New Properties of Holomorphic Sobolev-Hardy Spaces

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Dedicated to Steven G. Krantz

Abstract

We give new characterizations of the optimal data space for the $L^p(bD, \sigma)$-Neumann boundary value problem for the $\bar{\partial}$ operator associated to a bounded, Lipschitz domain $D \subset \mathbb{C}$. We show that the solution space is embedded (as a Banach space) in the Dirichlet space and that for $p = 2$, the solution space is a reproducing kernel Hilbert space.

1 Introduction

Let D be a bounded Lipschitz domain in $\mathbb{C}$ whose boundary bD is endowed with the induced Lebesgue measure σ. Let $\mathcal{H}^p(D)$ be the **holomorphic Hardy space**:

$$\mathcal{H}^p(D) := \{F \in \vartheta(D) : \ F^* \in L^p(bD, \sigma)\}, \qquad 0 < p \leq \infty$$

with $\vartheta(D)$ denoting the set of holomorphic functions on D and F^* the non-tangential maximal function of F. It is well-known that if D is simply connected, every element F of $\mathcal{H}^p(D)$ admits a nontangential limit $\dot{F}$ that lies in $L^p(bD, \sigma)$ (see [5, Theorem 10.3]). On the other hand, since Lipschitz domains are local epigraphs, any bounded Lipschitz domain must be finitely connected. Hence, an elementary localization argument shows that any $F \in \mathcal{H}^p(D)$ has a nontangential limit $\dot{F}$ defined σ-a.e. on bD. We will call the set of all such nontangential limits $h^p(bD)$. That is,

$$h^p(bD) := \left\{ \dot{F} \ : \ F \in \mathcal{H}^p(D) \right\} \subsetneq L^p(bD, \sigma).$$

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Let $\mathcal{H}^{1,p}(D)$ be the **holomorphic Sobolev-Hardy space**

$$\mathcal{H}^{1,p}(D) := \{G \in \vartheta(D) : G' \in \mathcal{H}^p(D)\}, \quad p > 0\,.$$

It is shown in [6] that, given $g \in L^p(bD, \sigma)$ subject to the compatibility condition: $\int\limits_{bD} g\, d\sigma = 0$, the **Neumann problem for the $\overline{\partial}$ operator**

$$\begin{cases} \bar{\partial} G & = & 0 & \text{in} \;\; D; \\ \dfrac{\partial G}{\partial n}(\zeta) & = & g(\zeta) & \text{for } \sigma\text{-a.e. } \zeta \in bD; \\ (G')^* & \in & L^p(bD, \sigma) & \end{cases} \tag{1}$$

is solvable if and only if the data g belongs to

$$\mathfrak{n}^p(bD) := \left\{ -iT(\zeta)(\dot{G'})(\zeta) : \; G \in \mathcal{H}^{1,p}(D) \right\}, \quad 1 \le p \le \infty, \tag{2}$$

where $\zeta \mapsto T(\zeta)$ is the unit tangent vector field for bD. Moreover, if $g \in \mathfrak{n}^p(bD)$ then all solutions of (1) belong to $\mathcal{H}^{1,p}(D)$. Any two solutions of (1) differ by an additive constant, hence for any fixed $\alpha \in D$ the space

$$\mathcal{H}^{1,p}_\alpha(D) := \{F \in \mathcal{H}^{1,p}(D) : F(\alpha) = 0\}$$

contains precisely one solution of (1). In the case when $p = 2$ and D is simply-connected, $\mathcal{H}^{1,2}_\alpha(D)$ is a Hilbert space with inner product

$$\langle F, G \rangle_{\mathcal{H}^{1,2}_\alpha(D)} := \int\limits_{bD} (\dot{F'})(\zeta)\, \overline{(\dot{G'})(\zeta)}\, d\sigma(\zeta).$$

In this paper we explore properties of $\mathcal{H}^{1,2}_\alpha(bD)$ and of $\mathfrak{n}^p(bD)$. Specifically, after recalling a few well-known basic properties of Lipschitz domains (Section 2), we show that the solution space $\mathcal{H}^{1,2}_\alpha(D)$ is a reproducing kernel Hilbert space (Theorem 3.1) and for $D = \mathbb{D}$ (the unit disc) we compute its reproducing kernel. Next we show that for $1 < p < \infty$ there is a Banach space embedding of $\mathcal{H}^{1,p}_\alpha(D)$ in the Dirichlet space $\mathcal{D}^p_\alpha(D)$ (Theorem 3.3). In Section 4 we give various characterizations of $\mathfrak{n}^p(bD)$ for simply connected D: in terms of $L^p(bD, \sigma)$-functions whose moments all vanish on bD; or in terms of the vanishing of the Cauchy integral over $\overline{D}^c$, the complement of the closure of D; as well as in terms of its conformal map (Theorem 4.1 and Theorem 4.3). Finally, in Section 5 we provide a characterization of $\mathfrak{n}^p(bD)$ for multiply connected D: in this case the aforementioned vanishing moment condition takes a more restrictive form, see Theorem 5.3.

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2 Preliminaries

2.1 Lipschitz domains

Throughout this paper the domains under consideration will be Lipschitz domains on $\mathbb{C}$, as defined below.

Definition 2.1. A bounded domain $D \subset \mathbb{C}$ with boundary bD is called a **Lipschitz domain** if there are finitely many rectangles $\{R_j\}_{j=1}^m$ with sides parallel to the coordinate axes, angles $\{\theta_j\}_{j=1}^m$, and Lipschitz functions $\phi_j : \mathbb{R} \to \mathbb{R}$ such that the collection $\{e^{-i\theta_j} R_j\}_{j=1}^m$ covers bD and $(e^{i\theta_j} D) \cap R_j = \{x + iy : y > \phi_j(x),\ x \in (a_j, b_j)\}$ for some $a_j < b_j < \infty$. We refer to such R_j's as *coordinate rectangles*.

