$\mathcal{N}_\beta(f) = \mathcal{N}_{\beta'}(\partial_j f) - \mathcal{S}_\beta(f).$$

**Proof of Lemma 5.4**: Making use of Stokes' Theorem and Corollary 3.1, we get for $x \in \mathbf{B}_R$,

$$\begin{aligned}
\mathcal{N}_\beta(f)(x) =& \lim_{\epsilon \to 0} \int_{\mathbf{B}_R - \mathbf{B}_\epsilon(x)} D_x^\beta \Gamma(x-y)\big(f(y) - T_k^x(f)(y)\big) dy \\
=& \lim_{\epsilon \to 0} \int_{\mathbf{B}_R - \mathbf{B}_\epsilon(x)} D_x^{\beta'} \Gamma(x-y) \partial_j \big(f(y) - T_k^x(f)(y)\big) dy \\
&- \lim_{\epsilon \to 0} \int_{\mathbf{B}_R - \mathbf{B}_\epsilon(x)} \partial_j \Big( D_x^{\beta'} \Gamma(x-y)\big(f(y) - T_k^x(f)(y)\big) \Big) dy \\
=& \lim_{\epsilon \to 0} \int_{\mathbf{B}_R - \mathbf{B}_\epsilon(x)} D_x^{\beta'} \Gamma(x-y)\big(\partial_j f(y) - T_{k-1}^x(\partial_j f)(y)\big) dy \\
&- \int_{\partial \mathbf{B}_R} D_x^{\beta'} \Gamma(x-y)\big(f(y) - T_k^x(f)(y)\big) \nu_j d\sigma_y \\
&+ \lim_{\epsilon \to 0} \int_{\partial \mathbf{B}_\epsilon(x)} D_x^{\beta'} \Gamma(x-y)\big(f(y) - T_k^x(f)(y)\big) \nu_j d\sigma_y
\end{aligned}$$

Since the third term in the last identity of the above expression is 0, the formula is thus proved. ∎

**Proposition 5.5.** *Let* $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$ *and let* $\beta, \beta'$ *be two multi-indices with* $|\beta| = k+2$ *and* $D^\beta = \partial_j D^{\beta'}$*. We have*

$$D^\beta \mathcal{N}(f) = \mathcal{N}_{\beta'}(\partial_j f) - \mathcal{S}_\beta(f) - \mathcal{T}_\beta(f).$$

## 5.2 Estimates of $\mathcal{S}_\beta$

We start by proving the following preparation lemma.

**Lemma 5.6.** *For any $x \in \mathbf{B}_1$, $0 < \alpha < 1$,*

$$\int_{|y|=1} \frac{1}{|x-y|^{n-\alpha}} d\sigma_y \leq C(1-|x|)^{\alpha-1}.$$

**Proof of Lemma 5.6**: Assume $x = (r, 0, \ldots, 0)$ after rotation if necessary. One can assume in addition that $r \geq \frac{1}{2}$.

Choose spherical coordinates $y_1 = \cos\theta_1, y_2 = \sin\theta_1 \cos\theta_2, \ldots, y_n = \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1}$, where $0 \leq \theta_i \leq \pi$ for $1 \leq i \leq n-2$, $0 \leq \theta_{n-1} \leq 2\pi$ and denote by $\partial \mathbf{B}_1^{n-1}$ the unit sphere in $\mathbb{R}^{n-1}$. We have,

$$\begin{aligned}
\int_{|y|=1} \frac{1}{|x-y|^{n-\alpha}} d\sigma_y =& \int_0^\pi \frac{\sin^{n-2}\theta_1}{[(\cos\theta_1 - r)^2 + \sin^2\theta_1]^{\frac{n-\alpha}{2}}} d\theta_1 \int_{\partial \mathbf{B}_1^{n-1}} d\sigma_z \\
=& C \int_0^\pi \frac{\sin^{n-2}\theta}{[1 - 2r\cos\theta + r^2]^{\frac{n-\alpha}{2}}} d\theta \\
=& C \int_0^\pi \frac{\sin^{n-2}\theta}{[(1-r)^2 + 4r\sin^2\frac{\theta}{2}]^{\frac{n-\alpha}{2}}} d\theta \\
\leq& C\Big(\int_0^{1-r} \frac{\sin^{n-2}\theta}{(1-r)^{n-\alpha}} d\theta + \int_{1-r}^\pi \frac{\sin^{n-2}\theta}{(2\sqrt{r}\sin\frac{\theta}{2})^{n-\alpha}} d\theta\Big) \\
=& A + B,
\end{aligned}$$

where $A := C\int_0^{1-r} \frac{\sin^{n-2}\theta}{(1-r)^{n-\alpha}} d\theta$ and $B := C\int_{1-r}^\pi \frac{\sin^{n-2}\theta}{(2\sqrt{r}\sin\frac{\theta}{2})^{n-\alpha}} d\theta$.

For $A$, since $\sin\theta \leq \theta$ when $\theta > 0$,

$$A \leq \frac{C}{(1-r)^{n-\alpha}} \int_0^{1-r} \theta^{n-2} d\theta = \frac{C}{(1-r)^{n-\alpha}} (1-r)^{n-1} = C(1-r)^{\alpha-1}.$$

For $B$, making use of $\sin\theta \geq C\theta$ when $0 \leq \theta \leq \frac{\pi}{2}$ and the assumption $r \geq \frac{1}{2}$, we get

$$B \leq C \int_{1-r}^\pi \frac{\sin^{n-2}\frac{\theta}{2}}{(\sin\frac{\theta}{2})^{n-\alpha}} d\theta = C \int_{1-r}^\pi \sin^{\alpha-2}\frac{\theta}{2} d\theta \leq C \int_{1-r}^\pi \left(\frac{\theta}{2}\right)^{\alpha-2} d\theta \leq C(1-r)^{\alpha-1} + C \leq C(1-r)^{\alpha-1}.$$

The lemma is thus concluded. ∎

The following lemma in Appendix 6.2a [NW] is quoted without proof.

**Lemma 5.7.** *[NW] If $z$ and $z'$ are two points of the open unit disk in $\mathbb{C}$, and $\gamma$ is the shorter segment of the circle through $z$ and $z'$ and orthogonal to the unit circle, then*

$$\int_\gamma \frac{|dw|}{(1 - w\bar{w})^{1-\alpha}} \leq \frac{2}{1-\alpha} |z - z'|^\alpha$$

*for $0 < \alpha < 1$.*

We are now in a position to show $\mathcal{S}_\beta$ is a bounded operator from $\mathcal{C}^{k,\alpha}(\mathbf{B}_R)$ into $\mathcal{C}^{\alpha}(\mathbf{B}_R)$.

