Unique continuation of Schrödinger-type equations for $\bar\partial$

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Abstract

The purpose of this paper is to study the unique continuation property for a Schrödinger-type equation $\bar\partial u = Vu$ on a domain in $\mathbb{C}^n$, where the solution u may be a scalar function, or a vector-valued function. While simple examples show that the unique continuation property fails in general if the potential $V \in L^p, p < 2n$, we first prove that, in the case when u is a scalar function, the unique continuation property holds when $V \in L^{2n}_{loc}$ and is $\bar\partial$-closed. For vector-valued smooth solutions, we establish the unique continuation property either when $V \in L^p_{loc}$, $p > 2n$ for $n \geq 3$, or when $V \in L^{2n}_{loc}$ for $n = 2$. Finally, we discuss the unique continuation property for some special cases where $V \notin L^{2n}_{loc}$, for instance, V is a constant multiple of $\frac{1}{|z|}$.

1 Introduction

Let Ω be a domain in $\mathbb{C}^n, n \geq 1$. Let $u : \Omega \to \mathbb{C}^N$ be a $H^1_{loc}(\Omega)$ solution to the following Schrödinger-type equation for the $\bar\partial$ operator:

$$\bar\partial u = Vu \quad \text{on} \quad \Omega \tag{1.1}$$

in the sense of distributions. Here the potential V is an $N \times N$ matrix of $(0, 1)$ forms with $L^p_{loc}(\Omega)$ coefficients for some $p \geq 1$, and the space $H^k_{loc}(\Omega) := W^{k,2}_{loc}(\Omega)$, where $W^{k,p}_{loc}(\Omega)$ is the standard Sobolev space of functions whose weak derivatives up to order k exist and belong to $L^p_{loc}(\Omega)$. The equation (1.1) arises naturally from various questions in CR and almost complex geometry and plays an important role, for instance, while studying the boundary regularity and uniqueness of CR-mappings, as well as uniqueness of J-holomorphic curves. See [2, 8] et al.

In this paper, we study the (strong) unique continuation property of (1.1). Namely, we investigate whether a solution to (1.1) vanishing to infinite order in the L^2 sense at one point vanishes identically. Here a function $u \in L^2_{loc}(\Omega)$ is said to vanish to infinite order (or, be flat) in the L^2 sense at a point $z_0 \in \Omega$ if for all $m \geq 1$,

$$\lim_{r \to 0} r^{-m} \int_{|z-z_0|<r} |u(z)|^2 dv_z = 0,$$

where dv_z is the Lebesgue measure in $\mathbb{C}^n$ with respect to the dummy variable z. Otherwise, u is said to vanish to a finite order in the L^2 sense at z_0.

As demonstrated by Example 2, the unique continuation property fails in general for (1.1) with L^p_{loc} potentials, $p < 2n = dim_{\mathbb{R}}\Omega$, the real dimension of the source domain Ω. On the other hand,

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it should be reminded that for the real Laplacian Δ, the unique continuation property has been thoroughly understood. In particular, the works of Chanillo-Sawyer [5] and Wolff [19, 20] have shown that for a domain $\Omega \subset \mathbb{R}^d$ and $V \in L^d_{loc}(\Omega)$, the unique continuation property for $H^2_{loc}(\Omega)$ solutions of the differential inequality

$$|\Delta u| \leq V|\nabla u| \quad \text{on} \quad \Omega \tag{1.2}$$

holds when $d = 2, 3, 4$, and fails in general when $d \geq 5$.

Surprisingly, due to the more rigid structure of $\bar{\partial}$, the unique continuation property of (1.1) holds for all $\bar{\partial}$-closed $L^{2n}_{loc}(\Omega)$ potentials, $n(= dim_{\mathbb{C}}\Omega) \geq 1$, as stated in the following theorem in the case when their solutions are scalar functions. Here a locally Lebesgue integrable form V on Ω is said to be $\bar{\partial}$-closed if $\int_\Omega V \wedge (\bar{\partial}\eta) = 0$ for any test form η. This dimension independence of the unique continuation property for $\bar{\partial}$ stands in stark contrast to the aforementioned result for Δ. In view of Example 2, the theorem fails in general if the assumption $V \in L^{2n}_{loc}(\Omega)$ is weakened to $V \in L^p_{loc}(\Omega)$ for $p < 2n$.

Theorem 1.1. *Let Ω be a domain in $\mathbb{C}^n$. Suppose $u : \Omega \to \mathbb{C}$ with $u \in H^1_{loc}(\Omega)$, and satisfies $\bar{\partial}u = Vu$ on Ω in the sense of distributions for some $\bar{\partial}$-closed $(0, 1)$ form $V \in L^{2n}_{loc}(\Omega)$. If u vanishes to infinite order in the L^2 sense at some $z_0 \in \Omega$, then u vanishes identically.*

It is worthwhile to point out that the $\bar{\partial}$-closedness assumption of V in the theorem is natural in order to guarantee the existence of solutions to (1.1), due to the following formal computation

$$0 = \bar{\partial}^2 u = u\bar{\partial}V - V \wedge \bar{\partial}u = u\bar{\partial}V - uV \wedge V = u\bar{\partial}V.$$

The $n = 1$ case of Theorem 1.1 was established in [16] (for arbitrary target dimension N, see also Theorem 4.1); related quantitative estimates for solutions to $\bar{\partial}u = Vu$ were obtained by Kenig-Wang [11]. The real-valued solution case has been proved lately in [3] concerning the gradient operator ∇, given the equivalence of $\bar{\partial}$ to ∇ on such solutions. See also Corollary 5.7 for a similar result for smooth functions satisfying the inequality $|\bar{\partial}u| \leq V|u|$ for $V \in L^{2n}_{loc}$. The proof of Theorem 1.1 relies on a classification result of weak solutions to (1.1) in the following theorem.

Theorem 1.2. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. Given a $\bar{\partial}$-closed $(0, 1)$ form $V \in L^{2n}_{loc}(\Omega)$, there exists a function $f \in W^{1,2n}_{loc}(\Omega)$ such that every $H^1_{loc}(\Omega)$ solution $u : \Omega \to \mathbb{C}$ to $\bar{\partial}u = Vu$ on Ω in the sense of distributions is of the form $e^f h$, for some holomorphic function h on Ω. In particular, $u \in W^{1,q}_{loc}(\Omega)$ for all $1 \leq q < 2n$.*

In the second part of the paper, we study the case when solutions to (1.1), or to the following general inequality, are vector-valued (i.e., the target dimension $N \geq 1$):

$$|\bar{\partial}u| \leq V|u| \quad \text{a.e. on} \quad \Omega. \tag{1.3}$$

Here the potential V is a nonnegative scalar function in $L^p_{loc}(\Omega)$ for some $p \geq 1$. With the help of a complex polar coordinate formula in Lemma 4.2, we convert the unique continuation problem on a source domain of dimension n to that on the complex plane, where [16] can readily take into effect. As a consequence of this, we prove in Section 4 that for smooth solutions of (1.3), the strong unique continuation property holds for L^p_{loc} potentials, $p > 2n$. Note that in smooth category, a function vanishes to infinite order in the L^2 sense at a point if and only if it vanishes to infinite order in the usual jet sense at that point, that is, all its derivatives vanish at that point, see Lemma 3.2.

Theorem 1.3. *Let Ω be a domain in $\mathbb{C}^n$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in C^\infty(\Omega)$, and satisfies $|\bar\partial u| \le V|u|$ a.e. on Ω for some $V \in L^p_{loc}(\Omega)$, $p > 2n$. If u vanishes to infinite order at some $z_0 \in \Omega$, then u vanishes identically.*

See also a very recent result by Shi [17] under some stronger assumptions on the potentials. Specifically, in the case when $n = 2$, we prove the unique continuation property of (1.3) for L^4_{loc} potentials, which, as indicated by Example 2, is sharp. The key to its proof in Section 7 incorporates a weighted estimate of the Cauchy integral established in [16] and a Carleman inequality Proposition 6.3 for $\bar\partial$.

Theorem 1.4. *Let Ω be a domain in $\mathbb{C}^2$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in C^\infty(\Omega)$, and satisfies $|\bar\partial u| \le V|u|$ a.e. on Ω for some $V \in L^4_{loc}(\Omega)$. If u vanishes to infinite order at some $z_0 \in \Omega$, then u vanishes identically.*

Due to Theorem 1.1 and Example 2, a natural question arises about whether the strong unique continuation property holds for (1.3) with L^{2n}_{loc} potentials in the vector-valued solutions case for any complex source dimension n. At this point we are only able to establish Theorem 1.4 for $n = 2$ (and in [16] for $n = 1$). It remains unclear whether this property continues to be true when $n \ge 3$, in particular, in view of Wolff's intricate counter-examples to (1.2) in higher dimensional cases (with the real source dimension $d \ge 5$). See Remark 7.2 for unsolved questions along this line in detail. However, it is noteworthy that the weak unique continuation property holds even for L^2_{loc} potentials, as shown in [16]. Namely, any solution to (1.3) vanishing on an open subset must vanish identically.