Definition 2.2. Let D be a bounded Lipschitz domain. For any $\zeta \in bD$, let $\{\Gamma(\zeta), \zeta \in D\}$ be a family of truncated (one-sided) open cones $\Gamma(\zeta)$ with vertex at ζ satisfying the following property: for each rectangle R_j in Definition 2.1, there exists two cones Δ_1 and Δ_2, each with vertex at the origin and axis along the y axis such that for $\zeta \in bD \cap e^{-i\theta_j} R_j$,

$$e^{-i\theta_j}\Delta_1 + \zeta \quad \subset \quad \Gamma(\zeta) \quad \subset \quad \overline{\Gamma(\zeta)} \setminus \{\zeta\} \quad \subset \quad e^{-i\theta_j}\Delta_2 + \zeta \quad \subset \quad D \ \cap \ e^{-i\theta_j} R_j.$$

It is well known that for Lipschitz D, $\Gamma(\zeta) \neq \emptyset$ for any $\zeta \in bD$; see e.g., [4] or [12, Section 0.4]. We will sometimes refer to $\Gamma(\zeta)$ as a *regular cone*, or a *coordinate cone*. For a function F on D and $\zeta \in bD$, we define the **nontangential maximal function** $F^*(\zeta)$ and the **nontangential limit** $\dot{F}(\zeta)$ as

$$F^*(\zeta) := \sup_{z \in \Gamma(\zeta)} |F(z)|, \qquad \text{and} \quad \dot{F}(\zeta) = \lim_{\substack{z \to \zeta \\ z \in \Gamma(\zeta)}} F(z) \quad \text{if such limit exists.}$$

We will need an approximation scheme of D by smooth subdomains constructed by Nečas in [10], which we refer to as a **Nečas exhaustion of** D. See also [8] and [12, Theorem 1.12]. (Recall that Lipschitz functions are differentiable almost everywhere; thus if D is Lipschitz and simply connected its boundary bD is a rectifiable Jordan curve that admits a (positively oriented) unit tangent vector $T(\zeta)$ as σ-a.e. $\zeta \in bD$.)

Lemma 2.3. *[10, p. 5][12, Theorem 1.12] Let D be a bounded Lipschitz domain. There exists a family $\{D_k\}_{k=1}^{\infty}$ of smooth domains with D_k compactly contained in D that satisfy the following:*

(a). For each k there exists a Lipschitz diffeomorphism Λ_k that takes D to D_k and extends to the boundaries: $\Lambda_k : bD \to bD_k$ with the property that

$$\sup\{|\Lambda_k(\zeta) - \zeta| : \zeta \in bD\} \leq C/k$$

for some fixed constant C. Moreover $\Lambda_k(\zeta) \in \Gamma(\zeta)$.

(b). There is a covering of bD by finitely many coordinate rectangles which also form a family of coordinate rectangles for bD_k for each k. Furthermore for every such rectangle R, if ϕ and ϕ_k denote the Lipschitz functions whose graphs describe the boundaries of D and D_k, respectively, in R, then $\|(\phi_k)'\|_\infty \leq \|\phi'\|_\infty$ for any k; $\phi_k \to \phi$ uniformly as $k \to \infty$, and $(\phi_k)' \to \phi'$ a.e. and in every $L^p((a, b))$ with $(a, b) \subset \mathbb{R}$ as in Definition 2.1.

(c). There exist constants $0 < m < M < \infty$ and positive functions (Jacobians) $w_k : bD \to [m, M]$ for any $k \in \mathbb{N}$, such that for any measurable set $F \subseteq bD$ and for any measurable function f_k on $\Lambda_k(F)$ the following change-of-variables formula holds:

$$\int_F f_k(\Lambda_k(\eta))\, w_k(\zeta)\, d\sigma(\eta) = \int_{\Lambda_k(F)} f_k(\eta_k)\, d\sigma_k(\eta_k).$$

where $d\sigma_k$ denotes arc-length measure on bD_k. Furthermore we have

$$w_k \to 1 \quad \sigma\text{-a.e. } bD \quad \text{and in every} \quad L^p(bD, \sigma) \quad \text{for any} \quad 1 \leq p < \infty\,.$$

(d). Let T_k denote the unit tangent vector for bD_k and T denote the unit tangent vector of bD. We have that

$$T_k \to T \quad \sigma\text{-a.e. } bD \quad \text{and in every} \quad L^p(bD, \sigma) \quad \text{for any} \quad 1 \leq p < \infty\,.$$

Note that in conclusions *(b)* through *(d)* the exponent $p = \infty$ cannot be allowed unless D is of class C^1. Nečas exhaustions can be used to transfer well-known results for holomorphic functions over domains with smooth boundaries to Hardy space functions on Lipschitz domains. In particular, one can use it to prove Cauchy's Theorem. See also [6, Lemma 2.7] for the proof.

Lemma 2.4. *Let D be a bounded Lipschitz domain. Then any $f \in h^1(bD)$ satisfies Cauchy's Theorem. That is*

$$\int_{bD} f(\zeta)\, d\zeta = 0 \qquad \textit{for any} \quad f \in h^1(bD).$$

Next we state some definitions and results involving Cauchy integrals and the Cauchy transform, which we first define:

Definition 2.5. Let $f : bD \to \mathbb{C}$. The **Cauchy integral $\mathbf{C}_D f$** of f is

$$\mathbf{C}_D f(z) := \frac{1}{2\pi i} \int_{bD} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \in D.$$

Similarly

$$\mathbf{C}_{\overline{D}^c} f(z) := \frac{1}{2\pi i} \int_{bD} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \overline{D}^c.$$

Finally, the **Cauchy transform $\mathcal{C}_D f$** of f is denoted by

$$\mathcal{C}_D f(\zeta) := (\dot{\mathbf{C}}_D f)(\zeta), \quad \zeta \in bD.$$

In both integrals bD is oriented counterclockwise (that is, in the positive direction for D).