**Lemma 5.8.** *let $\beta$ be a multi-index with $|\beta| = k+2$. The operator $\mathcal{S}_\beta$ sends $\mathcal{C}^{k,\alpha}(\mathbf{B}_R)$ into $\mathcal{C}^{\alpha}(\mathbf{B}_R)$. Moreover, for any $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$,*

$$\parallel \mathcal{S}_\beta(f) \parallel_\alpha \leq C \parallel f \parallel_\alpha^{(k)} .$$

**Proof of Lemma 5.8**: Write $g(y) := f(y) - T_k^x(f)(y)$.

(i) The estimate for $\parallel \mathcal{S}_\beta(f) \parallel$. Indeed, by Lemma 2.1, for $x \in \mathbf{B}_R$,

$$\begin{aligned} |\int_{\partial \mathbf{B}_R} D_x^{\beta'} \Gamma(x-y) g(y) \nu_j d\sigma_y| \leq & C \parallel f \parallel_\alpha^{(k)} R^{-\alpha} \int_{\partial \mathbf{B}_R} |y-x|^{2-n-k-1} |y-x|^{k+\alpha} d\sigma_y \\ = & C \parallel f \parallel_\alpha^{(k)} R^{-\alpha} \int_{\partial \mathbf{B}_R} |y-x|^{1-n+\alpha} d\sigma_y \\ = & C \parallel f \parallel_\alpha^{(k)} \int_{\partial \mathbf{B}_1} |y - \frac{x}{R}|^{1-n+\alpha} d\sigma_y \\ \leq & C \parallel f \parallel_\alpha^{(k)} . \end{aligned}$$

(ii) Given $x, x' \in \mathbf{B}_R$, we estimate $|\mathcal{S}_\beta(f)(x) - \mathcal{S}_\beta(f)(x')|$. Assume without loss of generality that $x, x'$ lie on the plane $\{y_3 = \cdots = y_n = 0\}$ and write $x = Rz, x' = Rz'$ with $z, z' \in \mathbf{B}_1$. Then

$$\begin{aligned} \mathcal{S}_\beta(f)(x) - \mathcal{S}_\beta(f)(x') = & \int_{\partial \mathbf{B}_R} \big( D_x^{\beta'} \Gamma(x-y) - D_x^{\beta'} \Gamma(x'-y) \big) g(y) \nu_j d\sigma_y \\ = & R^{-k} \int_{\partial \mathbf{B}_1} \big( D_z^{\beta'} \Gamma(z-y) - D_{z'}^{\beta'} \Gamma(z'-y) \big) g(Ry) \nu_j d\sigma_y . \end{aligned}$$

Let $\gamma(t) = (\gamma_1(t), \gamma_2(t), 0, \ldots, 0) : [0,1] \to \{y_3 = \cdots = y_n = 0\} \cong \mathbb{C}$ be a parametrization of the shorter segment of the circle through $z$ and $z'$ and orthogonal to the unit circle in $\mathbb{C}$ with $\gamma(0) = z', \gamma(1) = z$. We then have

$$\begin{aligned} \mathcal{S}_\beta(f)(x) - \mathcal{S}_\beta(f)(x') = & R^{-k} \int_{\partial \mathbf{B}_1} \int_0^1 \frac{d}{dt} \big( D_\gamma^{\beta'} \Gamma(\gamma(t)-y) \big) dt g(Ry) \nu_j d\sigma_y \\ = & R^{-k} \int_0^1 \sum_{k=1}^{2} \gamma_k'(t) dt \int_{\partial \mathbf{B}_1} \big( \partial_{\gamma_k} D_\gamma^{\beta'} \Gamma(\gamma(t)-y) \big) g(Ry) \nu_j d\sigma_y . \end{aligned}$$

Making use of Corollary 3.1, we have for any $0 \leq t \leq 1$,

$$\int_{\partial \mathbf{B}_1} \big( \partial_{\gamma_k} D_\gamma^{\beta'} \Gamma(\gamma(t)-y) \big) g(Ry) \nu_j d\sigma_y = \int_{\partial \mathbf{B}_1} \big( \partial_{\gamma_k} D_\gamma^{\beta'} \Gamma(\gamma(t)-y) \big) \big( g(Ry) - T_k^{R\gamma(t)}(g)(Ry) \big) \nu_j d\sigma_y ,$$

where $T_k^{R\gamma(t)}(g)(\cdot)$ is the $k$-th order power series expansion of $g$ at $R\gamma(t)$. Furthermore, by Lemma 2.1,

$$
\begin{aligned}
|g(Ry) - T_k^{R\gamma(t)}(g)(Ry)| &\leq C|Ry - R\gamma(t)|^{k+\alpha} \sum_{|\mu|=k} H_\alpha[D^\mu g] \\
&= CR^{k+\alpha}|y-\gamma(t)|^{k+\alpha} \sum_{|\mu|=k} H_\alpha[D^\mu f].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&|\mathcal{S}_\beta(f)(x) - \mathcal{S}_\beta(f)(x')| \\
\leq & C\big(\sum_{|\mu|=k} H_\alpha[D^\mu f]\big) R^{-k} \int_0^1 \sum_{k=1}^2 |\gamma_k'(t)| dt \int_{\partial \mathbf{B}_1} |\gamma(t) - y|^{2-n-k-2} R^{k+\alpha} |y-\gamma(t)|^{k+\alpha} d\sigma_y \\
\leq & C\big(\sum_{|\mu|=k} H_\alpha[D^\mu f]\big) R^{\alpha} \int_0^1 \sum_{k=1}^2 |\gamma_k'(t)| dt \int_{\partial \mathbf{B}_1} |\gamma(t) - y|^{-n+\alpha} d\sigma_y.
\end{aligned}
$$

Applying Lemma 5.6 to $\int_{\partial \mathbf{B}_1} |\gamma(t) - y|^{-n+\alpha} d\sigma_y$ in the last expression, we have

$$
\begin{aligned}
|\mathcal{S}_\beta(f)(x) - \mathcal{S}_\beta(f)(x')| \leq & C\big(\sum_{|\mu|=k} H_\alpha[D^\mu f]\big) R^{\alpha} \int_0^1 \sum_{k=1}^2 \frac{|\gamma_k'(t)|}{(1-|\gamma(t)|)^{1-\alpha}} dt \\
\leq & C\big(\sum_{|\mu|=k} H_\alpha[D^\mu f]\big) R^{\alpha} \int_0^1 \frac{|\gamma'(t)| dt}{(1-|\gamma(t)|^2)^{1-\alpha}} \\
= & C\big(\sum_{|\mu|=k} H_\alpha[D^\mu f]\big) R^{\alpha} \int_\gamma \frac{|dw|}{(1-w\bar{w})^{1-\alpha}}.
\end{aligned}
$$

Hence by Lemma 5.7,

$$
|\mathcal{S}_\beta(f)(x) - \mathcal{S}_\beta(f)(x')| \leq C\big(\sum_{|\mu|=k} H_\alpha[D^\mu f]\big) R^\alpha |z-z'|^\alpha = C\big(\sum_{|\mu|=k} H_\alpha[D^\mu f]\big) |x-x'|^\alpha.
$$

Namely,

$$
H_\alpha[\mathcal{S}_\beta(f)] \leq C\big(\sum_{|\mu|=k} H_\alpha[D^\mu f]\big).
$$

We finally have shown, combining (i) and (ii),

$$
\| \mathcal{S}_\beta(f) \|_\alpha \leq C \| f \|_\alpha^{(k)} . \blacksquare
$$

**Remark 5.9.** *In the proof of Lemma 5.8(ii) when carrying out the Hölder norm for $\mathcal{S}_\beta$, the estimate in Lemma 5.7 would fail if the curve $\gamma$ is chosen to be the straight line connecting $z$ and $z$ instead of the geodesic as in [NW].*

## 5.3 Estimate on $D^\beta\mathcal{N}$

Applying Lemma 5.4 and Lemma 5.8 inductively, we eventually obtain the following formula and estimate.