Finally, despite the general failure of the unique continuation property for (1.3) with L^p_{loc} potentials, $p < 2n$, we explore in Section 5 and Section 6 a special case where $V \notin L^{2n}_{loc}$, yet the unique continuation property may still be anticipated. More precisely, V here takes the form of a constant multiple of $\frac{1}{|z|}$. Interestingly, the cases of $N = 1$ and $N \ge 2$ under this context are rather distinct: the unique continuation property holds true for all positive constant multiple C when $N = 1$, while when $N \ge 2$, this property fails in general if C is large, see Example 5.

Theorem 1.5. *Let Ω be a domain in $\mathbb{C}^n$ and $0 \in \Omega$. Let $u : \Omega \to \mathbb{C}^N$ with $u \in C^\infty(\Omega)$, and satisfy $|\bar\partial u| \le \frac{C}{|z|}|u|$ a.e. on Ω. Assume u vanishes to infinite order at $0 \in \Omega$.*
1). If $N = 1$, then u vanishes identically.
2). If $N \ge 2$ and $C < \frac{1}{4}$, then u vanishes identically.

We point out that in the case when either $N = 1$ or $n = 1$, the smoothness assumption on u above can be relaxed to $u \in H^1_{loc}(\Omega)$, as established in Theorem 5.1 and Theorem 6.1. See also Theorem 5.5 for the unique continuation when the potentials include both powers of $\frac{1}{|z|}$ and Lebesgue integrable functions. As an application, it allows us to refine an earlier result in [3] in terms of ∇, which states that near any flat point of a smooth function u, either $\frac{|\nabla u|}{|u|} \notin L^{2n}$, or u vanishes identically there. More precisely, denote by $u^{-1}(0)$ the zero set of a smooth function u. We obtain in Section 5 the following blowing-up property in terms of $\bar\partial$ near a flat point of u.

Corollary 1.6. *Let Ω be a domain in $\mathbb{C}^n$. Suppose $u : \Omega \to \mathbb{C}$ with $u \in C^\infty(\Omega)$, and vanishes to infinite order at some $z_0 \in \Omega$. Then for every neighborhood U of z_0 in Ω, either $U \setminus u^{-1}(0) = \emptyset$, or*

$$\int_{U\setminus u^{-1}(0)} \frac{|\bar\partial u|^{2n}}{|u|^{2n}}\, dv = \infty. \tag{1.4}$$

Remark 1.7. *One can compare Corollary 1.6 with the following entertaining facts for compactly supported functions on real and complex Euclidean spaces.*
1. *[3, Theorem 2.7] For any $u : \mathbb{R}^d \to \mathbb{C}$ with $u \in C_c^\infty(\mathbb{R}^d), d \geq 2$,*

$$\int_{supp\ u} \frac{|\nabla u|^2}{|u|^2}\, dv = \infty.$$

2. *[16, Theorem 1.3] For any $u : \mathbb{C}^n \to \mathbb{C}$ with $u \in C_c^\infty(\mathbb{C}^n)$,*

$$\int_{supp\ u} \frac{|\bar{\partial} u|^2}{|u|^2}\, dv = \infty. \tag{1.5}$$

The power 2 in (1.5) is optimal, in view of an example $u_0 \in C_c^\infty(\mathbb{C})$ in [13] by Mandache, which satisfies for all $p < 2$,

$$\int_{supp\ u_0} \frac{|\bar{\partial} u_0|^p}{|u_0|^p}\, dv < \infty.$$

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2 Moser-Trudinger inequality and applications

Let Ω be a bounded domain in $\mathbb{R}^d$. One of the technical aspects to prove our main theorems is a chain rule for weak derivatives of the exponential of $W^{1,d}$ functions. In this section, we shall show

Proposition 2.1. *Let Ω be a domain in $\mathbb{R}^d$ and $f \in W^{1,d}_{loc}(\Omega)$. Then $e^f \in W^{1,q}_{loc}(\Omega)$ for all $1 \leq q < d$. Moreover, $\nabla e^f = e^f \nabla f$ in the sense of distributions.*

The $W^{1,d}$ space is the critical Sobolev space where the Sobolev embedding theorem fails, and instead is substituted by the classical **Moser-Trudinger inequality**. Recall that the Moser-Trudinger inequality states (see [14]) that for a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, there exists a positive constant C_{MT} depending only on d such that

$$\sup_{u \in W_0^{1,d}(\Omega),\quad \|\nabla u\|_{L^d(\Omega)} \leq 1} \int_\Omega e^{\alpha_d |u|^{\frac{d}{d-1}}} dv \leq C_{MT} |\Omega|.$$

Here $\alpha_d := d w_{d-1}^{\frac{1}{d-1}}$, with w_{d-1} the surface area of the unit sphere in $\mathbb{R}^d$, and $|\Omega|$ the volume of Ω. It turns out that the Moser-Trudinger inequality is exactly the key to prove Proposition 2.1. Before proceeding to its proof, we first make use of the inequality to show that the exponential of $W^{1,d}$ functions belongs to L^p for all $p < \infty$.

Lemma 2.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $f \in W^{1,d}(\Omega)$. Then for any $1 \leq p < \infty$, $e^{|f|} \in L^p(\Omega)$ with*

$$\left\| e^{|f|} \right\|^p_{L^p(\Omega)} \leq 2|\Omega| \left(e^{\frac{C_\Omega p^d \|f\|^d_{W^{1,d}(\Omega)}}{\alpha_d^{d-1}}} + C_{MT} \right), \tag{2.1}$$

for some constant C_Ω dependent only on Ω. In particular, $e^f \in L^p(\Omega)$. Equivalently, if $\log|g| \in W^{1,d}(\Omega)$ for some function g on Ω, then $g, \frac{1}{g} \in L^p(\Omega)$ for all $1 \leq p < \infty$.

Proof. Extend f to be a function $\tilde{f}$ on a bounded Lipschitz domain $\tilde{\Omega}$, such that $\Omega \subset\subset \tilde{\Omega}$, $|\tilde{\Omega}| \leq 2|\Omega|$, $\tilde{f} \in W_0^{1,d}(\tilde{\Omega})$ with

$$\|\nabla \tilde{f}\|^d_{L^d(\tilde{\Omega})} \leq \|\tilde{f}\|^d_{W^{1,d}(\tilde{\Omega})} \leq C_\Omega \|f\|^d_{W^{1,d}(\Omega)}, \tag{2.2}$$

with C_Ω dependent only on Ω. See [7, pp. 268]. Denoting by $\mu := \frac{p^{d-1}\|\nabla \tilde{f}\|^d_{L^d(\tilde{\Omega})}}{\alpha_d^{d-1}}$, then

$$\int_\Omega e^{p|f|}dv \leq \int_{\tilde{\Omega}} e^{p|\tilde{f}|}dv = \int_{x\in\tilde{\Omega}, |\tilde{f}(x)|\leq\mu} e^{p|\tilde{f}|}dv + \int_{x\in\tilde{\Omega}, |\tilde{f}(x)|>\mu} e^{p|\tilde{f}|}dv.$$

Since

$$\left\{x \in \tilde{\Omega} : |\tilde{f}(x)| > \mu\right\} = \left\{x \in \tilde{\Omega} : p|\tilde{f}(x)| \leq \frac{\alpha_d |\tilde{f}(x)|^{\frac{d}{d-1}}}{\|\nabla \tilde{f}\|^{\frac{d}{d-1}}_{L^d(\tilde{\Omega})}}\right\} = \left\{x \in \tilde{\Omega} : p|\tilde{f}(x)| \leq \alpha_d |f_1(x)|^{\frac{d}{d-1}}\right\},$$

where $f_1 := \frac{\tilde{f}}{\|\nabla \tilde{f}\|_{L^d(\tilde{\Omega})}}$, we further have

$$\int_\Omega e^{p|f|}dv \leq e^{p\mu}|\tilde{\Omega}| + \int_{\tilde{\Omega}} e^{\alpha_d |f_1|^{\frac{d}{d-1}}}dv.$$

Note that $f_1 \in W_0^{1,d}(\tilde{\Omega})$ and $\|\nabla f_1\|_{L^d(\tilde{\Omega})} = 1$. Applying the Moser-Trudinger inequality to f_1 in the last inequality and making use of (2.2), we get

$$\int_\Omega e^{p|f|}dv \leq |\tilde{\Omega}|\left(e^{p\mu} + C_{MT}\right) \leq 2|\Omega| \left(e^{\frac{C_\Omega p^d \|f\|^d_{W^{1,d}(\Omega)}}{\alpha_d^{d-1}}} + C_{MT} \right).$$

(2.1) is proved. Since $|e^f| \leq e^{|f|}$, we further have $e^f \in L^p(\Omega)$. That $g, \frac{1}{g} \in L^p(\Omega)$ follows immediately from the facts that $|g| = e^f$ and $\frac{1}{|g|} = e^{-f}$ with $f := \log|g| \in W^{1,d}(\Omega)$. $\square$

It is worthwhile to note that the integrability assumption $f \in W^{1,d}(\Omega)$ in Lemma 2.2 is optimal in view of the following example. Denote by B_r the ball in $\mathbb{R}^d$ centered at 0 with radius r.