In this paper we will use the fact that a function f in $L^p(bD, \sigma)$ lies in $h^p(bD, \sigma)$ if and only if the Cauchy integral of f vanishes on $\overline{D}^c$. This latter fact is well-known for domains with smooth boundaries; here we prove it for Lipschitz domains, see Lemma 2.6 below. We first recall the Plemelj formulas for $f \in L^p(bD, \sigma)$, $1 < p < \infty$:

$$\mathcal{C}_D f(\zeta) = \frac{1}{2} f(\zeta) + \frac{1}{2} \mathcal{H}\mathfrak{C}_{bD} f(\zeta), \qquad \text{for } \sigma\text{-a.e. } \zeta \in bD, \tag{3}$$

and

$$\lim_{\substack{z \to \zeta \\ z \in \Gamma(\zeta, \overline{D}^c)}} \mathbf{C}_{\overline{D}^c} f(z) = -\frac{1}{2} f(\zeta) + \frac{1}{2} \mathcal{H}\mathfrak{C}_{bD} f(\zeta) \text{ for } \sigma\text{-a.e. } \zeta \in bD. \tag{4}$$

Here

$$\mathcal{H}\mathfrak{C}_{bD} f(\zeta) := \frac{1}{2\pi i}\, \text{P.V.} \int_{bD} \frac{f(w)}{w - \zeta}\, dw, \quad \text{for } \sigma\text{-a.e. } \zeta \in bD,$$

with bD oriented counterclockwise, and $\Gamma(\zeta, \overline{D}^c)$ is defined as in Definition 2.2, with D in there replaced by $\overline{D}^c$. Note that a Lipschitz domain D satisfies the exterior cone condition (see [7]) so the limit in (4) is well-defined. A deep result of Coifman, McIntosh, and Meyer [3] states that on bounded Lipschitz domains, $\mathcal{HC}_{bD}$ is indeed well-defined (i.e. the principal value integral exists σ-a.e.) and is bounded on $L^p(bD,\sigma)$, $1<p<\infty$. Thus, by the result of [2], the Plemelj formulas (3) and (4) hold (for more on Plemelj formulas, also see [9]).

Lemma 2.6. *Let D be a bounded simply connected Lipschitz domain and $1<p<\infty$. Assume $f\in L^p(bD,\sigma)$. Then $f\in h^p(bD,\sigma)$ if and only if $\mathbf{C}_{\overline{D}^c}f(z)=0$ for all $z\in\overline{D}^c$.*

Proof. First assume that $\mathbf{C}_{\overline{D}^c}f(z)=0$ for all $z\in\overline{D}^c$. By Equation (4), we have

$$0=\lim_{\substack{z\to\omega\\ z\in\Gamma(\zeta,\overline{D}^c)}} C_{\overline{D}^c}f(z)=\frac{1}{2\pi i}\text{P.V.}\int_{bD}\frac{f(\zeta)}{\zeta-\omega}d\zeta-\frac{1}{2}f(\omega).$$

That is, $\frac{1}{2}f(\omega)=\frac{1}{2\pi i}\text{P.V.}\int_{bD}\frac{f(\zeta)}{\zeta-\omega}$ for σ-a.e. $\omega\in bD$. Now, using Equation (3), we have for σ-a.e. $\omega\in bD$,

$$\mathcal{C}_Df(\omega)=\frac{1}{2\pi i}\text{P.V.}\int_{bD}\frac{f(\zeta)}{\zeta-\omega}d\zeta+\frac{1}{2}f(\omega)=\frac{1}{2}f(\omega)+\frac{1}{2}f(\omega)=f(\omega).$$

Thus, f is in the range of the Cauchy transform. Since the range of the Cauchy transform equals $h^p(bD,\sigma)$ when D is bounded and simply connected and $1<p<\infty$ (see [8]), the backward direction is proven. For the forward direction suppose $f\in h^p(bD,\sigma)$. Then there exists $F\in\mathcal{H}^p(D)$ such that $\dot{F}=f$. Let $z\in\overline{D}^c$ be arbitrary and consider the function $G_z(w):=(w-z)^{-1}$. Then G_z is holomorphic on D and is continuous on $\overline{D}$. Moreover, $\|(FG_z)^*\|_{L^p(bD,\sigma)}\le\|F^*\|_{L^p(bD,\sigma)}\|G_z^*\|_{L^\infty(bD,\sigma)}<\infty$. Thus $FG_z\in\mathcal{H}^p(D)$ and by Cauchy's Theorem (Lemma 2.4) we have

$$0=\int_{bD}(\dot{FG_z})(\zeta)d\zeta=\mathbf{C}_{\overline{D}^c}f(z),$$

as desired. ∎

3 Properties of $\mathcal{H}^{1,2}_\alpha(D)$ for simply connected D

In this section we show that $\mathcal{H}^{1,2}_\alpha(D)$ is a reproducing kernel Hilbert space and that it is a subset of the Dirichlet space.

3.1 $\mathcal{H}_\alpha^{1,2}(D)$ is a reproducing kernel Hilbert space

Theorem 3.1. *Let D be a bounded simply connected Lipschitz domain. Then for any base point $\alpha \in D$:*

(a) $\mathcal{H}_\alpha^{1,2}(D)$ is a Hilbert space with inner product

$$\langle F, G\rangle_{\mathcal{H}_\alpha^{1,2}(D)} := \langle (\dot{F'}), (\dot{G'})\rangle_{L^2(bD,\sigma)}.$$

(b) For any $z \in D$, the pointwise evaluation: $G \mapsto E_z(G) := G(z)$ is a bounded linear functional on $\mathcal{H}_\alpha^{1,2}(D)$. Hence $\mathcal{H}_\alpha^{1,2}(D)$ is a reproducing kernel Hilbert space (RKHS) with reproducing kernel $K_\alpha^z(\cdot) = K_\alpha(\cdot, z)$. Namely, for any $z \in D$, we have that

$$(\dot{K_\alpha^z})'(\zeta) \quad \equiv \quad \lim_{\substack{w\to\zeta \\ w\in\Gamma(\zeta)}} (K_\alpha^z)'(w)$$

exists for almost all $\zeta \in bD$ and for $F \in \mathcal{H}_\alpha^{1,2}(D)$ we have

$$F(z) = \int_{bD} (\dot{F'})(\zeta)\, \overline{(\dot{K_\alpha^z})'(\zeta)}\, d\sigma(\zeta), \quad z \in D. \tag{5}$$