**Theorem 5.10.** *For any $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$, $\mathcal{N}(f) \in \mathcal{C}^{k+2,\alpha}(\mathbf{B}_R)$. Moreover, given a multi-index $\beta$ with $|\beta| = k+2$, let $\{\beta^{(j)}\}$ be a continuously increasing nesting for $\beta$ of length $k+2$ and $\beta^{(j)'}$ be the dual of $\beta^{(j)}$ with respect to $\beta$ for $2 \le j \le k+2$, and $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$, we have*

$$D^\beta\mathcal{N}(f) = \mathcal{N}_{\beta^{(2)}}(D^{\beta^{(2)'}}f) - \sum_{j=3}^{k+2}\mathcal{S}_{\beta^{(j)}}(D^{\beta^{(j)'}}f) - \mathcal{T}_\beta(f),$$

*in $\mathbf{B}_R$ with*

$$\| \mathcal{N}(f) \|_\alpha^{(k+2)} \le C \| f \|_\alpha^{(k)} .$$

**Proof of Theorem 5.10**: By Lemma 5.5 together with (20),

$$\begin{aligned}
D^\beta\mathcal{N}(f) &= \mathcal{N}_{\beta^{(k+1)}}(D^{\beta^{(k+1)'}}f) - \mathcal{S}_{\beta^{(k+2)}}(f) - \mathcal{T}_\beta(f)\\
&= \mathcal{N}_{\beta^{(k)}}(D^{\beta^{(k)'}}f) - \mathcal{S}_{\beta^{(k+1)}}(D^{\beta^{(k+1)'}}f) - \mathcal{S}_{\beta^{(k+2)}}(f) - \mathcal{T}_\beta(f)\\
&= \cdots\\
&= \mathcal{N}_{\beta^{(2)}}(D^{\beta^{(2)'}}f) - \sum_{j=3}^{k+2}\mathcal{S}_{\beta^{(j)}}(D^{\beta^{(j)'}}f) - \mathcal{T}_\beta(f)\\
&= D^{\beta^{(2)}}\mathcal{N}(D^{\beta^{(2)'}}f) - \sum_{j=3}^{k+2}\mathcal{S}_{\beta^{(j)}}(D^{\beta^{(j)'}}f) - \mathcal{T}_\beta(f) - \mathcal{T}_{\beta^{(2)}}(D^{\beta^{(2)'}}f)
\end{aligned}$$

Hence for any $f \in \mathcal{C}^{k,\alpha}(\mathbf{B}_R)$,

$$\begin{aligned}
\| \mathcal{N}(f) \|_\alpha^{(k+2)} &:= \sup_{|\beta|=k+2} \| D^\beta\mathcal{N}(f) \|_\alpha\\
&\le C \sup_{|\beta|=k+2} \Big[ \| D^{\beta^{(2)}}\mathcal{N}(D^{\beta^{(2)'}}f) \|_\alpha + \sum_{j=3}^{k+2} \| \mathcal{S}_{\beta^{(j)}}(D^{\beta^{(j)'}}f) \|_\alpha + \| f \|_\alpha^{(k)} \Big].
\end{aligned}$$

Since $|\beta^{(j)}| = j$ and $|\beta^{(j)'}| = k+2-j$ from definition, by Theorem 4.4 and Lemma 5.8, we get

$$\begin{aligned}
\| \mathcal{N}(f) \|_\alpha^{(k+2)} &\le C \sup_{|\beta|=k+2} \Big[ \| \mathcal{N}(D^{\beta^{(2)'}}f) \|_\alpha^{(2)} + \sum_{j=3}^{k+2} \| D^{\beta^{(j)'}}f \|_\alpha^{(j-2)} + \| f \|_\alpha^{(k)} \Big]\\
&\le C \sup_{|\beta|=k+2} \Big[ \| D^{\beta^{(2)'}}f \|_\alpha + \| f \|_\alpha^{(k+2-j+j-2)} + \| f \|_\alpha^{(k)} \Big]\\
&\le C \| f \|_\alpha^{(k)} . \blacksquare
\end{aligned}$$

**Proof of Theorem 1.3**: The theorem is a direct consequence of Theorem 5.10. ∎

## 6 Construction of the contraction map

Recall that the nonlinear system under investigation is given by

$$\triangle^m u(x) = a(x, u, \nabla u, \dots, \nabla^{2m} u).$$

Assume $a \in \mathcal{C}^2$ in the above system first. For any vector-valued function $f \in (C_0^{2m,\alpha}(\mathbf{B}_R))^N$, introduce $\omega^{(1)}(f) := (\omega_1^{(1)}(f), \dots, \omega_N^{(1)}(f))$ with

$$\omega_j^{(1)}(f)(x) = \int_{\mathbf{B}_R} \Gamma(x-y) a_j(y, f(y), \nabla f(y), \dots, \nabla^{2m} f(y)) dy$$

for $1 \le j \le N$ in $\mathbf{B}_R$. According to Theorem 5.10, $\omega^{(1)}(f) \in (C^{2,\alpha}(\mathbf{B}_R))^N$ and

$$\| \omega_j^{(1)}(f) \|_\alpha^{(2)} \le C \| a_j(\cdot, f, \dots, \nabla^{2m} f) \|_\alpha .$$

$\omega^{(l)}(f) = (\omega_1^{(l)}(f), \dots, \omega_N^{(l)}(f))$ are inductively defined for $1 \le l \le m$ as follows. For each $1 \le j \le N$ and $x \in \mathbf{B}_R$,

$$\omega_j^{(l)}(f)(x) := \mathcal{N}(\omega_j^{(l-1)}(f))(x).$$

Note that, in terms of the Newtonian potential,

$$\omega_j^{(l)}(f) = \mathcal{N}^l\big(a_j(\cdot, f, \dots, \nabla^{2m} f)\big).$$

Therefore, by Theorem 5.10, $\omega^{(l)}(f) \in (C^{2l,\alpha}(\mathbf{B}_R))^N$ and

$$\| \omega_j^{(l)}(f) \|_\alpha^{(2l)} \le \| a_j(\cdot, f, \dots, \nabla^{2m} f) \|_\alpha . \tag{21}$$

Next, define $\theta(f) := (\theta_1(f), \dots, \theta_N(f))$ from $\omega^{(m)}(f)$ by truncating degree less than $2m$ terms and part of the degree $2m$ terms in its power series expansion at 0. Precisely speaking, for $1 \le j \le N$ and $x \in \mathbf{B}_R$,

$$\theta_j(f)(x) = \omega_j^{(m)}(f)(x) - T_{2m-1}\big(\omega_j^{(m)}(f)\big)(x) - \sum_{\beta \in \Lambda} \frac{D^\beta(\omega_j^{(m)}(f))(0)}{\beta!} x^\beta, \tag{22}$$

where $T_{2m-1}(\omega_j^{(m)}(f))$ is the $(2m-1)$-th power series expansion of $\omega_j^{(m)}(f)$ at 0, $\Lambda = \{\beta : |\beta| = 2m, \ \text{and at least one of } \beta_j \text{ is odd for } 1 \le j \le n\}$.

From the construction, it is immediate to see that for any $f \in (C_0^{2m,\alpha}(\mathbf{B}_R))^N$, $\omega^{(m)}(f) \in (C^{2m,\alpha}(\mathbf{B}_R))^N$ and so $\theta(f) \in (\mathcal{C}_0^{2m,\alpha}(\mathbf{B}_R))^N$. Moreover, $\triangle^m \theta(f)(x) = a(x, f(x), \nabla f(x), \dots, \nabla^{2m} f(x))$ when $x \in \mathbf{B}_R$.