Example 1. *For each $k \in \mathbb{N}$, let*

$$f = -\ln|x|^{2k}, x \in B_{\frac{1}{2}} \subset \mathbb{R}^d, d \geq 2.$$

A direct computation shows that for each $1 \leq p < \infty$, $f \in L^p(B_{\frac{1}{2}})$ and $\nabla f = -\frac{2kx}{|x|^2}$ on $B_{\frac{1}{2}} \setminus \{0\}$. By a result of Harvey-Polking in [9], we have $\nabla f = -\frac{2kx}{|x|^2}$ on $B_{\frac{1}{2}}$ in the sense of distributions. Consequently, $\nabla f \in L^q(B_{\frac{1}{2}})$ for all $q < d$, and thus

$$f \in W^{1,q}(B_{\frac{1}{2}}) \ \text{for all} \ q < d.$$

On the other hand, since $e^f = \frac{1}{|x|^{2k}}$ on $B_{\frac{1}{2}}$,

$$e^f \notin L^1(B_{\frac{1}{2}}) \ \text{if} \ k \geq \frac{d}{2}.$$

Proof of Proposition 2.1: Firstly, according to Lemma 2.2, $e^f \in L^p_{loc}(\Omega)$ for all $1 \le p < \infty$. By Hölder's inequality, we have $e^f \nabla f \in L^q_{loc}(\Omega)$ for each $1 \le q < d$, and for every Lipschitz subdomain $\tilde{\Omega} \subset\subset \Omega$,

$$\|e^f \nabla f\|_{L^q(\tilde{\Omega})} \le \|\nabla f\|_{L^d(\tilde{\Omega})} \|e^f\|_{L^{q^*}(\tilde{\Omega})} < \infty, \tag{2.3}$$

where $q^* := \frac{dq}{d-q}$. We next show that $\nabla e^f = e^f \nabla f$ in the sense of distributions. If so, then $e^f \in W^{1,q}_{loc}(\Omega)$ by (2.3) for all $1 \le q < d$, completing the proof.

$\nabla e^f = e^f \nabla f$ is trivially true if $f \in C^\infty(\Omega)$ as a consequence of the classical chain rule. For general $f \in W^{1,d}_{loc}(\Omega)$ and any Lipschitz subdomain $\tilde{\Omega} \subset\subset \Omega$, let $f_k \in C^\infty(\tilde{\Omega})$ converge to f in the $W^{1,d}(\tilde{\Omega})$ norm. By Sobolev embedding theorem, for all $1 \le p < \infty$,

$$\|f_k - f\|_{L^p(\tilde{\Omega})} \to 0 \tag{2.4}$$

as $k \to \infty$. Moreover, applying Lemma 2.2 to f and f_k, we have $e^f, e^{f_k} \in L^p(\tilde{\Omega})$ for all $1 \le p < \infty$, with

$$\|e^f\|_{L^p(\tilde{\Omega})} + \|e^{f_k}\|_{L^p(\tilde{\Omega})} \le C \tag{2.5}$$

for some constant C dependent only on $\|f\|_{W^{1,d}(\tilde{\Omega})}$, $\tilde{\Omega}$ and p.

We claim that $e^{f_k} \to e^f$ in the L^p norm, $1 \le p < \infty$, and $\nabla e^{f_k} \to e^f \nabla f$ in the L^q norm, $1 \le q < d$. We shall need the following elementary inequality as a consequence of the mean-value theorem: for $z_1, z_2 \in \mathbb{C}$,

$$|e^{z_1} - e^{z_2}| \le \sup_{t \in [0,1]} \left| e^{tz_1 + (1-t)z_2} \right| |z_1 - z_2| \le \left| e^{|z_1|} + e^{|z_2|} \right| |z_1 - z_2|.$$

Making use of this inequality, Hölder's inequality and (2.4)-(2.5), we have for every $1 \le p < \infty$,

$$\|e^{f_k} - e^f\|_{L^p(\tilde{\Omega})} \le \|(e^{|f_k|} + e^{|f|})|f_k - f|\|_{L^p(\tilde{\Omega})} \le \|e^{|f_k|} + e^{|f|}\|_{L^{2p}(\tilde{\Omega})} \|f_k - f\|_{L^{2p}(\tilde{\Omega})} \to 0 \tag{2.6}$$

as $k \to \infty$. Moreover, for each $q < d$, noting that $\nabla e^{f_k} = e^{f_k} \nabla f_k$, by (2.5) and (2.6)

$$\begin{aligned}\|\nabla e^{f_k} - e^f \nabla f\|_{L^q(\tilde{\Omega})} &\le \|(e^{f_k} - e^f)\nabla f_k\|_{L^q(\tilde{\Omega})} + \|e^f(\nabla f_k - \nabla f)\|_{L^q(\tilde{\Omega})} \\ &\le \|e^{f_k} - e^f\|_{L^{q^*}(\tilde{\Omega})} \|\nabla f_k\|_{L^d(\tilde{\Omega})} + \|e^f\|_{L^{q^*}(\tilde{\Omega})} \|\nabla f_k - \nabla f\|_{L^d(\tilde{\Omega})} \to 0\end{aligned}$$

as $k \to \infty$. The claim is proved. In particular, it immediately gives $e^{f_k} \to e^f$ and $\nabla e^{f_k} \to e^f \nabla f$ on $\tilde{\Omega}$ in the sense of distributions, and thus $\nabla e^f = e^f \nabla f$ on $\tilde{\Omega}$ in the sense of distributions. □

At the end of the section, we discuss another immediate application of Lemma 2.2. We say a function f to be Hölder at a point x_0 if there exists some $\alpha \in (0, 1]$ and a constant $C > 0$ such that for all x near x_0,

$$|f(x) - f(x_0)| \le C|x - x_0|^\alpha.$$

The following corollary states that the logarithms of such functions are never in $W^{1,d}$ near x_0. This also generalizes a similar result in [3] for Lipschitz functions (i.e., $\alpha = 1$).

Corollary 2.3. *Let f be a function near x_0 in $\mathbb{R}^d$ and be Hölder at x_0. Then $\ln|f(x) - f(x_0)| \notin W^{1,d}$ near x_0.*

Proof. Supposing not. Then by Lemma 2.2, for all $1 \le p < \infty$,

$$\frac{1}{|f(x) - f(x_0)|} = e^{-\ln|f(x)-f(x_0)|} \in L^p$$

near x_0. However, by the Hölder property of f at x_0, this would imply that there exist some constants $0 < \alpha < 1$ and $C > 0$, such that

$$\frac{1}{|f(x) - f(x_0)|} \ge \frac{1}{C|x - x_0|^\alpha} \in L^p$$

near x_0, which is absurd when $p \ge \frac{d}{\alpha}$. $\square$

3 Unique continuation for the target dimension $N = 1$

In this section, we prove the classification Theorem 1.2 of weak solutions to $\bar\partial$, and the unique continuation Theorem 1.1 for scalar solutions ($N = 1$) in a domain $\Omega \subset \mathbb{C}^n, n \ge 1$.

The following lemma concerning the local ellipticity of $\bar\partial$ for $(0,1)$ data with $W^{k,p}_{loc}$ coefficients is well-known for $p = 2$ (see, for instance, [4, Theorem 4.5.1]). However, it seems difficult to find a reference for general $p, 1 < p < \infty$. Since this property will be repeatedly used in the paper, we present a proof below.

Lemma 3.1. *Let Ω be a domain in $\mathbb{C}^n$, and $1 < p < \infty$. Let $V \in L^p_{loc}(\Omega)$ be a $\bar\partial$-closed $(0,1)$ form on Ω. Then every solution to $\bar\partial f = V$ on Ω in the sense of distributions belongs to $W^{1,p}_{loc}(\Omega)$. Furthermore, if $V \in W^{k,p}_{loc}(\Omega)$, $k \in \mathbb{Z}^+$, then every solution to $\bar\partial f = V$ on Ω in the sense of distributions belongs to $W^{k+1,p}_{loc}(\Omega)$.*

Proof. Suppose the $\bar\partial$-closed $(0,1)$ form V belongs to $W^{k,p}_{loc}(\Omega)$ for some $k \in \mathbb{Z}^+ \cup \{0\}$. Since the lemma is purely local, and every other solution is differed only by a holomorphic function, it suffices to show that for any $z_0 \in \Omega$, there exist a neighborhood U of z_0 and a solution f_0 to $\bar\partial f = V$ on U in the sense of distributions, such that $f_0 \in W^{k+1,p}_{loc}(U)$. For simplicity, let $z_0 = 0$ and $B_{2r} \subset\subset \Omega$ for some $r > 0$. Let η be a compactly supported function on B_{2r} such that $\eta = 1$ on B_r.