(c) Let $p \geq 2$ and $g \in \mathfrak{n}^p(bD)$. Then for any $\alpha \in D$ the solution of the holomorphic Neumann problem (1) *with boundary data g has the representation*

$$G_\alpha(z) = i \int\limits_{\zeta\in bD} g(\zeta)\, \overline{T(\zeta)\, (\dot{K_\alpha^z})'(\zeta)}\, d\sigma(\zeta), \quad z \in D.$$

Proof. To verify *(a)*, note that $\langle\cdot,\cdot\rangle_{\mathcal{H}_\alpha^{1,2}(D)}$ is a sesquilinear form and $\langle F, F\rangle_{\mathcal{H}_\alpha^{1,2}(D)} = \|F\|^2_{\mathcal{H}_\alpha^{1,2}(D)}$. A straightforward argument (whose details can be found in [6, Lemma 3.4]) shows that for $1 \leq p \leq \infty$ the set $\mathcal{H}_\alpha^{1,p}(D)$ is a Banach space with the norm defined as

$$\|F\|_{\mathcal{H}_\alpha^{1,p}(D)} = \|(\dot{F'})\|_{L^p(bD,\sigma)}.$$

Thus $\mathcal{H}_\alpha^{1,2}(D)$ is complete under the norm $\|\cdot\|_{\mathcal{H}_\alpha^{1,2}(D)}$, and so $\mathcal{H}_\alpha^{1,2}(D)$ is a Hilbert space.

Next we prove *(b)*. Fix $z \in D$ and consider the pointwise evaluation operator E_z. For any $\alpha \in D$ and a smooth path $\gamma_\alpha^z \subset D$ that connects α to z we have

$$|E_z(G)| = |G(z)| = |G(z) - G(\alpha)| = \left| \int\limits_{\gamma_\alpha^z} G'(w)dw \right| \leq |\gamma_\alpha^z| \sup_{w\in\gamma_\alpha^z} |G'(w)|\,.$$

Furthermore, for any $w \in \gamma_\alpha^z$, Cauchy formula and Hölder inequality give

$$|G'(w)| = \frac{1}{2\pi}\left|\int\limits_{bD} \frac{(\dot{G'})(\zeta)}{w-\zeta}d\zeta\right| \leq \frac{|bD|^{\frac{1}{2}}}{2\pi k_z}\|(\dot{G'})\|_{L^2(bD,\sigma)} = \frac{|bD|^{\frac{1}{2}}}{2\pi k_z}\|G\|_{\mathcal{H}_\alpha^{1,2}(D)},$$

where $k_z := \mathrm{dist}(\gamma_\alpha^z, bD) > 0$. Combining all of the above we see that for any $z \in D$, E_z is a bounded linear functional on $\mathcal{H}_\alpha^{1,2}(D)$; Hilbert space theory now grants the existence of the reproducing kernel function

$$K_\alpha^z \in \mathcal{H}_\alpha^{1,2}(D) \quad \text{with} \quad G(z) = \langle G, K_\alpha^z\rangle_{\mathcal{H}_\alpha^{1,2}(D)}.$$

Finally we verify *(c)*. Let $p \geq 2$ and $g \in \mathfrak{n}^p(bD)$. Suppose $G_\alpha \in \mathcal{H}_\alpha^{1,p}(D)$ is the solution to the Neumann problem (1) with datum g. Thus $(\dot{G'_\alpha}) = i\overline{T}g$ and $G_\alpha \in \mathcal{H}_\alpha^{1,2}(D)$. Hence for any $z \in D$ we have

$$G_\alpha(z) = \langle G_\alpha, K_\alpha^z\rangle_{\mathcal{H}_\alpha^{1,2}(D)} = \int\limits_{bD} (\dot{G'_\alpha})(\zeta)\overline{(\dot{K_\alpha^z})'(\zeta)}d\sigma(\zeta) = i\int\limits_{bD} g(\zeta)\overline{T(\zeta)(\dot{K_\alpha^z})'(\zeta)}d\sigma(\zeta),$$

as desired. ∎

In the case of the unit disc $\mathbb{D}$ we obtain explicit formulas and recover the full range of $1 \leq p \leq \infty$:

Theorem 3.2. *1. The reproducing kernel associated to $\mathcal{H}_\alpha^{1,2}(\mathbb{D})$ is given by*

$$K_\alpha^z(w) = \sum_{k=1}^{\infty} \frac{(w^k - \alpha^k)\overline{(z^k - \alpha^k)}}{2\pi k^2}, \qquad z, w \in \mathbb{D}. \tag{6}$$

2. Given $g \in \mathfrak{n}^p(b\mathbb{D})$, $1 \leq p \leq \infty$ and $\alpha := 0$, the unique solution $G \in \mathcal{H}_0^{1,p}(\mathbb{D})$ to the holomorphic Neumann problem (1) admits the following representation

$$G(z) = \frac{1}{2\pi}\int\limits_{b\mathbb{D}} g(\zeta)\mathrm{Log}\frac{1}{1 - z\overline{\zeta}}\, d\sigma(\zeta), \tag{7}$$

where Log *denotes the principal branch of the complex logarithm.*

Proof. To prove part *1.*, note that since $\mathbb{D}$ is simply connected every holomorphic function on $\mathbb{D}$ has an antiderivative. Thus the mapping $G \mapsto G'$ is an isometric isomorphism from $\mathcal{H}^{1,2}_{\alpha}(\mathbb{D})$ onto $\mathcal{H}^2(\mathbb{D})$. Since $\{\frac{1}{\sqrt{2\pi}} z^{k-1}\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}^2(\mathbb{D})$, the set of antiderivatives $\{\frac{z^k-\alpha^k}{\sqrt{2\pi}k}\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}^{1,2}_{\alpha}(\mathbb{D})$. Thus, by the theory of reproducing kernel Hilbert spaces, K_α as given in Equation (6) is the reproducing kernel for $\mathcal{H}^{1,2}_{\alpha}(\mathbb{D})$.