Recall $(\mathcal{C}_0^{2m,\alpha}(\mathbf{B}_R), \| \cdot \|_\alpha^{(2m)})$ is a Banach space. We now have constructed an operator between two Banach spaces as follows.

$$\theta : (C_0^{2m,\alpha}(\mathbf{B}_R))^N \to (C_0^{2m,\alpha}(\mathbf{B}_R))^N$$

with the corresponding norm

$$\| f \|_\alpha^{(2m)} = \max_{1\le j\le N} \| f_j \|_\alpha^{(2m)} .$$

The ball of radius $\gamma$ in $(C_0^{2m,\alpha}(\mathbf{B}_R))^N$ is denoted by

$$\mathcal{B}(R,\gamma) := \{f \in C_0^{2m,\alpha}(\mathbf{B}_R))^N : \| f \|_\alpha^{(2m)} < \gamma\}.$$

On the other hand, recall a function $u \in \mathcal{C}^{2k}$ is called $k$-harmonic if $\triangle^k u = 0$. Given $h = (h_1, \ldots, h_N)$ with $h_j$ any homogeneous $m$-harmonic polynomial of degree $2m$ and for any $f \in (\mathcal{C}_0^{2m,\alpha}(\mathbf{B}_R))^N$, consider

$$\theta_h(f) = h + \theta(f).$$

Then $\theta_h(f) \in (\mathcal{C}_0^{2m,\alpha}(\mathbf{B}_R))^N$, $\triangle^m \theta_h(f)(x) = \triangle^m \theta(f)(x) = a(x, f(x), \nabla f(x), \ldots, \nabla^{2m} f(x))$ in $\mathbf{B}_R$ while part of $2m$ jets $D^\beta \theta_h(f)(0)$ with $\beta \in \Lambda$ coincide with those of the given parameter $h$.

We will seek the solutions to (1) by making use of the fixed point theorem. Indeed, we first show there exists $\gamma > 0$ and $R > 0$, such that $\theta : \mathcal{B}(R,\gamma) \to \mathcal{B}(R, \frac{\gamma}{2})$ and $\theta$ is a contraction map. We then pick some nontrivial $h$ as above with $h \in \mathcal{B}(R, \frac{\gamma}{2})$ and consider the corresponding operator $\theta_h$. Consequently, $\theta_h : \mathcal{B}(R,\gamma) \to \mathcal{B}(R,\gamma)$ and is a contraction map. As an application of the fixed point theorem, there exists some $u \in (\mathcal{C}_0^{2m,\alpha}(\mathbf{B}_R))^N$ such that $\theta_h(u) = u$. This $u$ clearly satisfies $\triangle^m u = \triangle^m \theta_h(u) = a(\cdot, u, \nabla u, \ldots, \nabla^{2m} u)$ in $\mathbf{B}_R$. To obtain infinitely many solutions, one only needs to notice there are infinitely many choices for such an $h$. For instance, one can pick $h(x) = cx^\beta$ with $\beta \in \Lambda$ and $c$ any nonzero small constant. Each such different input $h$ results in a different solution out of the fixed point theorem from the construction.

**Remark 6.1.** *Since the solution $u$ is of vanishing order precisely $2m$ from the construction, $u$ is not a trivial solution.*

We divide our proof into two steps. In each step, we shall use Theorem 5.10.

## 6.1 Estimate of $\| \theta(f) - \theta(g) \|_\alpha^{(2m)}$

First, we note from (22) that for $1 \le j \le N$, for any $f, g \in \mathcal{B}(R,\gamma)$,

$$\begin{aligned}
\| \theta_j(f) - \theta_j(g) \|_\alpha^{(2m)} \le & \| \omega_j^{(m)}(f) - \omega_j^{(m)}(g) \|_\alpha^{(2m)} + \| \nabla^{2m}(\omega_j^{(m)}(f) - \omega_j^{(m)}(g)) \| \\
\le & 2 \| \omega_j^{(m)}(f) - \omega_j^{(m)}(g) \|_\alpha^{(2m)} \\
= & 2 \| \mathcal{N}(\omega_j^{(m-1)}(f)) - \mathcal{N}(\omega_j^{(m-1)}(g)) \|_\alpha^{(2m)} \\
= & 2 \| \mathcal{N}\big(\omega_j^{(m-1)}(f) - \omega_j^{(m-1)}(g)\big) \|_\alpha^{(2m)} \\
= & 2 \| \mathcal{N}^m \big(a_j(\cdot, f(\cdot), \nabla f(\cdot), \ldots, \nabla^{2m} f(\cdot)) - a_j(\cdot, g(\cdot), \nabla g(\cdot), \ldots, \nabla^{2m} g(\cdot))\big) \|_\alpha^{(2m)} .
\end{aligned}$$

Making use of Theorem 5.10 into the above expression, we have then

$$\| \theta_j(f) - \theta_j(g) \|_\alpha^{(2m)} \le C \| a_j(\cdot, f(\cdot), \nabla f(\cdot), \ldots, \nabla^{2m} f(\cdot)) - a_j(\cdot, g(\cdot), \nabla g(\cdot), \ldots, \nabla^{2m} g(\cdot)) \|_\alpha . \tag{23}$$

We next proceed to prove an estimate of (23). Note that due to Lemma 2.2, when $f \in \mathcal{B}(R,\gamma)$, then $\| \nabla^j f \| \leq CR^{2m-j}\gamma$ for $0 \leq j \leq 2m$. Therefore, the variables $(p_{-1}, p_0, p_1, \ldots, p_{2m})$ of the vector-valued function $a$ takes value in $E := \{p_{-1} \in \mathbf{B}_R, p_j \in \mathbf{B}_{CR^{2m-j}\gamma}, 0 \leq j \leq 2m\}$ when $u \in \mathcal{B}(R,\gamma)$.

Denote by $A_j := \sup_E |\nabla_{p_j} a|$, $Q_{jk} := \sup\big\{\frac{|\nabla_{p_j} a(p_{-1},\ldots,p_{k-1},p_k,p_{k+1},\cdots,p_{2m}) - \nabla_{p_j} a(p_{-1},\ldots,p_{k-1},p'_k,p_{k+1},\ldots,p_{2m})|}{|p_k - p'_k|^\alpha} : (p_{-1},\ldots,p_{k-1},p_k,p_{k+1},\cdots,p_{2m}), (p_{-1},\ldots,p_{k-1},p'_k,p_{k+1},\cdots,p_{2m}) \in E, p_k \neq p'_k\big\}$ and $L_j := \sup_E(\nabla^2_{p_j p_{2m}} a)$ with $-1 \leq j \leq 2m$. Therefore, for $-1 \leq j,k \leq 2m$,

$$\begin{aligned} &A_j \leq C \| \nabla_{p_j} a \|_{\mathcal{C}^0(E)} \leq C \| a \|_{\mathcal{C}^{1,\alpha}(E)}, \\ &Q_{jk} \leq C \| a \|_{\mathcal{C}^{1,\alpha}(E)}, \\ &L_j \leq C \| \nabla_{p_{2m}} a \|_{\mathcal{C}^1(E)} \leq C \| a \|_{\mathcal{C}^2(E)} . \end{aligned} \tag{24}$$

Here $\| a \|_{\mathcal{C}^{1,\alpha}(E)} = \| a \|_{\mathcal{C}^1} + H_\alpha[\nabla a]$.

**Lemma 6.2.** *For any $f, g \in \mathcal{B}(R,\gamma)$, if $a \in \mathcal{C}^2$,*

$$\| a(\cdot, f(\cdot), \nabla f(\cdot), \ldots, \nabla^{2m} f(\cdot)) - a(\cdot, g(\cdot), \nabla g(\cdot), \ldots, \nabla^{2m} g(\cdot)) \|_\alpha \leq \delta(R,\gamma) \| f - g \|_\alpha^{(2m)},$$

*where*

$$\delta(R,\gamma) = C \sum_{j=0}^{2m} R^{2m-j}\big(A_j + R^\alpha(Q_{j(-1)} + \sum_{k=0}^{2m-1} Q_{jk} R^{(2m-k-1)\alpha}\gamma^\alpha) + \gamma L_j\big). \tag{25}$$

*Moreover, if $a$ is independent of $p_{2m}$, then when $a \in \mathcal{C}^{1,\alpha}$,*

$$\delta(R,\gamma) = C \sum_{j=0}^{2m-1} R^{2m-j}\big(A_j + R^\alpha(Q_{j(-1)} + \sum_{k=0}^{2m-1} Q_{jk} R^{(2m-k-1)\alpha}\gamma^\alpha)\big). \tag{26}$$

The proof of Lemma 6.2 is similar to that of Lemma 4.3 in [Pan2]. For the completeness of the paper, we provide with a sketch of the proof as follows.