Given a mollifier ϕ on $\mathbb{C}^n$, we have $V_\epsilon := V * \phi_\epsilon \in C^\infty(B_{2r})$, V_ϵ is $\bar\partial$-closed on B_{2r} and $V_\epsilon \to V$ in the $L^p(B_{2r})$ norm. Applying the Bochner-Martinelli representation formula to ηV_ϵ on B_{2r}, one has

$$\eta(z)V_\epsilon(z) = -\int_{B_{2r}} \bar\partial(\eta(\zeta)V_\epsilon(\zeta)) \wedge B_1(\zeta, z) - \bar\partial \int_{B_{2r}} \eta(\zeta)V_\epsilon(\zeta) \wedge B_0(\zeta, z), \quad z \in B_{2r},$$

where for $q = 0, 1$,

$$B_q(\zeta, z) = - * \partial_\zeta \overline{\Gamma_q(\zeta, z)}$$

with

$$\Gamma_q(\zeta, z) = \frac{(n-2)!}{q!2^{q+1}\pi^n} \frac{1}{|\zeta - z|^{2n-2}} \left(\sum_{j=1}^n d\bar\zeta_j dz_j \right)^q.$$

See, for instance, [12, Chapter I]. Note that $\bar\partial(\eta V_\epsilon) = \bar\partial\eta \wedge V_\epsilon$ on B_{2r} and supp $\bar\partial\eta \subset B_{2r} \setminus B_r$. Then restricting on B_r,

$$V_\epsilon(z) = -\int_{B_{2r} \setminus B_r} \bar\partial\eta(\zeta) \wedge V_\epsilon(\zeta) \wedge B_1(\zeta, z) - \bar\partial \int_{B_{2r}} \eta(\zeta)V_\epsilon(\zeta) \wedge B_0(\zeta, z), \quad z \in B_r. \tag{3.1}$$

By Young's convolution inequality, there exists some $C > 0$ such that

$$\left\| \int_{B_{2r}\setminus B_r} \bar\partial\eta(\zeta) \wedge V_\epsilon(\zeta) \wedge B_1(\zeta, z) - \int_{B_{2r}\setminus B_r} \bar\partial\eta(\zeta) \wedge V(\zeta) \wedge B_1(\zeta, z) \right\|_{L^1(B_r)} \le C\|V_\epsilon - V\|_{L^1(B_{2r})},$$

which goes to 0 as $\epsilon \to 0$. Similarly,

$$\left\| \int_{B_{2r}} \eta(\zeta) \wedge V_\epsilon(\zeta) \wedge B_0(\zeta, z) - \int_{B_{2r}} \eta(\zeta) \wedge V(\zeta) \wedge B_0(\zeta, z) \right\|_{L^1(B_r)} \to 0$$

as $\epsilon \to 0$. Thus passing $\epsilon \to 0$ in (3.1), we obtain

$$V(z) = -\int_{B_{2r}\setminus B_r} \bar\partial\eta(\zeta) \wedge V(\zeta) \wedge B_1(\zeta, z) - \bar\partial \int_{B_{2r}} \eta(\zeta) V(\zeta) \wedge B_0(\zeta, z) \quad \text{on} \quad B_r \tag{3.2}$$

in the sense of distributions.

Note that

$$I := -\int_{B_{2r}\setminus B_r} \bar\partial\eta(\zeta) \wedge V(\zeta) \wedge B_1(\zeta, z) \in C^\infty(B_r),$$

and I is $\bar\partial$-closed on B_r by (3.2). By ellipticity of $\bar\partial$ for smooth data (see [4, Theorem 4.5.1]), there exists a function $v_0 \in C^\infty(B_r)$ such that $\bar\partial v_0 = I$ on B_r. On the other hand, according to the classical potential theory for the fundamental solution of Laplacian,

$$u_0 := -\int_{B_r} \eta(\zeta) V(\zeta) \wedge B_0(\zeta, z) \in W^{k+1,p}(B_r).$$

Letting

$$f_0 := v_0 + u_0, \tag{3.3}$$

we have $f_0 \in W^{k+1,p}_{loc}(B_r)$, and $\bar\partial f_0 = V$ on B_r in the sense of distributions by (3.2). □

Proof of Theorem 1.2: Since $V \in L^2_{loc}(\Omega)$ is $\bar\partial$-closed and Ω is pseudoconvex, by Hörmander's L^2 theory (see [4, Theorem 4.3.5]), there exists $f \in L^2_{loc}(\Omega)$ satisfying $\bar\partial f = V$ on Ω. Noting that $\bar\partial$ is an elliptic operator of order one for $(0, 1)$ data by Lemma 3.1 and $V \in L^{2n}_{loc}(\Omega)$, we further get $f \in W^{1,2n}_{loc}(\Omega)$. Hence we can apply Proposition 2.1 to $\tilde u := e^{-f}$, and obtain that $\tilde u \in W^{1,q}_{loc}(\Omega)$ for any $q < 2n$, and $\bar\partial\tilde u = -V\tilde u$ on Ω in the sense of distributions. In particular, $\tilde u \in L^p_{loc}(\Omega)$ for every $p < \infty$ by Sobolev embedding theorem.

For each solution $u \in H^1_{loc}(\Omega)$ to $\bar\partial u = Vu$, consider $h := u\tilde u$ on Ω. Similarly as in the proof of Proposition 2.1, we verify that $\bar\partial h = 0$ on Ω in the sense of distributions. In detail, let $u_k \in C^\infty(\Omega) \to u$ in the H^1_{loc} norm. Then for every subdomain $\tilde\Omega \subset\subset \Omega$, one has

$$\|\bar\partial u_k - Vu\|_{L^2(\tilde\Omega)} = \|\bar\partial u_k - \bar\partial u\|_{L^2(\tilde\Omega)} \to 0 \tag{3.4}$$

as $k \to \infty$. Moreover, by Sobolev embedding theorem

$$\|u_k - u\|_{L^{\frac{2n}{n-1}}(\tilde\Omega)} \le \|u_k - u\|_{H^1(\tilde\Omega)} \to 0 \tag{3.5}$$

as $k \to 0$. On the other hand, by Hölder's inequality, for each $k \ge 0$, $h_k := u_k\tilde u$ satisfies

$$\|h_k - h\|_{L^1(\tilde\Omega)} = \|(u_k - u)\tilde u\|_{L^1(\tilde\Omega)} \le \|u_k - u\|_{L^2(\tilde\Omega)}\|\tilde u\|_{L^2(\tilde\Omega)} \to 0$$

as $k \to \infty$. Since $u_k \in C^\infty(\Omega)$, the product rule applies to give $\bar\partial h_k = \bar\partial u_k \tilde u - V u_k \tilde u$ on Ω in the sense of distributions. Consequently, we use Hölder's inequality again to infer

$$\begin{aligned}\|\bar\partial h_k\|_{L^1(\tilde\Omega)} =&\|\bar\partial u_k \tilde u - V u_k \tilde u\|_{L^1(\tilde\Omega)} \le \|(\bar\partial u_k - Vu)\tilde u\|_{L^1(\tilde\Omega)} + \|V(u_k-u)\tilde u\|_{L^1(\tilde\Omega)}\\ \le&\|\bar\partial u_k - Vu\|_{L^2(\tilde\Omega)}\|\tilde u\|_{L^2(\tilde\Omega)} + \|V\|_{L^{2n}(\tilde\Omega)}\|u_k-u\|_{L^{\frac{2n}{n-1}}(\tilde\Omega)}\|\tilde u\|_{L^2(\tilde\Omega)} \to 0\end{aligned}$$

as $k \to \infty$ by (3.4)-(3.5). In particular, $h_k \to h$ and $\bar\partial h_k \to 0$ on Ω in the sense of distributions. Hence $\bar\partial h = 0$ on Ω in the sense of distributions. Altogether, $u = \tilde u^{-1} h = e^f h$ for some function $f \in W^{2n}_{loc}(\Omega)$, and some holomorphic function h on Ω. Moreover, since $e^f \in W^{1,q}_{loc}(\Omega)$ for all $1 \le q < 2n$ by Proposition 2.1, so does u. □

Recall that a smooth function vanishes to infinite order (or, is flat) in the jet sense at one point if all its derivatives vanish at that point. We verify below that for smooth functions, flatness in the jet sense and flatness in the L^2 sense are equivalent to each other.

Lemma 3.2. *Let h be a smooth function near $x_0 \in \mathbb{R}^d$. Then h vanishes to infinite order in the L^2 sense at x_0 if and only if h vanishes to infinite order in the jet sense at x_0. In particular, if h is real-analytic near x_0, then either $h \equiv 0$ near x_0, or h vanishes to a finite order in the L^2 sense at x_0.*

Proof. Without loss of generality let $x_0 = 0$. If h vanishes to infinite order in the jet sense at 0, then for any $m \ge 1$, there exists a constant C dependent on m such that $|h(x)| \le C|x|^m$ for $|x| << 1$. Thus

$$r^{-m}\int_{|x|<r} |h(x)|^2 dv_x \le Cr^{-m}\int_0^r t^{d-1+2m}dt \le Cr^{m+d} \to 0$$

as r goes to 0. Namely, h vanishes to infinite order in the L^2 sense at 0.