For the proof of part *2.*, note that the reproducing kernel for $\mathbb{D}$ satisfies

$$\overline{(K_0^z)'(w)} = \frac{\partial}{\partial \overline{w}} \sum_{k=1}^{\infty} \frac{z^k \overline{w}^k}{2\pi k^2} = \sum_{k=1}^{\infty} \frac{z^k \overline{w}^{k-1}}{2\pi k} = \frac{1}{2\pi \overline{w}} \mathrm{Log} \frac{1}{1 - z\overline{w}}, \quad w \in \mathbb{D}.$$

Hence for every $\zeta \in b\mathbb{D}$, and since $T(\zeta) = i\zeta$, we have

$$\overline{T(\zeta)(\dot{K_0^z})'(\zeta)} \quad = \quad \frac{1}{2\pi i} \mathrm{Log} \frac{1}{1 - z\overline{\zeta}}, \quad \zeta \in b\mathbb{D}.$$

So for $g \in \mathfrak{n}^2(b\mathbb{D})$ we have that Equation (7) follows from the above and Theorem 3.1 part *(c)*.

For $g \in \mathfrak{n}^p(b\mathbb{D})$, $1 \leq p \leq \infty$, define G as in (7). Then $G \in \vartheta(\mathbb{D})$ and

$$G'(z) = \frac{1}{2\pi} \int_{b\mathbb{D}} \frac{g(\zeta)\overline{\zeta}}{1 - z\overline{\zeta}}\, d\sigma(\zeta) = \frac{1}{2\pi} \int_{b\mathbb{D}} \frac{g(\zeta)}{\zeta - z}\, d\sigma(\zeta) = \frac{1}{2\pi i} \int_{b\mathbb{D}} \frac{i\overline{T(\zeta)}g(\zeta)}{\zeta - z}\, d\zeta = \mathbf{C}_{\mathbb{D}}(i\overline{T}g)(z), \;\; z \in \mathbb{D}.$$

Here we used the facts that $\zeta\overline{\zeta} = 1$ and $d\sigma(\zeta) = \overline{T(\zeta)}d\zeta$ on $\mathbb{D}$. Consequently, $(G')^* \in L^p(b\mathbb{D}, \sigma)$ by the mapping property of the Cauchy integral $\mathbf{C}_{\mathbb{D}}$ and Cauchy transform $\mathcal{C}_{\mathbb{D}}$. Moreover, from the above we also have that

$$(\dot{G'})(\zeta) = \mathcal{C}_{\mathbb{D}}(i\overline{T}g)(\zeta), \quad \text{a.e. } \zeta \in b\mathbb{D}.$$

But $\overline{T}g \in h^p(b\mathbb{D})$ because $g \in \mathfrak{n}^p(b\mathbb{D})$, and $\mathcal{C}_{\mathbb{D}}$ is the identity on $h^p(b\mathbb{D})$, thus

$$\frac{\partial G}{\partial n}(\zeta) = -iT(\zeta)(\dot{G'})(\zeta) = -iT(\zeta)\mathcal{C}_{\mathbb{D}}(i\overline{T}g)(\zeta) = g(\zeta), \;\; \zeta \in b\mathbb{D} \;\; \sigma - a.e..$$

That is, G solves (1) for $1 \leq p \leq \infty$. (Uniqueness was proved in [6].) ∎

3.2 $\mathcal{H}^{1,p}_\alpha(D)$ is embedded in the Dirichlet Space

In [1], Axler and Shields introduced the **Dirichlet space** $\mathcal{D}^2_\alpha(D)$ for a general domain D, namely

$$\mathcal{D}^2_\alpha(D) := \left\{ F \in \vartheta(D) : F(\alpha) = 0, \int\limits_D |F'|^2 (z)\, dV(z) < \infty \right\}, \quad \alpha \in D,$$

which is a Hilbert space with inner product

$$\langle F, G \rangle_{\mathcal{D}^2_\alpha(D)} := \int\limits_D F'(z)\, \overline{G'(z)}\, dV(z).$$

(Here dV is the Lebesgue measure for $\mathbb{C}$.) The analogous definition of $\mathcal{D}^p_\alpha(D)$ with $1 \le p \le \infty$ yields a Banach space with norm

$$\|F\|_{\mathcal{D}^p_\alpha(D)} := \int\limits_D |F'(z)|^p\, dV(z).$$

Theorem 3.3. *Let D be a bounded simply connected Lipschitz domain and $1 < p < \infty$. Suppose that $F \in \mathcal{H}^{1,p}_\alpha(D)$. Then $F \in \mathcal{D}^p_\alpha(D)$ and*

$$\|F\|_{\mathcal{D}^p_\alpha(D)} \lesssim \|F\|_{\mathcal{H}^{1,p}_\alpha(D)}.$$

That is, the holomorphic Sobolev-Hardy space is embedded in the Dirichlet space.

To prove Theorem 3.3 we need the following result:

Lemma 3.4. *Let D be a bounded simply connected Lipschitz domain and $1 < p < \infty$. Suppose that $F \in \mathcal{H}^p(D)$. Then $F \in \vartheta(D) \cap L^p(D)$ and*

$$\|F\|_{L^p(D)} \lesssim \|\dot{F}\|_{L^p(bD,\sigma)}.$$

That is, the holomorphic Hardy space is embedded in the Bergman space.

Proof. In [6, Lemma 2.8] it is shown that if $1 < p < \infty$ and D is a simply connected and bounded Lipschitz domain, then for $F \in \mathcal{H}^p(D)$ quantities $\|F^*\|_{L^p(bD,\sigma)}$ and $\|\dot{F}\|_{L^p(bD,\sigma)}$ are comparable. Thus it suffices to show that $\|F\|_{L^p(D)} \lesssim \|F^*\|_{L^p(bD,\sigma)}$.