**Proof of Lemma 6.2**: For simplicity of notation, we shall skip the dots in $f(\cdot)$ and $g(\cdot)$.

$$\begin{aligned} &\| a(\cdot, f, \nabla f, \ldots, \nabla^{2m} f) - a(\cdot, g, \nabla g, \ldots, \nabla^{2m} g) \|_\alpha \\ = &\| \int_0^1 \frac{d}{dt} a(\cdot, tf + (1-t)g, \nabla(tf + (1-t)g), \ldots, \nabla^{2m}(tf + (1-t)g)) dt \|_\alpha \\ \leq &\sum_{j=0}^{2m} \| \mathcal{A}_j^t \cdot \nabla^j (f-g) \|_\alpha \\ \leq &\sum_{j=0}^{2m} \| \mathcal{A}_j^t \|_\alpha \| f - g \|_\alpha^{(j)} . \end{aligned}$$

Here $\mathcal{A}_j^t(\cdot) := \int_0^1 \nabla_{p_j} a(\cdot, tf + (1-t)g, \nabla(tf + (1-t)g), \ldots, \nabla^{2m}(tf + (1-t)g))dt$. If $a$ is independent of $p_{2m}$, then the summation in the above expression runs over 1 through $2m-1$ instead of $2m$.

Making use of Lemma 2.2 together with the fact that $f, g \in \mathcal{B}(R, \gamma)$, we obtain

$$\| a(\cdot, f, \nabla f, \ldots, \nabla^{2m} f) - a(\cdot, g, \nabla g, \ldots, \nabla^{2m} g) \|_\alpha \leq C \sum_{j=0}^{2m} R^{2m-j} \| \mathcal{A}_j^t \|_\alpha \| f - g \|_\alpha^{(2m)} . \quad (27)$$

Next, we shall estimate $\| \mathcal{A}_j^t \|_\alpha$. Indeed, $\| \mathcal{A}_j^t \| \leq A_j$ by definition. To show its Hölder estimates, we denote $tf + (1-t)g$ by $\tilde{f}^t$, or even simpler but without confusion, by $\tilde{f}$. Then $\tilde{f} \in \mathcal{B}(R, \gamma)$ and hence $\| \tilde{f} \|_\alpha^{(2m)} \leq \gamma$. For $x, x' \in \mathbf{B}_R$, by an elementary triangle inequality argument,

$$\begin{aligned}
&|\mathcal{A}_j^t(x) - \mathcal{A}_j^t(x')| \\
\leq& C\big(Q_{j(-1)}|x - x'|^\alpha + \sum_{k=0}^{2m-1} Q_{jk} |\nabla^k(\tilde{f}(x) - \tilde{f}(x'))|^\alpha + L_j |\nabla^{2m}(\tilde{f}(x) - \tilde{f}(x'))|\big) \\
\leq& C\big(Q_{j(-1)}|x - x'|^\alpha + \sum_{k=0}^{2m-1} Q_{jk} (\| \nabla^{k+1} \tilde{f} \| \, |x - x'|)^\alpha + L_j R^{-\alpha} \| \tilde{f} \|_\alpha^{(2m)} |x - x'|^\alpha\big) \\
\leq& C\big(Q_{j(-1)}|x - x'|^\alpha + \sum_{k=0}^{2m-1} Q_{jk} R^{(2m-k-1)\alpha} (\| \tilde{f} \|_\alpha^{(2m)})^\alpha |x - x'|^\alpha + L_j R^{-\alpha} \| \tilde{f} \|_\alpha^{(2m)} |x - x'|^\alpha\big).
\end{aligned}$$

We note that if $a$ is independent of $p_{2m}$, then $L_j$ term does not show up in the expression and hence $C^2$ regularity of $a$ is not needed due to the above estimates. We conclude that

$$\| \mathcal{A}_j^t \|_\alpha \leq C\big(A_j + R^\alpha (Q_{j(-1)} + \sum_{k=0}^{2m-1} Q_{jk} R^{(2m-k-1)\alpha} \gamma^\alpha) + \gamma L_j\big). \quad (28)$$

The lemma follows consequently by combining (28) and (27). ∎

We then have obtained from (23), by using Lemma 6.2 that

$$\| \theta(f) - \theta(g) \|_\alpha^{(2m)} \leq \delta(R, r) \| f - g \|_\alpha^{(2m)}, \quad (29)$$

with $\delta(R, \gamma)$ given in (25) or (26).

## 6.2 Estimate of $\| \theta(f) \|_\alpha^{(2m)}$

Similarly, for $f \in \mathcal{B}(R, \gamma)$, $1 \leq j \leq N$,

$$\begin{aligned}
\| \theta_j(f) \|_\alpha^{(2m)} \leq& \| \omega_j^{(m)}(f) \|_\alpha^{(2m)} + \| \nabla^{2m}(\omega_j^{(m)}(f)) \| \\
\leq& 2 \| \omega_j^{(m)}(f) \|_\alpha^{(2m)} \\
=& 2 \| \mathcal{N}^m\big(a_j(\cdot, f(\cdot), \nabla f(\cdot), \ldots, \nabla^{2m} f(\cdot))\big) \|_\alpha^{(2m)} \\
\leq& C \| a_j(\cdot, f(\cdot), \nabla f(\cdot), \ldots, \nabla^{2m} f(\cdot)) \|_\alpha .
\end{aligned} \quad (30)$$

The following lemma leads to an estimate of (30) for the purpose of a contraction map.

**Lemma 6.3.** *For any $f \in \mathcal{B}(R,\gamma)$, if $a \in \mathcal{C}^2$,*

$$\| a(\cdot, f(\cdot), \nabla f(\cdot), \dots, \nabla^{2m} f(\cdot)) \|_\alpha \leq \eta(R,r),$$

*where*

$$\eta(R,\gamma) = |a(0)| + C\Big( R\big(A_{-1} + R^\alpha (Q_{(-1)(-1)} + \sum_{k=0}^{2m-1} Q_{(-1)k} R^{(2m-k-1)\alpha}\gamma^\alpha + \gamma L_{-1})\big) + \gamma\delta(R,\gamma)\Big) \tag{31}$$

*with $\delta(R,\gamma)$ given in (25).*

*Moreover, if $a$ is independent of $p_{2m}$, then when $a \in \mathcal{C}^{1,\alpha}$,*

$$\eta(R,\gamma) = |a(0)| + C\Big( R\big(A_{-1} + R^\alpha (Q_{(-1)(-1)} + \sum_{k=0}^{2m-1} Q_{(-1)k} R^{(2m-k-1)\alpha}\gamma^\alpha)\big) + \gamma\delta(R,\gamma)\Big). \tag{32}$$