Conversely, suppose that h vanishes to infinite order in the L^2 sense, but vanishes to a finite order in the jet sense at 0. Let $k > 0$ be the smallest integer such that the k-th order homogeneous Taylor polynomial p_k of h is nonzero, and write $h(x) = p_k(x) + q_{k+1}(x)$, where $q_{k+1}(x)$ is the remaining term of the Taylor expansion of h. By definition of p_k,

$$c_0 := r^{-2k}\int_{|x|=1} |p_k(x)|^2 dS_x > 0$$

and is independent of r since p_k is a homogeneous polynomial of degree k. Let r_0 be such that for all $r < r_0$,

$$r^{-2k}\int_{|x|=1} |q_{k+1}(x)|^2 dS_x \le \frac{c_0}{8}.$$

Then for any $0 < r < r_0$, making use of the inequality $|a+b|^2 \ge \frac{3}{4}|a|^2 - 3|b|^2$ for $a, b \in \mathbb{R}$, we have

$$\begin{aligned}\int_{|x|<r} |h(x)|^2 dv_x &= \int_0^r t^{d-1}\int_{|x|=1} |p_k(x)+q_{k+1}(x)|^2 dS_x dt\\ &\ge \int_0^r t^{d-1}\int_{|x|=1} \frac{3}{4}|p_k(x)|^2 - 3|q_{k+1}(x)|^2 dS_x dt\\ &\ge \frac{3c_0}{8}\int_0^r t^{d-1}t^{2k}dt = \frac{3c_0}{8(d+2k)} r^{d+2k}.\end{aligned}$$

In particular,

$$\lim_{r\to 0} r^{-(d+2k)}\int_{|x|<r}|h(x)|^2dv_x \geq \frac{3c_0}{8(d+2k)}.$$

Contradiction!

If h is real-analytic near 0, then either $h\equiv 0$ near 0, or h vanishes to a finite order in the jet sense at 0, which is further equivalent to vanishing to a finite order in the L^2 sense at 0. The proof is complete. $\square$

Proof of Theorem 1.1: Again, let $z_0=0$, and $r_0>0$ be small such that $B_{r_0}\subset\Omega$. For each solution $u(\not\equiv 0)$ to $\bar\partial u = Vu$ on B_{r_0} in the sense of distributions, by Theorem 1.2 there exists a holomorphic function $h(\not\equiv 0)$ on B_{r_0} and a function $f\in W^{1,2n}_{loc}(B_{r_0})$ such that $u=e^fh$ on B_{r_0}. Applying Lemma 2.2 to $-f$, we further have $h=ue^{-f}$ with $e^{-f}\in L^2_{loc}(B_{r_0})$ in particular. Consequently, there exist some constants $C_1, C_2>0$ such that

$$\sup_{|z|<\frac{r_0}{2}}|h|\leq C_1 \quad\text{and}\quad \int_{|z|<\frac{r_0}{2}}|e^{-f}|^2dv_z\leq C_2.$$

Making use of Hölder's inequality, we have for any $0<r<\frac{r_0}{2}$,

$$\begin{aligned}\left(\int_{|z|<r}|h|^2dv_z\right)^2 &\leq C_1^2\left(\int_{|z|<r}|h|dv_z\right)^2\leq C_1^2\int_{|z|<r}|u|^2dv_z\int_{|z|<r}|e^{-f}|^2dv_z\\ &\leq C_1^2C_2\int_{|z|<r}|u|^2dv_z.\end{aligned}$$

Since h vanishes to a finite order in the L^2 sense at 0 according to Lemma 3.2, the same holds true for u. This completes the proof. $\square$

The assumption $V\in L^{2n}_{loc}$ in Theorem 1.1 can not be relaxed in the following sense. For each $p<2n$, there exists a differential equation $\bar\partial u = Vu$ with $V\in L^p_{loc}$, and this equation has a nontrivial solution that vanishes to infinite order at a specific point.

Example 2. *For each $1\leq p<2n$, let $\epsilon\in(0,\frac{2n}{p}-1)$ and consider*

$$\bar\partial u = \frac{\epsilon z d\bar z}{2|z|^{\epsilon+2}}u \quad on \quad B_1\subset\mathbb{C}^n.$$

It is straightforward to verify that $V:=\frac{\epsilon zd\bar z}{2|z|^{\epsilon+2}}\in L^p(B_1)$. On the other hand, $u_0=e^{-\frac{1}{|z|^\epsilon}}$ is a nontrivial solution to the above equation that vanishes to infinite order in the L^2 sense at 0.

The proof to Theorem 1.1 indicates that solutions to (1.1) inherit the unique continuation property from that of holomorphic functions. However, from the perspective of zero set, due to the presence of the other factor e^f, such solutions could exhibit a much larger zero set than holomorphic functions do. In fact, the following example constructs a global potential $V\in L^{2n}(\mathbb{C}^n)$, such that the "zero set" of every weak solution to (1.1) with this potential contains a countable dense set in $\mathbb{C}^n$. It is noteworthy that the exponential factor e^f no longer contributes zeros if $V\in L^p_{loc}, p>2n$, since in this case f becomes continuous by Sobolev embedding theorem.

Example 3. *Let $\phi(z) = -\ln(-\ln|z|)$ on $B_{\frac{1}{3}} \subset \mathbb{C}^n$, and $\chi(\geq 0) \in C_c^\infty(B_{\frac{1}{3}})$ such that $\chi = 1$ on $B_{\frac{1}{6}}$. Then $\psi := \chi\phi \in W^{1,2n}(\mathbb{C}^n)$ with $\psi \leq 0$ on $\mathbb{C}^n$. Given a countable dense set $S := \cup_{j=1}^\infty \{a_j\} \subset \mathbb{C}^n$, consider*

$$f(z) := \sum_{j=1}^\infty 2^{-j}\psi(z - a_j), \quad z \in \mathbb{C}^n.$$

Clearly, $f \in W^{1,2n}(\mathbb{C}^n)$. Further let

$$V := \bar{\partial} f \quad on \quad \mathbb{C}^n.$$

(Note that $\operatorname{supp} V = \mathbb{C}^n$.) According to the construction of V and Theorem 1.2, any solution u to $\bar{\partial} u = Vu$ on $\mathbb{C}^n$ in the sense of distributions is of the from $e^f h$, for some holomorphic function h on $\mathbb{C}^n$. Note that near every $a_j \in S$,

$$|e^f| \leq \frac{1}{|\ln|z - a_j||^{2^{-j}}}.$$

Hence for all $a_j \in S$,

$$\lim_{z \to a_j} u = 0.$$

On the other hand, u vanishes to a finite order in the L^2 sense at these points, as a consequence of Theorem 1.1.

As mentioned in the introduction, when the solutions are real-valued, Theorem 1.1 can be reduced to a unique continuation property in [3] for the inequality $|\nabla u| \leq V|u|$ with $V \in L^{2n}_{loc}$, due to the equivalence of $\bar{\partial}$ and ∇ for real-valued functions. The following example constructed in [8] by Gong and Rosay carries a family of continuous solutions to $\bar{\partial} u = Vu$ with some $V \in L^{2n}_{loc}$, yet $\frac{|\nabla u|}{|u|} \notin L^{2n}_{loc}$, as a result of which [3] fails to apply. Instead, we may use Theorem 1.1 to conclude that none of the nontrivial solutions vanishes to infinite order at any point in the L^2 sense.

Example 4. *Let $\{a_j\}_{j=1}^\infty$ be a sequence of distinct points in $B_{\frac{1}{4}} \subset \mathbb{C}^n$ convergent to 0 and consider*

$$\bar{\partial} u = Vu \qquad on \quad B_{\frac{1}{4}} \tag{3.6}$$

with

$$V := \frac{z d\bar{z}}{|z|^2 |\ln|z|^2|^2 \Pi_{j=1}^\infty |\ln|z - a_j|^2|^{\frac{1}{j^2}}} + \sum_{k=1}^\infty \frac{(z - a_k) d\bar{z}}{k^2 |z - a_k|^2 \ln|z|^2 \, |\ln|z - a_k|^2|^{\frac{1}{k^2}+1} \, \Pi_{j \neq k} |\ln|z - a_j|^2|^{\frac{1}{j^2}}}.$$

Then $V \in L^{2n}(B_{\frac{1}{4}})$. Indeed, since $|\ln|z - a_j|^2| \geq 1, |\ln|z|^2| \geq 1$ on $B_{\frac{1}{4}}$, and $g(z) := \frac{1}{|z| \ln|z|} \in L^{2n}(B_{\frac{1}{2}})$, one has

$$\|V\|_{L^{2n}(B_{\frac{1}{4}})} \lesssim \|g\|_{L^{2n}(B_{\frac{1}{4}})} + \sum_{k=1}^\infty \frac{1}{k^2} \|g\|_{L^{2n}(B_{\frac{1}{2}})} < \infty.$$

(3.6) *possesses a family of nontrivial solutions. In fact, for every holomorphic function h on $B_{\frac{1}{4}}$,*

$$u^h(z) := \frac{h(z)}{\ln|z|^2 \Pi_{j=1}^\infty |\ln|z - a_j|^2|^{\frac{1}{j^2}}}$$

is continuous on $B_{\frac{1}{4}}$ and satisfies $\bar\partial u = Vu$ on $B_{\frac{1}{4}}\setminus\{\cup_{j=1}^{\infty}\{a_j\}\cup\{0\}\}$. Applying a general removable singularity result in [9], one further has $\bar\partial u^h = Vu^h$ to hold on $B_{\frac{1}{4}}$. Note that the zero set $(u^h)^{-1}(0) = \cup_{j=1}^{\infty}\{a_j\}\cup h^{-1}(0)\cup\{0\}$. According to Theorem 1.1, none of the nontrivial solutions to (3.6) vanishes to infinite order in the L^2 sense at any of these zero points.