Consider a Nečas exhaustion $\{D_k\}$ of D. Then there are finitely many coordinate rectangles $R_j := [a_j, b_j] \times (c_j, d_j)$ with Lipschitz functions ϕ^j_k and ϕ^j whose graphs determine D_k and D, respectively, on R_j and ϕ^j_k converges uniformly to ϕ^j. For any $k \in \mathbb{N}$,

$x \in [a_j, b_j]$ and $y \in \left(\phi^j(x), \phi^j_k(x)\right]$, the point $x+iy$ lies directly above $x+i\phi^j(x)$ and thus $x+iy \in \Delta_1 + (x+i\phi^j(x))$, where Δ_1 is the cone in Definition 2.2. And so $e^{-i\theta_j}(x+iy)$ lies in $e^{-i\theta_j}(\Delta_1 + (x+i\phi^j(x))) \subseteq \Gamma(e^{-i\theta_j}(x+i\phi^j(x)))$. Fix $k \in \mathbb{N}$ so that for each j we have $\|\phi^j_k - \phi^j\|_\infty < 1$. Then we have for $F \in \mathcal{H}^p(D)$

$$
\begin{aligned}
\iint_{D-D_k} |F(z)|^p dA(z) &\leq \sum_j \iint_{e^{-i\theta_j}R_j \cap (D-D_k)} |F(z)|^p dA(z) \\
&= \sum_j \int_{a_j}^{b_j} \int_{\phi^j(x)}^{\phi^j_k(x)} |F(e^{-i\theta_j}(x+iy))|^p dy dx \\
&\leq \sum_j \int_{a_j}^{b_j} \int_{\phi^j(x)}^{\phi^j_k(x)} F^*(e^{-i\theta_j}(x+i\phi_j(x)))^p dy dx \\
&\leq \sum_j \int_{a_j}^{b_j} F^*(e^{-i\theta_j}(x+i\phi_j(x)))^p dx \\
&\leq \sum_j \int_{a_j}^{b_j} F^*(e^{-i\theta_j}(x+i\phi_j(x)))^p \left|e^{i\theta_j}(1+i\phi_j'(x))\right| dx \\
&= \sum_j \int_{bD \cap e^{-i\theta_j}R_j} F^*(\zeta)^p d\sigma(\zeta) \lesssim \|F^*\|^p_{L^p(bD,\sigma)}.
\end{aligned}
$$

Since D_k is compactly contained in D and k is fixed, $\operatorname{dist}(D_k, bD) > d$ for some constant d depending on D. So, similar to the argument of the proof of part *(b)* of Theorem 3.1, by the Cauchy integral formula we have

$$\iint_{D_k} |F(z)|^p dA(z) \lesssim \|F^*\|^p_{L^p(bD,\sigma)},$$

completing the proof to $\|F\|_{L^p(D)} \lesssim \|F^*\|_{L^p(bD,\sigma)}$. ∎

Proof of Theorem 3.3. Let $F \in \mathcal{H}^{1,p}_\alpha(D)$. Then $F(\alpha) = 0$ and $F' \in \mathcal{H}^p(D)$. By Lemma 3.4, we also have $F' \in \vartheta(D) \cap L^p(D)$ giving that $F \in \mathcal{D}^p_\alpha(D)$, as desired. ∎

4 Characterizations of $\mathfrak{n}^p(bD)$ for simply connected D

Theorem 4.1. *Let D be a bounded simply connected Lipschitz domain and $1 \le p \le \infty$. Then $\mathfrak{n}^p(bD)$ defined as in (2) is closed in the $L^p(bD,\sigma)$-norm. Moreover, for*

$$\begin{aligned}
\mathfrak{n}_1 &:= \{Tg : g \in h^p(bD)\},\\
\mathfrak{n}_2 &:= \left\{ f \in L^p(bD,\sigma) : \int_{bD} \zeta^k f(\zeta)d\sigma(\zeta) = 0 \text{ for all } k = 0,1,2,\dots \right\},\\
\mathfrak{n}_3 &:= \left\{ f \in L^p(bD,\sigma) : \mathbf{C}_{\overline{D}^c}(\overline{T}f) = 0 \right\}
\end{aligned}$$

we have that $\mathfrak{n}^p(bD) = \mathfrak{n}_1 = \mathfrak{n}_2$. If $1 < p < \infty$, then we also have $\mathfrak{n}^p(bD) = \mathfrak{n}_3$.

Proof. The inclusion $\mathfrak{n}^p(bD) \subseteq \mathfrak{n}_1$ is immediate from (2). The reverse inclusion holds because D is simply connected and thus all holomorphic functions on D have antiderivatives. As $h^p(bD)$ is closed in the $L^p(bD,\sigma)$-norm, we see that $\mathfrak{n}_1$, and thus $\mathfrak{n}^p(bD)$ is also closed. Next, the identity $\mathfrak{n}^p(bD) = \mathfrak{n}_2$ follows from the fact that $T(\zeta)d\sigma(\zeta) = d\zeta$ and the well-known result of Smirnov that $g \in L^p(bD,\sigma)$ lies in $h^p(bD)$ if and only if

$$\int_{bD} \zeta^k g(\zeta)d\zeta = 0 \quad \text{for} \quad k = 0,1,2,\dots$$

See, for example, [5, Theorem 10.4]. Finally, the identity $\mathfrak{n}^p(bD) = \mathfrak{n}_3$ for $1 < p < \infty$ follows from Lemma 2.6. ∎

We may also characterize the elements of $\mathfrak{n}^p(bD)$ for a bounded simply connected Lipschitz domain D via its Riemann maps. We shall need the following description of the tangent vector.