*with $\delta(R,\gamma)$ given in (26).*

**Proof of Lemma 6.3**: The proof is similar to that of Lemma 6.2. Indeed, for $-1 \leq j \leq 2m$, write $\mathcal{B}_j(\cdot) := \mathcal{B}_j^t(\cdot) = \int_0^1 \nabla_{p_j} a(t\cdot, tf, t\nabla f, \dots, t\nabla^{2m} f)dt$, then

$$\begin{aligned}
&\| a(\cdot, f, \nabla f, \dots, \nabla^{2m} f) \|_\alpha \\
&= \| a(0) + \int_0^1 \frac{d}{dt} a(t\cdot, tf, t\nabla f, \dots, t\nabla^{2m} f)dt \|_\alpha \\
&\leq |a(0)| + \| \mathcal{B}_{-1} \cdot x \|_\alpha + \sum_{j=0}^{2m} \| \mathcal{B}_j \cdot \nabla^j f \|_\alpha \\
&\leq |a(0)| + C\big(R \| \mathcal{B}_{-1} \|_\alpha + \sum_{j=0}^{2m} \| \mathcal{B}_j \|_\alpha \| f \|_\alpha^{(j)} \big) \\
&\leq |a(0)| + C\big(R \| \mathcal{B}_{-1} \|_\alpha + \sum_{j=0}^{2m} R^{2m-j} \| \mathcal{B}_j \|_\alpha \| f \|_\alpha^{(2m)} \big).
\end{aligned}$$

If $a$ is independent of $p_{2m}$, the summation in the above expression runs over 1 through $2m-1$ instead. On the other hand, using exactly the same argument as in the estimate of $\| \mathcal{A}_j^t \|_\alpha$ in Lemma 6.2, we have for $-1 \leq j \leq 2m$, $\| \mathcal{B}_j \| \leq A_j$, and

$$\| \mathcal{B}_j \|_\alpha \leq C\big(A_j + R^\alpha (Q_{j(-1)} + \sum_{k=0}^{2m-1} Q_{jk} R^{(2m-k-1)\alpha}\gamma^\alpha) + \gamma L_j\big).$$

As before, when $a$ is independent of $p_{2m}$, $L_j$ term does not show up in the above inequality. The proof of the lemma is thus complete. ∎

Combining Lemma 6.3 and (30), we have

$$\parallel \theta(f) \parallel_\alpha^{(2m)} \leq \eta(R,\gamma), \tag{33}$$

with $\eta(R,\gamma)$ given in (31) or in (32).

# 7 Proof of Theorem 1.4 - 1.6

We first prove a slightly more general result following [Pan2].

**Theorem 7.1.** *Let $a \in \mathcal{C}^2$ and $a(0) = 0$. There is a constant $\delta(<1)$ depending only on $n, N$ and $\alpha$, such that when*

$$|\nabla_{p_{2m}} a(0)| + |\nabla^2_{p_{2m}p_{2m}} a(0)| \leq \delta,$$

*the system (1) has infinitely many solutions in $\mathcal{C}^{2m,\alpha}$ of vanishing order $2m$ at the origin in some small neighborhood.*

**Proof of Theorem 7.1**: Our goal is to show $\theta$ sends $\mathcal{B}(R,\gamma)$ into $\mathcal{B}(R,\frac{\gamma}{2})$ for some positive $R$ and $\gamma$ and is a contraction map between $\mathcal{B}(R,\gamma)$. In other words, we show there exist $\gamma > 0$ and $R > 0$ such that for any $f, g \in \mathcal{B}(R,\gamma)$,

$$\parallel \theta(f) - \theta(g) \parallel_\alpha^{(2m)} \leq c \parallel f - g \parallel_\alpha^{(2m)} \quad \text{with } c < 1$$

and

$$\parallel \theta(f) \parallel_\alpha^{(2m)} < \frac{\gamma}{2}.$$

From (29) and (33), it boils down to show there exist $\gamma > 0$ and $R > 0$ such that

$$\begin{aligned} \delta(R,\gamma) &\leq c < 1 \\ \eta(R,\gamma) &< \frac{\gamma}{2}. \end{aligned} \tag{34}$$

Denote by $\tau := |\nabla_{p_{2m}} a(0)| + |\nabla^2_{p_{2m}p_{2m}} a(0)|$, use $\epsilon_\gamma(R)$ to represent a constant converging to 0 as $R \to 0$ for each fixed $\gamma$, and $\epsilon(R+\gamma)$ to represent a constant converging to 0 as both $R$ and $\gamma$ go to 0. Then by continuity of $a$,

$$A_{2m} \leq \tau + \epsilon(R+\gamma), \quad L_{2m} \leq \tau + \epsilon(R+\gamma).$$

(25) and (31) can hence be written as

$$\delta(R,\gamma) = C_a\tau(1+\gamma) + \epsilon_\gamma(R) + \epsilon(R+\gamma), \tag{35}$$

$$\eta(R,\gamma) = C\gamma\delta(R,\gamma) + \epsilon_\gamma(R). \tag{36}$$

with $C_a$ dependent on $\| a \|_{\mathcal{C}^2(E)}$.

First, for each $\gamma$, choose $R_0$ such that $\epsilon_\gamma(R) \leq \frac{\gamma}{4}$ when $R \leq R_0$ in (36). (35) and (36) will suffice if we can choose $\gamma$ and $R$ small enough so that $\delta(R,\gamma) \leq c := \min\{\frac{1}{4C\gamma}, \frac{1}{2}\} < 1$. Indeed, by choosing $\gamma(\leq 1)$ and $R$ small, we can make $\epsilon_\gamma(R) + \epsilon(R+\gamma) < \frac{c}{2}$ in (35) and hence

$$\delta(R,\gamma) < 2C_a\tau + \frac{c}{2}.$$

When $\tau \leq \frac{c}{8C_a}$, we thus have (34) holds.

Now recall $\Lambda = \{\beta : |\beta| = 2m, \text{ and at least one of } \beta_j \text{ is odd for } 1 \leq j \leq n\}$. For $R$ and $\gamma$ chosen as above, Pick $h(x) = bx^\beta$ with $\beta \in \Lambda$, and make $b > 0$ small enough such that $\| h \|_\alpha^{(2m)} < \frac{\gamma}{2}$ and hence $h \in \mathcal{B}(R, \frac{\gamma}{2})$. Consider the operator $\theta_h(f) := h + \theta(f)$. Then $\theta_h : \mathcal{B}(R,\gamma) \to \mathcal{B}(R,\gamma)$ forms a contraction map from the construction. By fixed point theorem for Banach spaces, there is some $u \in \mathcal{B}(R,\gamma)$ such that $\theta_h(u) = u$. $u$ thus solves the system (1) in the class $\mathcal{C}^{2m,\alpha}$ and is of vanishing order $2m$ by the construction. ∎

**Remark 7.2.** *None of the solutions constructed in the proof of Theorem 7.1 is radially symmetric. This means, even if the system (1) can be reduced into an ODE system with respect to the radial variable $r = |x|$, there exist infinitely many non-radial solutions. Indeed, if the solution $u(x) = u(r) \in \mathcal{C}_0^{2m,\alpha}$, then near 0, $u(r) = er^{2m} + o(|r|^{2m})$ for some constant $e$. In particular, $D^\beta u(0) = 0$ for all $\beta \in \Lambda$. This apparently can not happen because from the construction, $h(x) = bx^{\beta_0}$ with some $\beta_0 \in \Lambda$ and $D^{\beta_0}u(0) = D^{\beta_0}h(0) \neq 0$.*

**Proof of Theorem 1.5**: Theorem 1.5 is a consequence of Theorem 7.1 and Remark 7.2. ∎

**Proof of Theorem 1.4**: When $c_j = 0, 0 \leq j \leq 2m-1$ and $a$ is independent of $p_{2m}, A_{2m}, Q_{(2m)j}$ and $L_j(-1 \leq j \leq 2m)$ are all 0 and so (29) and (33) becomes

$$\begin{aligned} &\delta(R,\gamma) \leq \epsilon_\gamma(R), \\ &\eta(R,\gamma) \leq |a(0)| + \epsilon_\gamma(R). \end{aligned}$$

Here we only need $\mathcal{C}^{1,\alpha}$ regularity for $a$ from the estimates (26) and (32). Now we choose some positive $\gamma_0$ so that $\gamma_0 > 4|a(0)|$. Consequently, we choose $R$ sufficiently small so $\epsilon_{\gamma_0}(R) \leq c := \min\{\frac{1}{2}, \frac{\gamma_0}{4}\} < 1$. Hence

$$\begin{aligned} &\delta(R,\gamma_0) \leq c < 1; \\ &\eta(R,\gamma_0) < \frac{\gamma_0}{2}. \end{aligned}$$

Applying the same strategy as in the proof of Theorem 7.1, we can find a solution $u \in \mathcal{B}(R,\gamma_0)$ to the ODE system (2) which is not radially symmetric.