In particular, if h in the expression of u^h is a holomorphic function of one variable with zeroes on $B_{\frac{1}{4}}$ (say, $h(z) = z_1$), then

$$|\nabla u^h| \approx |\bar\partial u^h| + |\partial u^h| \approx \left(V + \frac{|\partial h|}{|h|}\right)|u^h|.$$

Since $\frac{\partial h}{h}\notin L^2(B_{\frac{1}{4}})$ near any zero of h (see [3, Proposition 5.2]), we have

$$\frac{|\nabla u^h|}{|u^h|}\notin L^{2n}_{loc}(B_{\frac{1}{4}}),$$

where [3] fails to apply.

One can compare Theorem 1.1 with a uniqueness result for Lipschitz functions in [3]: if u is Lipschitz and satisfies $\nabla u = Vu$ for some $V\in L^{2n}(B_{\frac{1}{4}})$ and $u(0) = 0$, then $u\equiv 0$. Note that if the holomorphic function h in Example 4 is nowhere zero on $B_{\frac{1}{4}}$, then the continuous function u^h satisfies $\nabla u^h = Vu^h$ for some $V\in L^{2n}_{loc}(B_{\frac{1}{4}})$, and has infinite many zeros on $B_{\frac{1}{4}}$. The uniqueness property fails for u^h since it does not belong to $Lip(B_{\frac{1}{4}})$.

4 Unique continuation for L^p potentials with $p\ge 2n$

In this section we prove Theorem 1.3 for smooth vector-valued solutions (where the target dimension $N\ge 1$) to $|\bar\partial u|\le V|u|$ a.e. on a domain in $\mathbb{C}^n$ with L^p potentials, $p\ge 2n$. When $N\ge 1$, this inequality with a solution $u = (u_1,\dots,u_N)$ reads as

$$|\bar\partial u| := \left(\sum_{j=1}^{n}\sum_{k=1}^{N}|\bar\partial_j u_k|^2\right)^{\frac{1}{2}} \le V\left(\sum_{k=1}^{N}|u_k|^2\right)^{\frac{1}{2}} := V|u|.$$

The following unique continuation properties have been proved for L^2_{loc} potentials.

Theorem 4.1. *[16] Let Ω be a domain in $\mathbb{C}^n$. Suppose $u:\Omega\to\mathbb{C}^N$ with $u\in H^1_{loc}(\Omega)$, and satisfies $|\bar\partial u|\le V|u|$ a.e. on Ω for some $V\in L^2_{loc}(\Omega)$.*
1). The weak unique continuation holds: if u vanishes in an open subset of Ω, then u vanishes identically.
2). If $n = 1$, then the (strong) unique continuation holds: if u vanishes to infinite order in the L^2 sense at some $z_0\in\Omega$, then u vanishes identically.

In particular, since all the potentials under consideration in this paper belong to L^2_{loc} away from the reference point z_0, their unique continuation properties can be reduced to demonstrating that solutions vanish near a neighborhood of z_0, in view of the above weak unique continuation property.

The complex polar coordinates change formula below will play a crucial rule throughout the rest of the paper. One can find the formula that was used in [10, pp. 260] without proof. For the convenience of the reader, we provide its proof below. Let S^d be the unit sphere in $\mathbb{R}^{d+1}$, and D_{r_0} be the disk centered at 0 with radius r_0 in $\mathbb{C}$. Recall that B_{r_0} is the ball centered at 0 with radius r_0 in $\mathbb{C}^n$.

Lemma 4.2. *Let $u \in L^1(B_{r_0})$. Then for a.e. $\zeta \in S^{2n-1}$, $|w|^{2n-2}u(w\zeta)$ as a function of $w \in D_{r_0}$ is in $L^1(D_{r_0})$, with*

$$\int_{|z|<r_0} u(z)dv_z = \frac{1}{2\pi}\int_{|\zeta|=1}\int_{|w|<r_0} |w|^{2n-2}u(w\zeta)dv_w dS_\zeta.$$

Proof. First, by the standard polar coordinates change,

$$\begin{aligned} 2\pi\int_{|z|<r_0} u(z)dv_z =& 2\pi\int_0^{r_0} r^{2n-1}\int_{|\xi|=1} u(r\xi)dS_\xi dr \\ =& \int_0^{r_0} r^{2n-1}\int_{|\eta|=1}\int_{|\xi|=1} u(r\xi)dS_\xi dS_\eta dr. \end{aligned}$$

Here dS_η and dS_ξ are the Lebesgue measures over S^1 and S^{2n-1}, respectively. Since for each $\eta \in S^1$,

$$\int_{|\xi|=1} u(r\xi)dS_\xi = \int_{|\zeta|=1} u(r\zeta\eta)|\eta|dS_\zeta = \int_{|\zeta|=1} u(r\zeta\eta)dS_\zeta,$$

we further have by Fubini's theorem,

$$\begin{aligned} 2\pi\int_{|z|<r_0} u(z)dv_z &= \int_0^{r_0} r^{2n-1}\int_{|\eta|=1}\int_{|\zeta|=1} u(r\zeta\eta)dS_\zeta dS_\eta dr \\ &= \int_{|\zeta|=1}\int_0^{r_0} r^{2n-1}\int_{|\eta|=1} u(r\zeta\eta)dS_\eta dr dS_\zeta \\ &= \int_{|\zeta|=1}\int_0^{r_0} r\int_{|\eta|=1} |r\eta|^{2n-2}u(r\zeta\eta)dS_\eta dr dS_\zeta \\ &= \int_{|\zeta|=1}\int_{|w|<r_0} |w|^{2n-2}u(w\zeta)dv_w dS_\zeta. \end{aligned}$$

□

As seen below, Lemma 4.2 allows us to transform (1.3) with $L^p_{loc}, p > 2n$ potentials into new ones with L^2_{loc} potentials along almost all complex one-dimensional radial directions. On the other hand, when the solutions under consideration are smooth, the flatness of these solutions at a point naturally extends to their restrictions along those radial directions. Thus one can completely convert the unique continuation property in the higher source dimension case to that on the complex one dimension, where Theorem 4.1 can be applied.

Proof of Theorem 1.3: Without loss of generality, let $z_0 = 0$ and $r > 0$ be small such that $V \in L^p(B_r)$. For each fixed $\zeta \in S^{2n-1}$, let $\tilde{V}(w) := |w|^{\frac{2n-2}{p}}V(w\zeta)$ and $v(w) := u(w\zeta), w \in D_r$. Since all jets of u vanish at 0 by assumption, the same holds true for all jets of v. Thus v vanishes to infinite order at 0 in the L^2 sense by Lemma 3.2. Moreover, v satisfies

$$|\bar{\partial}v(w)| = |\zeta\cdot\bar{\partial}u(w\zeta)| \le V(w\zeta)|u(w\zeta)| = |w|^{-\frac{2n-2}{p}}\tilde{V}(w)|v(w)|, \quad w \in D_r.$$

Here for the solution $u=(u_1,\dots,u_N)$ and $\zeta\in S^{2n-1}$ realized as a complex vector $(\zeta_1,\dots,\zeta_n)\in\mathbb{C}^n$, the dot product $|\zeta\cdot\bar\partial u(w\zeta)|$ is understood as $\sum_{k=1}^N\left|\sum_{j=1}^n\zeta_j\bar\partial_j u_k(w\zeta)\right|$.