Lemma 4.2. *Let D be a bounded simply connected Lipschitz domain and $\psi : D \to \mathbb{D}$ be a conformal map. Then the tangent vector T of bD (which is defined a.e.) can be written as*

$$T = i\frac{(\dot{\psi'})}{|(\dot{\psi'})|}\dot{\psi}, \qquad \sigma\text{-a.e. on } bD.$$

Proof. Let $\phi : \mathbb{D} \to D$ be defined as $\phi = \psi^{-1}$. Since bD is Lipschitz, it is a Jordan curve so by Carathéodory's theorem ϕ extends to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{D}$. By [5, Theorem 3.13], we have that $\phi' \in \mathcal{H}^1(\mathbb{D})$ so that $(\dot{\phi'})$ exists σ-a.e., ϕ is absolutely continuous on $b\mathbb{D}$, and

$$\frac{d}{dt}\phi(e^{it}) = ie^{it}(\dot{\phi'})(e^{it}). \tag{8}$$

Thus we can write the unit tangent vector T via $(\dot{\phi'})$ for almost all $\zeta \in \partial D$. To do so, first note that for $r < 1$

$$\psi'(\phi(re^{it})) = \frac{1}{\phi'(re^{it})}.$$

Since ϕ is conformal and $(\dot{\phi'})$ exists and is nonzero a.e., we see that the nontangential limit $(\dot{\psi'})$ exists a.e. and satisfies

$$(\dot{\psi'})(\zeta) = \frac{1}{(\dot{\phi'})(\psi(\zeta))}. \tag{9}$$

Choose t_0 so that $\zeta = \phi(e^{it_0})$. Then by Equations (8) and (9) we have

$$T(\zeta) = \left.\frac{\frac{d}{dt}\phi(e^{it})}{|\frac{d}{dt}\phi(e^{it})|}\right|_{t=t_0} = \frac{(\dot{\phi'})(e^{it_0})ie^{it_0}}{|(\dot{\phi'})(e^{it_0})|} = i\frac{(\dot{\phi'})(\psi(\zeta))\psi(\zeta)}{|(\dot{\phi'})(\psi(\zeta))|} = i\frac{|(\dot{\psi'})(\zeta)|\psi(\zeta)}{(\dot{\psi'})(\zeta)}\sigma\text{-a.e.},$$

as desired. ∎

Theorem 4.3. *Let D is a bounded simply connected Lipschitz domain, and $1 \le p \le \infty$. Let $\psi : D \to \mathbb{D}$ be a conformal map with $\alpha := \psi^{-1}(0) \in D$. Then*

$$\mathfrak{n}^p(bD) = \left\{ \frac{(\dot{\psi'})}{|(\dot{\psi'})|}\dot{F} : F \in \mathcal{H}^p(D), F(\alpha) = 0 \right\}.$$

Proof. First by Proposition 4.1, one has

$$\mathfrak{n}^p(bD) = \left\{ T\dot{G} : G \in \mathcal{H}^p(D) \right\}.$$

Making use of Lemma 4.2, we further obtain

$$\mathfrak{n}^p(bD) = \left\{ \frac{(\dot{\psi'})}{|(\dot{\psi'})|}\dot{\psi}\dot{G} : G \in \mathcal{H}^p(D) \right\}.$$

Note that ψ is conformal on D and continuous on $\overline{D}$. In particular, ψ has only one zero at α and that zero is simple. Letting $F := \psi G$, then

$$G \in \mathcal{H}^p(D) \quad \text{if and only if} \quad F \in \mathcal{H}^p(D), F(\alpha) = 0.$$

The proof is complete. ∎

Note that for $D = \mathbb{D}$ we can choose $\psi(z) = z$, in which case Theorem 4.3 takes an especially simple form, namely

$$\mathfrak{n}^p(b\mathbb{D}) = \{\dot{F} : F \in \mathcal{H}^p(\mathbb{D}), F(0) = 0\}.$$

5 A characterization of $\mathfrak{n}^p(bD)$ for multiply connected D

Let D be a bounded Lipschitz domain. Then there exists $N \geq 1$, such that the boundary bD consists of N closed rectifiable curves. Here and throughout we denote by $\gamma_1, \gamma_2, \ldots, \gamma_N$ those closed curves of bD endowed with the positive orientation, with γ_N denoting the outer curve of bD (that is, D lies in the set of points inside of γ_N).

In order to characterize $\mathfrak{n}^p(bD)$ we need to understand which elements of $\mathcal{H}^p(D)$ admit holomorphic antiderivatives. According to classical complex analysis theory, a continuous complex-valued function has an antiderivative in a domain D (which may be simply or multiply-connected) if and only if the line integral of the function along every closed contour (i.e. piecewise C^1 path) in D is zero. See, for instance, [11, Thereom 6.44]. This leads us to the following:

Proposition 5.1. *Let D be a bounded Lipschitz domain and let the boundary of D be denoted as above. For $1 \leq p \leq \infty$ and $F \in \mathcal{H}^p(D)$ we have that F is the complex derivative of a holomorphic function on D if and only if*

$$\int_{\gamma_j} \dot{F}(\zeta) d\zeta = 0 \quad \text{for all} \quad j = 1, \ldots, N. \tag{10}$$

Proof. Let $\{D_k\}$ be Nečas exhaustion of D as defined in Lemma 2.3. We will use the notation of Lemma 2.3 throughout this proof. For each k and $1 \leq j \leq N$, let γ_j^k denote portion of bD_k such that $\Lambda_k(\gamma_j^k) = \gamma_j$.

First, assume F is a derivative of a holomorphic function on D. For each k the curve γ_j^k is a closed contour in D. Thus, by the Fundamental Theorem of Calculus, we have

$$\int_{\gamma_j^k} F(\zeta) d\zeta = 0, \quad j = 1, \ldots, N.$$

Thus

$$\begin{aligned} 0 &= \lim_{k\to\infty} \int_{\gamma_j^k} F(\zeta) d\zeta = \lim_{k\to\infty} \int_{\gamma_j^k} F(\eta_k) T_k(\eta_k) d\sigma_k(\eta_k) \\ &= \lim_{k\to\infty} \int_{\gamma_j} F(\Lambda_k(\eta)) T_k(\Lambda_k(\eta)) w_k(\eta) d\sigma(\eta) = \int_{\gamma_j} \dot{F}(\eta) T(\eta) d\sigma(\eta) = \int_{\gamma_j} \dot{F}(\zeta) d\zeta, \end{aligned}$$

where we used the Dominated Convergence Theorem with the dominating function $M|F^*|$ (here we are using the fact that $F \in \mathcal{H}^p(D)$ so that $F^* \in L^1(bD, \sigma)$), obtaining (10).