For general given jets $c_\beta$'s with multi-indices $\beta$, we write $T_{2m-1}(x) := \sum_{j=0}^{2m-1} \frac{c_\beta}{\beta!} x^\beta$. Consider the new system

$$\begin{aligned}
&\triangle^m \tilde{u}(x) = a(x, \tilde{u} + T_{2m-1}(x), \nabla(\tilde{u} + T_{2m-1}(x)), \ldots, \nabla^{2m-1}(\tilde{u} + T_{2m-1}(x));\\
&D^\beta \tilde{u}(0) = 0, \quad 0 \leq |\beta| \leq 2m-1.
\end{aligned}$$

This is a system with all the initial values equal to 0. We then obtain some solution $\tilde{u}$ in the class of $\mathcal{C}^{2m,\alpha}$ in some small neighborhood of 0. Then $u = \tilde{u} + T_{2m-1}$ solves the system (2) in the class of $\mathcal{C}^{2m,\alpha}$ in some small neighborhood of 0. Apparently, the solution obtained in this way is of vanishing order at most $2m$. Moreover, $u$ is not radially symmetric since $\tilde{u}$ is not. ∎

**Proof of Theorem 1.6**: Since $a$ is independent of $x$ and $a(0) = 0$, $A_{-1}$, $Q_{(-1)j}$, $Q_{j(-1)}$ and $L_{-1}$ are 0 and hence in (31),

$$\eta(R, \gamma) \leq C\gamma\delta(R, \gamma).$$

In order to prove Theorem 1.6, we need to show for any fixed $R > 0$, there exists some $\gamma_0 > 0$ such that

$$\begin{aligned}
\delta(R, \gamma_0) &< 1;\\
\eta(R, \gamma_0) &< \frac{\gamma_0}{2},
\end{aligned}$$

which is equivalent to showing

$$\delta(R, \gamma_0) \leq c := \min\{\frac{1}{2}, \frac{1}{2C}\} < 1. \tag{37}$$

Indeed, since $\nabla a(0) = 0$, we have for $0 \leq j \leq 2m$,

$$A_j \leq C \parallel \nabla_{p_j} a \parallel_{\mathcal{C}^1(E)} R^{2m-j}\gamma \leq C \parallel a \parallel_{\mathcal{C}^2(E)} R^{2m-j}\gamma.$$

Furthermore, for $0 \leq j, k \leq 2m$,

$$\begin{aligned}
Q_{jk} &\leq C \parallel \nabla_{p_j} a \parallel_{\mathcal{C}^1(E)} (R^{2m-k}\gamma)^{1-\alpha} \leq C \parallel a \parallel_{\mathcal{C}^2(E)} (R^{2m-k}\gamma)^{1-\alpha},\\
L_j &\leq C \parallel \nabla_{p_j} a \parallel_{\mathcal{C}^1(E)} \leq C \parallel a \parallel_{\mathcal{C}^2(E)}.
\end{aligned}$$

Therefore, (25) can be simplified as

$$\delta(R, \gamma) = \epsilon_R(\gamma),$$

where $\epsilon_R(\gamma)$ represents some function converging to 0 as $\gamma$ goes to 0 for each fixed $R > 0$. (37) is thus true and the proof of Theorem 1.6 is complete. ∎

# 8 Remark and examples

Although the existence domain of solutions in Theorem 1.6 can be made arbitrarily large, the neighborhood in Theorem 1.4 where the solutions exist is necessarily small in general, as indicated by the following example of Osserman [Os].

**Remark 8.1.** *Consider the system in $\mathbb{R}^n$ $(n \geq 3)$ with prescribed 1-jet:*

$$\begin{aligned} \triangle u &= |u|^{\frac{n+2}{n-2}}; \\ u(0) &= c_0; \\ \nabla u(0) &= c_1. \end{aligned}$$

*Theorem 1.4 applies to obtain some $\mathcal{C}^{2,\alpha}$ solution over a small neighborhood of 0, say, $\{x \in \mathbb{R}^n : |x| < R\}$. On the other hand, by a result of [Os], if the solution exists in $\{x \in \mathbb{R}^n : |x| < R\}$ and $c_0 > 0$, then $R \leq nu(0)^{-\frac{2}{n-2}} = nc_0^{-\frac{2}{n-2}}$. Consequently, $R \to 0$ as $c_0 \to +\infty$. This does not contradict with Theorem 1.6 though, since the solutions constructed in Theorem 1.6 are of vanishing order $2m \geq 2$ and hence $c_0 = 0$.*

As a matter of fact, a large class of the systems fit into one or more of the three solvability theorems. In particular, we establish the solvability of the following systems.

**Example 8.2.** *For any $p > 1$ and any given $R > 0$, the system*

$$\triangle^m u = \pm |u|^p$$

*has infinitely many $\mathcal{C}^{2m,\alpha}$ non-radial solutions over $\{x \in \mathbb{R}^n : |x| < R\}$, as a consequence of Theorem 1.6. Here $\alpha = \min\{1 - \epsilon, p - 1\}$ with $\epsilon$ any arbitrarily small positive number. Those solutions are necessarily smooth after a standard bootstrap argument.*

The following system has been well studied in the literature and our solvability results suit it as well.

**Example 8.3.** *Let $H \in \mathcal{C}^3$ and $H'(0) = 0$. Consider the system*

$$\triangle u = \nabla\big(H(u)\big).$$

*According to Theorem 1.6, for any $R > 0$, the above system has infinitely many non-radial solutions in $\mathcal{C}^{2,\alpha}(\{x \in \mathbb{R}^n : |x| < R\})$ for any $0 < \alpha < 1$.*

Indeed, a straightforward computation shows in the above example that $a(u, \nabla u) = \nabla\big(H(u)\big) = H'(u)\nabla u$, $\nabla_{p_0}\big(a(u, \nabla u)\big) = H''(u)\nabla u$ and $\nabla_{p_1}\big(a(u, \nabla u)\big) = H'(u)$ and hence the system satisfies $a \in \mathcal{C}^2$ and $a(0) = \nabla a(0) = 0$. By Theorem 1.6, for any $R > 0$, there exist infinitely many solutions in the class of $\mathcal{C}^{2,\alpha}(\{x \in \mathbb{R}^n : |x| < R\})$ and none of them is radially symmetric.

One similarly can obtain solvability for the following $m$-th order Poisson type system.