We claim that $|w|^{-\frac{2n-2}{p}}\tilde V(w)\in L^2(D_r)$ for a.e. $\zeta\in S^{2n-1}$. In fact, according to Lemma 4.2,

$$\int_{|z|<r}|V(z)|^p dv_z=\frac{1}{2\pi}\int_{|\zeta|=1}\int_{|w|<r}|w|^{2n-2}|V(w\zeta)|^p dv_w dS_\zeta=\frac{1}{2\pi}\int_{|\zeta|=1}\int_{|w|<r}|\tilde V(w)|^p dv_w dS_\zeta.$$

In particular, $\tilde V\in L^p(D_r)$ for a.e. $\zeta\in S^{2n-1}$. By Hölder's inequality

$$\begin{aligned}\int_{|w|<r}\left||w|^{-\frac{2n-2}{p}}\tilde V(w)\right|^2 dv_w&\le\left(\int_{|w|<r}|\tilde V(w)|^p dv_w\right)^{\frac{2}{p}}\left(\int_{|w|<r}|w|^{-\frac{2n-2}{p}\frac{2p}{p-2}}dv_w\right)^{\frac{p-2}{p}}\\&=\left(\int_{|w|<r}|\tilde V(w)|^p dv_w\right)^{\frac{2}{p}}\left(\int_{|w|<r}|w|^{-\frac{4n-4}{p-2}}dv_w\right)^{\frac{p-2}{p}}.\end{aligned}$$

Since $p>2n$, we have $\frac{4n-4}{p-2}<2$ and thus $\int_{|w|<r}|w|^{-\frac{4n-4}{p-2}}dv_w<\infty$. This, combined with the fact that $\tilde V\in L^p(D_r)$, gives the desired claim. Hence we can make use of Theorem 4.1 part 2) to obtain $v=0$ on D_r for a.e. $\zeta\in S^{2n-1}$. Thus $u=0$ on B_r. The weak unique continuation property in Theorem 4.1 part 1) further applies to give $u\equiv 0$ on Ω. □

5 Unique continuation for potentials involving $\frac{1}{|z|}$ for $N=1$

Let Ω be a domain in $\mathbb{C}^n$ containing 0. Let $u:\Omega\to\mathbb{C}^N$ be an $H^1_{loc}(\Omega)$ solution to the inequality

$$|\bar\partial u|\le\frac{C}{|z|}|u|\quad\text{a.e. on}\quad\Omega,\tag{5.1}$$

where C is some positive constant. Note that the potential $\frac{C}{|z|}\notin L^{2n}_{loc}(\Omega)$. The goal of this section is to show the unique continuation property for (5.1) if the target dimension $N=1$ as stated below.

Theorem 5.1. *Let Ω be a domain in $\mathbb{C}^n$ and $0\in\Omega$. Let $u:\Omega\to\mathbb{C}$ with $u\in H^1_{loc}(\Omega)$, and satisfies $|\bar\partial u|\le\frac{C}{|z|}|u|$ a.e. on Ω for some constant $C>0$. If u vanishes to infinite order in the L^2 sense at 0, then u vanishes identically.*

To prove Theorem 5.1, we need a few preparation lemmas.

Lemma 5.2. *Let Ω be a domain in $\mathbb{C}^n$ containing 0. Let $u\in L^2_{loc}(\Omega)$ and vanishes to infinite order in the L^2 sense at 0. Then for each $M>0$, $\frac{u}{|z|^M}\in L^2_{loc}(\Omega)$, and vanishes to infinite order in the L^2 sense at 0.*

Proof. For each $m\ge 1$ and $\epsilon>0$, by the L^2 flatness of u at 0, there exists $\delta>0$ such that for $0<r\le\delta$,

$$\int_{|z|<r}|u|^2 dv_z\le\epsilon r^{m+2M}.$$

Then for $0 < r \le \delta$,

$$\begin{aligned}\int_{|z|<r} \frac{|u|^2}{|z|^{2M}} dv_z =& \sum_{j=1}^{\infty} \int_{\frac{r}{2^j}<|z|<\frac{r}{2^{j-1}}} \frac{|u|^2}{|z|^{2M}} dv_z \le \sum_{j=1}^{\infty} \frac{2^{2Mj}}{r^{2M}} \int_{\frac{r}{2^j}<|z|<\frac{r}{2^{j-1}}} |u|^2 dv_z \\ \le& \sum_{j=1}^{\infty} \frac{2^{2Mj}}{r^{2M}} \int_{|z|<\frac{r}{2^{j-1}}} |u|^2 dv_z \le \epsilon \sum_{j=1}^{\infty} \frac{2^{2Mj}}{r^{2M}} \frac{r^{m+2M}}{2^{(m+2M)(j-1)}} \\ =& \epsilon 2^{2M} r^m \sum_{j=1}^{\infty} 2^{-m(j-1)} \le \epsilon 2^{2M+1} r^m.\end{aligned}$$

In particular, $\frac{u}{|z|^M} \in L^2$ near 0, and vanishes to infinite order in the L^2 sense at 0. $\square$

Lemma 5.3. *Let Ω be a domain in $\mathbb{C}^n$ and $0 \in \Omega$. Assume that $u \in L^2_{loc}(\Omega)$ and is holomorphic in $\Omega \setminus \{0\}$. If u vanishes to infinite order in the L^2 sense at 0, then $u \equiv 0$.*

Proof. Write u in terms of the Laurent expansion $u(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha$ near 0 for some constants a_α, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$. Then for each $0 < r << 1$,

$$\int_{|z|<r} |u|^2 dv_z = \int_{|z|<r} \sum_{\alpha,\beta \in \mathbb{Z}^n} a_\alpha \overline{a}_\beta z^\alpha \overline{z}^\beta dv_z = \sum_{\alpha \in \mathbb{Z}^n} |a_\alpha|^2 \int_{|z|<r} |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} dv_z.$$

The L^2 integrability of u near 0 leads to $a_\alpha = 0$ for any $\alpha = (\alpha_1, \dots, \alpha_n)$ with some $\alpha_j < 0$. Thus u is holomorphic on Ω. By Lemma 3.2, we further see $u \equiv 0$. $\square$

Lemma 5.4. *Let Ω be a bounded domain in $\mathbb{C}^n$ and $0 \in \Omega$. For any $\alpha, \beta > 0$ with $\alpha + \beta = 2n$, there exists some constant $C > 0$ such that*

$$\int_\Omega \frac{dv_\zeta}{|\zeta|^\alpha |\zeta - z|^\beta} \le C\left(1 + |\ln|z||\right), \quad z \in \Omega.$$

Proof. Fix $z \in \Omega$ and let $t := |z|$. Let $r_0 > 0$ be large such that $\Omega \subset B_{r_0}$. For all $\zeta \in \Omega \setminus B_{2t}$, $|\zeta - z| \ge |\zeta| - t \ge |\zeta| - \frac{1}{2}|\zeta| = \frac{1}{2}|\zeta|$. Hence

$$\int_{\Omega \setminus B_{2t}} \frac{dv_\zeta}{|\zeta|^\alpha |\zeta - z|^\beta} \le 2^\beta \int_{\Omega \setminus B_{2t}} \frac{dv_\zeta}{|\zeta|^{\alpha+\beta}} \le 2^\beta \left(\int_{2t}^{r_0} \frac{dr}{r}\right) w_{2n-1} \le C_1\left(1 + |\ln t|\right)$$

for some constant $C_1 > 0$ independent of z. On the other hand, writing $z = tz_0$ with $z_0 \in S^{2n-1}$ and applying a change of coordinates $\zeta = t\eta$, we have

$$\begin{aligned}\int_{B_{2t}} \frac{dv_\zeta}{|\zeta|^\alpha |\zeta - z|^\beta} =& \frac{1}{t^{\alpha+\beta-2n}} \int_{B_2} \frac{dv_\eta}{|\eta|^\alpha |\eta - z_0|^\beta} = \int_{B_2} \frac{dv_\eta}{|\eta|^\alpha |\eta - z_0|^\beta} \\ =& \int_{B_{\frac{1}{2}}} \frac{dv_\eta}{|\eta|^\alpha |\eta - z_0|^\beta} + \int_{B_2 \setminus B_{\frac{1}{2}}} \frac{dv_\eta}{|\eta|^\alpha |\eta - z_0|^\beta} \\ \le& 2^\beta \int_{B_{\frac{1}{2}}} \frac{dv_\eta}{|\eta|^\alpha} + 2^\alpha \int_{B_2 \setminus B_{\frac{1}{2}}} \frac{dv_\eta}{|\eta - z_0|^\beta} \le C_2\end{aligned}$$

for some constant $C_2 > 0$ independent of z. Altogether, we get the desired inequality. $\square$

Let us begin with the case when the source dimension $n = 1$. In fact, we shall prove the unique continuation for a larger class of potentials, which take on a hybrid form involving both powers of $\frac{1}{|z|}$ and Lebesgue integrable functions. Note that none of these potentials below belongs to L^2_{loc}.

Theorem 5.5. *Let Ω be a domain in $\mathbb{C}$ containing 0 and $1 < \beta < \infty$. Suppose $u : \Omega \to \mathbb{C}$ with $u \in H^1_{loc}(\Omega)$, and satisfies*

$$|\bar{\partial} u| \le \frac{\tilde{V}}{|z|^{\frac{\beta-1}{\beta}}}|u| \quad a.e. \ on \ \ \Omega$$

for some $\tilde{V} \in L^{2\beta}_{loc}(\Omega)$. If u vanishes to infinite order in the L^2 sense at 0, then u vanishes identically.

Proof. Firstly, $u \in L^p_{loc}(\Omega)$ for all $p < \infty$ by Sobolev embedding theorem. Since $\tilde{V} \in L^{2\beta}_{loc}(\Omega)$ with $2\beta > 2$, we have $\tilde{V}|z|^{\frac{1-\beta}{\beta}} u \in L^{p_0}_{loc}(\Omega \setminus \{0\})$ for some $p_0 > 2$ by Hölder's inequality. Thus by the ellipticity Lemma 3.1, $u \in W^{1,p_0}_{loc}(\Omega \setminus \{0\}) \subset C^0(\Omega \setminus \{0\})$, the space of continuous functions on $\Omega \setminus \{0\}$, as a consequence of Sobolev embedding theorem.