Conversely, assume (10) holds. Fixing a point $a \in D$, we shall show that for any $z \in D$, and any contour η in D connecting a and z, the following line integral

$$\int_{\eta} F(\zeta) d\zeta$$

is independent of the choice of the path.

Indeed, let η_1 and η_2 be two contours joining a and z and let $\beta = \eta_1 \cup (-\eta_2)$ be the closed contour starting and ending at a (here $-\eta_2$ is η_2 oriented in the opposite direction). Without loss of generality, suppose β is oriented counterclockwise and has no self-intersections. If the domain bounded by β is a subset of D, then $\int_\beta F(\zeta)d\zeta = 0$ by Cauchy's theorem. Else, for some m between 1 and N there are m components of bD, say, $\gamma_1, \ldots, \gamma_m$, that lie inside the domain bounded by β, while the remaining components $\gamma_{m+1}, \ldots, \gamma_N$ lie outside of such domain. With same notation as before, for a Nečas exhaustion $\{D_k\}$, we choose k large enough so that D_k contains β, $\gamma_1^k, \ldots, \gamma_m^k$ lie inside of β, and $\gamma_{m+1}^k, \ldots, \gamma_N^k$ lie outside of β. By a generalized version of Cauchy's theorem (see, for example, [11, Thereom 8.9]),

$$\int_\beta F(\zeta) d\zeta = \sum_{\ell=1}^{m} \int_{\gamma_\ell^k} F(\zeta) d\zeta \quad \text{for any large } k.$$

By an argument similar to the proof of Equation (11) we have

$$\int_\beta F(\zeta) d\zeta = \lim_{k\to\infty} \sum_{\ell=1}^{m} \int_{\gamma_\ell^k} F(\zeta) d\zeta = \sum_{\ell=1}^{m} \int_{\gamma_\ell} \dot{F}(\zeta) d\zeta = 0,$$

where we used (10) in the last equality. Equivalently,

$$\int_{\eta_1} F(\zeta) d\zeta = \int_{\eta_2} F(\zeta) d\zeta,$$

thus

$$H(z) := \int_{\eta} F(\zeta) d\zeta$$

is well defined and is a holomorphic antiderivative of F on D. ∎

Remark 5.2. By Cauchy's theorem (in Lemma 2.4), we have

$$\sum_{j=1}^{N} \int_{\gamma_j} \dot{F}(\zeta) d\zeta = \int_{bD} \dot{F}(\zeta) d\zeta = 0,$$

for any $F \in \mathcal{H}^p(D)$. Then we can refine the statement of Proposition 5.1 by requiring that only $(N-1)$-many terms in Equation (10) vanish. Without loss of generality, we choose the first $(N-1)$ terms. Hence, Equation (10) is equivalent to

$$\int_{\gamma_j} \dot{F}(\zeta)d\zeta = 0 \quad \text{for all} \quad j = 1, \dots, N-1. \tag{11}$$

Theorem 5.3. *Let D be a bounded Lipschitz domain and $1 \le p \le \infty$. Then with $\mathfrak{n}^p(bD)$ as in* (2) *we have*

$$\mathfrak{n}^p(bD) = \left\{ Tf : \; f \in h^p(bD), \quad \int_{\gamma_j} f(\zeta)\, d\zeta = 0, \quad 1 \le j \le N-1 \right\}. \tag{12}$$

If D is simply connected then the above identity reads $\mathfrak{n}^p(bD) = \mathfrak{n}_1$, see Theorem 4.1 (we should perhaps point out that the congruence of $\mathfrak{n}^p(bD)$ with the two spaces $\mathfrak{n}_2$ and $\mathfrak{n}_3$ proved therein relies upon results that are classically stated for simply connected D).

Proof. Let

$$L^p_0(bD, \sigma) := \left\{ g \in L^p(bD, \sigma) : \int_{bD} g(\zeta) d\sigma(\zeta) = 0 \right\}$$

and

$$L^p_{00}(bD, \sigma) := \left\{ g \in L^p(bD, \sigma) : \int_{\gamma_j} g(\zeta) d\sigma(\zeta) = 0, \quad 1 \le j \le N \right\}.$$

Obviously $L^p_{00}(bD, \sigma) \subset L^p_0(bD, \sigma)$. We claim that

$$\mathfrak{n}^p(bD) = \{ g \in L^p_{00}(bD, \sigma) : \overline{T} g \in h^p(bD) \}. \tag{13}$$

Indeed, if $g \in \mathfrak{n}^p(bD)$ there exists a $G \in \vartheta(D)$ with $G' \in \mathcal{H}^p(D)$ such that $g = -iT(\dot{G'})$, see (2); hence $\bar{T} g = -i(\dot{G'}) \in h^p(bD)$. Moreover Proposition 5.1 gives that

$$\int_{\gamma_j} g(\zeta) d\sigma(\zeta) = -i \int_{\gamma_j} (\dot{G'})(\zeta) d\zeta = 0, \quad j = 1, \dots, N-1$$

proving that $g \in L^p_{00}(bD,\sigma)$ and concluding the proof of the forward inclusion. For the reverse inclusion, suppose $g \in L^p_{00}(bD,\sigma)$ and $g = T\dot{F}$ for some $F \in \mathcal{H}^p(D)$. Then

$$\int_{\gamma_j} \dot{F}(\zeta)d\zeta = \int_{\gamma_j} g(\zeta)d\sigma(\zeta) = 0, \quad j = 1,\ldots,N-1$$

and it follows from Proposition 5.1 and Equation (11) that F has an antiderivative $G \in \vartheta(D)$. Note that $iG \in \mathcal{H}^{1,p}(D)$ by definition. Thus, $g = T\dot{F} = -iT(i\dot{G}') \in \mathfrak{n}^p(bD)$. The proof of (13) is concluded. Equation (12) now follows since for g as above we have $g = Tf$ with $f := \overline{T}g$. ∎

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