**Example 8.4.** *Let $H \in \mathcal{C}^3$ and $H'(0) = 0$. Then for any $R > 0$,*

$$\triangle^m u = \nabla\big(H(u, \nabla u, \ldots, \nabla^{2m-2} u)\big)$$

*has infinitely many non-radial smooth solutions in $\mathcal{C}^{2m,\alpha}(\{x \in \mathbb{R}^n : |x| < R\})$ for any $0 < \alpha < 1$.*

To see the solvability of the above example, a similar computation shows

$$a(u, \nabla u, \ldots, \nabla^{2m-1} u) = \nabla\big(H(u, \nabla u, \ldots, \nabla^{2m-2} u)\big) = \sum_{j=0}^{2m-2} \nabla_j H(u, \nabla u, \ldots, \nabla^{2m-2} u) \nabla^{j+1} u,$$

where $\nabla_j H$ is the derivative of $H$ with respect to $\nabla^j u$ variable. Furthermore,

$$\nabla_{p_0}\big(a(u, \nabla u, \ldots, \nabla^{2m-1} u)\big) = \sum_{j=0}^{2m-2} \nabla_j \nabla_0 H(u, \nabla u, \ldots, \nabla^{2m-2} u) \nabla^{j+1} u$$

and for $k \geq 1$,

$$\begin{aligned}\nabla_{p_k}\big(a(u, \nabla u, \ldots, \nabla^{2m-1} u)\big) =& \nabla_{p_k}\Big(\sum_{j=0}^{2m-2} \nabla_j H(u, \nabla u, \ldots, \nabla^{2m-2} u) \nabla^{j+1} u\Big) \\ =& \sum_{0 \leq j,k \leq 2m-2} \nabla_j \nabla_k H(u, \nabla u, \ldots, \nabla^{2m-2} u) \nabla^{j+1} u \\ &+ \nabla_{k-1} H(u, \nabla u, \ldots, \nabla^{2m-2} u).\end{aligned}$$

Hence $a \in \mathcal{C}^2$ and $a(0) = \nabla a(0) = 0$. By Theorem 1.6, for any $R > 0$, there exist infinitely many solutions in the class of $\mathcal{C}^{2,\alpha}(\{x \in \mathbb{R}^n : |x| < R\})$ and none of them is radially symmetric.

# Appendix: Computation of $\mathcal{I}_{B_1}(0, 0, 1)$

We will compute $\mathcal{I}_{B_1}(0, 0, 1)(x) := \int_{\partial \mathbf{B}_1} \Gamma(x - y)\nu_1 d\sigma_y$ for $x \in \mathbf{B}_1$. The constant $c_n$ in $\Gamma$ is omitted here for simplicity.

Write $x = U \cdot [a, 0, \ldots, 0]^t$, where $U = (u_{ij})_{1 \leq i,j \leq n}$ is some unitary matrix and $a = |x|$, and then make a change of coordinates by letting $y = U \cdot \tilde{y}$ in the expression of $\mathcal{I}_{B_1}(0, 0, 1)$. We then get

$$\begin{aligned}\mathcal{I}_{B_1}(0, 0, 1) :=& \int_{\partial \mathbf{B}_1} \frac{\sum_{0 \leq j \leq n} u_{1j} \tilde{y}_j}{\sqrt{(a - \tilde{y}_1)^2 + \tilde{y}_2^2 + \cdots + \tilde{y}_n^2}^{\,n-2}} d\sigma_{\tilde{y}} \\ =& u_{11} \int_{\partial \mathbf{B}_1} \frac{y_1}{\sqrt{(a - y_1)^2 + y_2^2 + \cdots + y_n^2}^{\,n-2}} d\sigma_y \\ &+ \sum_{2 \leq j \leq n} u_{1j} \int_{\partial \mathbf{B}_1} \frac{y_j}{\sqrt{(a - y_1)^2 + y_2^2 + \cdots + y_n^2}^{\,n-2}} d\sigma_y\end{aligned}$$

Since $\frac{y_j}{\sqrt{(a-y_1)^2+y_2^2+\cdots+y_n^2}^{n-2}}$ is odd with respect to $y_j$ when $j \geq 2$,

$$\int_{\partial \mathbf{B}_1} \frac{y_j}{\sqrt{(a-y_1)^2+y_2^2+\cdots+y_n^2}^{n-2}} d\sigma_y = 0$$

when $j \geq 2$ and hence

$$\mathcal{I}_{B_1}(0,0,1) := u_{11} \int_{\partial \mathbf{B}_1} \frac{y_1}{\sqrt{(a-y_1)^2+y_2^2+\cdots+y_n^2}^{n-2}} d\sigma_y.$$

Next, rewrite the above integral in terms of the spherical coordinates, then we obtain

$$\begin{aligned} \mathcal{I}_{B_1}(0,0,1) =& \omega_{n-1} u_{11} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin t \cos^{n-2} t}{\sqrt{(a-\sin t)^2+\cos^2 t}^{n-2}} dt \\ =& \omega_{n-1} u_{11} \int_{-1}^{1} \frac{u(1-u^2)^{\frac{n-3}{2}}}{(1-2au+a^2)^{\frac{n-2}{2}}} du \end{aligned} \tag{38}$$

Here $\omega_{n-1}$ is the surface area of the unit sphere in $\mathbb{R}^{n-1}$.

In order to compute (38), we make use of Gegenbauer polynomials. Recall for each fixed $\rho$, the Gegenbauer polynomials are $\{C_n^{(\rho)}(x)\}$ in $[-1,1] \subset \mathbb{R}$ satisfying

$$\frac{1}{(1-2xt+t^2)^\rho} = \sum_{n=0}^{\infty} C_n^{(\rho)}(x) t^n$$

in $(-1,1)$. In particular,

$$\begin{aligned} C_0^{(\rho)}(x) &= 1, \\ C_1^{(\rho)}(x) &= 2\rho x, \\ C_n^{(\rho)}(x) &= \frac{1}{n}[2x(n+\rho-1)C_{n-1}^{(\rho)}(x) - (n+2\rho-2)C_{n-2}^{(\rho)}(x)]. \end{aligned}$$

Moreover, $\{C_n^{(\rho)}(x)\}$ are orthogonal polynomials on the interval [-1,1] with respect to the weight function $(1-x^2)^{\rho-\frac{1}{2}}$. In other words,

$$\begin{aligned} &\int_{-1}^{1} C_n^{(\rho)}(x) C_m^{(\rho)}(x)(1-x^2)^{\rho-\frac{1}{2}} dx = 0, m \neq n, \\ &\int_{-1}^{1} [C_n^{(\rho)}(x)]^2 (1-x^2)^{\rho-\frac{1}{2}} dx = \frac{\pi 2^{1-2\rho}\Gamma(n+2\rho)}{n!(n+\rho)\Gamma(\rho)^2}. \end{aligned}$$

Letting $\rho = \frac{n-2}{2}$, then $\frac{1}{(1-2au+a^2)^{\frac{n-2}{2}}} = \sum_{n=0}^{\infty} C_n^{(\rho)}(u)a^n$ and $u = \frac{C_1^{(\rho)}(u)}{2\rho}$. (38) can hence be written as

$$\begin{aligned}\mathcal{I}_{B_1}(0,0,1) =& \omega_{n-1}u_{11}\int_{-1}^{1}\sum_{n=0}^{\infty} C_n^{(\rho)}(u)a^n \cdot \frac{C_1^{(\rho)}(u)}{2\rho}(1-u^2)^{\rho-\frac{1}{2}}du \\ =& \omega_{n-1}u_{11}\int_{-1}^{1} C_1^{(\rho)}(u)a \cdot \frac{C_1^{(\rho)}(u)}{2\rho}(1-u^2)^{\rho-\frac{1}{2}}du \\ =& \omega_{n-1}u_{11}\frac{a}{2\rho}\frac{\pi 2^{1-2\rho}\Gamma(1+2\rho)}{(1+\rho)\Gamma(\rho)^2} \\ =& \frac{4\pi^{\frac{n}{2}}}{n\Gamma(\frac{n-2}{2})}x_1. \blacksquare\end{aligned}$$

Making use of the same approach as the above, one can practically compute $\mathcal{I}_{\mathbf{B}_R}(\beta,\mu,j)$ for all $(\beta,\mu,j)$.

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