For any subdomain $\tilde{\Omega} \subset\subset \Omega$ containing 0, set $S := \{z \in \tilde{\Omega} \setminus \{0\} : u(z) = 0\}$ and let

$$V := \begin{cases} \frac{\bar{\partial} u}{u}, & \text{on} \ \ (\tilde{\Omega} \setminus \{0\}) \setminus S; \\ 0, & \text{on} \ \ S \cup \{0\}. \end{cases}$$

Then

$$\bar{\partial} u = V u \ \ \text{on} \ \ (\tilde{\Omega} \setminus \{0\}) \setminus S \tag{5.2}$$

in the sense of distributions. Since $|V| \le \tilde{V}|z|^{\frac{1-\beta}{\beta}} \in L^q(\tilde{\Omega})$ for all $1 \le q < 2$, letting

$$f(z) := \frac{1}{\pi} \int_{\Omega} \frac{V(\zeta)}{\zeta - z} dv_\zeta, \quad z \in \tilde{\Omega},$$

one has

$$\bar{\partial} f = V \ \ \text{on} \ \ \Omega$$

in the sense of distributions. See, for instance, [18]. Moreover, by Hölder's inequality and Lemma 5.4 with $n = 1$,

$$|f(z)| \le \frac{1}{\pi} \|\tilde{V}\|_{L^{2\beta}(\tilde{\Omega})} \left(\int_{\tilde{\Omega}} \frac{dv_\zeta}{|\zeta|^{\frac{2\beta-2}{2\beta-1}} |\zeta - z|^{\frac{2\beta}{2\beta-1}}} \right)^{\frac{2\beta-1}{2\beta}} \le M \left(1 + |\ln|z||\right), \quad z \in \tilde{\Omega}$$

for some constant $M > 0$. Hence there exists some $C > 0$ such that

$$\left|e^{-f}\right| \le \frac{C}{|z|^M} \ \ \text{on} \ \ \tilde{\Omega}. \tag{5.3}$$

On the other hand, restricting on $\tilde{\Omega} \setminus \{0\}$, we have $V \in L^{2\beta}_{loc}(\tilde{\Omega} \setminus \{0\})$. Hence $f \in W^{1,2\beta}_{loc}(\tilde{\Omega} \setminus \{0\})$ by Lemma 3.1, and further $f \in C^0(\tilde{\Omega} \setminus \{0\})$. Applying Proposition 2.1 to f on $\tilde{\Omega} \setminus \{0\}$, we get

$$\bar{\partial} e^{-f} = -e^{-f} V \ \ \text{on} \ \ \tilde{\Omega} \setminus \{0\} \tag{5.4}$$

in the sense of distributions.

Let $h := ue^{-f}$ on $\tilde\Omega \setminus \{0\}$. Then $h \in C^0(\tilde\Omega \setminus \{0\})$. Repeating a similar argument as in the proof of Theorem 1.2 to h on $(\tilde\Omega \setminus \{0\}) \setminus S$, and using (5.2) and (5.4), one can further show that $\bar\partial h = 0$ on $(\tilde\Omega \setminus \{0\}) \setminus S$. Noting that $f \in C^0(\tilde\Omega \setminus \{0\})$, we have $S = h^{-1}(0)$ and so $\bar\partial h = 0$ on $(\tilde\Omega \setminus \{0\}) \setminus h^{-1}(0)$. By Rado's theorem, h is holomorphic on $\tilde\Omega \setminus \{0\}$. On the other hand, since u vanishes to infinite order in the L^2 sense at 0, by (5.3) and Lemma 5.2, $h \in L^2_{loc}(\tilde\Omega)$ and vanishes to infinite order in the L^2 sense at 0 as well. As a consequence of Lemma 5.3, $h = 0$ and thus $u = 0$ on $\tilde\Omega$. $\square$

When $\beta = 1$, the unique continuation holds due to Theorem 4.1 part 2). When $\beta = \infty$, by employing exactly the same argument as in the proof of Theorem 5.5, with the index $\frac{\beta-1}{\beta}$ replaced by 1 $(= \lim_{\beta\to\infty} \frac{\beta-1}{\beta})$, and $\tilde V$ by a positive constant C, one can obtain the following unique continuation property. As a result, it resolves Theorem 5.1 for $n = 1$.

Theorem 5.6. *Let Ω be a domain in $\mathbb{C}$ containing 0. Suppose $u : \Omega \to \mathbb{C}$ with $u \in H^1_{loc}(\Omega)$, and satisfies*

$$|\bar\partial u| \le \frac{C}{|z|}|u| \quad a.e.\ on\ \ \Omega$$

for some constant $C > 0$. If u vanishes to infinite order in the L^2 sense at 0, then u vanishes identically.

Next, we address the case where the source dimension $n \ge 1$. We first explore some direct applications of Theorem 5.5 to the unique continuation problem for smooth solutions.

Proof of Corollary 1.6: Assume that (1.4) fails, say for $z_0 = 0$. Then there exists some $V \in L^{2n}_{loc}(U)$ such that $|\bar\partial u| \le V|u|$ on U. When $n = 1$, $u \equiv 0$ on U due to Theorem 4.1 part 2). So we assume $n \ge 2$. Let r be small such that $B_r \subset U$. For each $\zeta \in S^{2n-1}$, let $\tilde V(w) := |w|^{\frac{n-1}{n}} V(w\zeta)$ and $v(w) := u(w\zeta), w \in D_r$. Then v vanishes to infinite order at 0 in the L^2 sense by Lemma 3.2, and satisfies

$$|\bar\partial v(w)| = |\zeta \cdot \bar\partial u(w\zeta)| \le V(w\zeta)|u(w\zeta)| = |w|^{-\frac{n-1}{n}} \tilde V(w)|v(w)|, \quad w \in D_r.$$

Moreover, for a.e. $\zeta \in S^{2n-1}$, $\tilde V \in L^{2n}_{loc}(D_r)$ by Lemma 4.2. Theorem 5.5 with $\beta = n$ applies to give $v = 0$ on D_r for a.e. $\zeta \in S^{2n-1}$. Hence $u = 0$ on B_r. The weak unique continuation property Theorem in 4.1 part 1) further leads to $u \equiv 0$ on U. $\square$

Theorem 5.5 also allows us to recover a similar result as in Theorem 1.1 for smooth solutions without imposing the $\bar\partial$-closedness assumption on the potential.

Corollary 5.7. *Let Ω be a domain in $\mathbb{C}^n$. Suppose $u : \Omega \to \mathbb{C}$ with $u \in C^\infty(\Omega)$, and satisfies $|\bar\partial u| \le V|u|$ a.e. on Ω for some $V \in L^{2n}_{loc}(\Omega)$. If u vanishes to infinite order at some $z_0 \in \Omega$, then u vanishes identically.*

Proof. Let r be small such that $B_r \subset \Omega$. As in the proof of Corollary 1.6, for each fixed $\zeta \in S^{2n-1}$, let $\tilde V(w) := |w|^{\frac{n-1}{n}} V(w\zeta)$ and $v(w) := u(w\zeta), w \in D_r$. Then v vanishes to infinite order at 0 in the L^2 sense, and satisfies

$$|\bar\partial v(w)| \le |w|^{-\frac{n-1}{n}} \tilde V(w)|v(w)|, \quad w \in D_r.$$

For a.e. $\zeta \in S^{2n-1}$, we apply Theorem 5.5 to get $v = 0$ on D_r. Hence $u = 0$ on B_r. The weak unique continuation property further gives $u \equiv 0$. $\square$

We are now in a position to prove Theorem 5.1 for H^1_{loc} solutions. We shall use the following special case of [8, Proposition A] concerning the zero set of solutions to (1.3) with bounded potentials.

Theorem 5.8. *[8] Let Ω be a domain in $\mathbb{C}^n$. Suppose $u : \Omega \to \mathbb{C}$ with $u \in C^0(\Omega)$, and satisfies $|\bar\partial u| \le C|u|$ a.e. on Ω for some constant $C > 0$. Then the zero set $u^{-1}(0)$ of u is a complex analytic variety.*

Proof of Theorem 5.1: The $n = 1$ case was proved in Theorem 5.6. So we assume $n \ge 2$. According to Sobolev embedding theorem, $u \in L^{q^*}_{loc}(\Omega)$ for $q^* = \frac{2n}{n-1}$. The ellipticity Lemma 3.1 of $\bar\partial$ and the inequality $|\bar\partial u| \le \frac{C}{|z|}|u| \in L^{q^*}(\Omega \setminus \{0\})$ further ensures $u \in W^{1,q^*}_{loc}(\Omega \setminus \{0\})$, and thus $u \in L^{q^{**}}$ with $q^{**} = \frac{2n}{n-2}$. Employing a boot-strap argument eventually gives $u \in W^{1,q_0}_{loc}(\Omega \setminus \{0\})$ for some $q_0 > 2n$. In particular, $u \in C^0(\Omega \setminus \{0\})$.

Set $S := \{z \in \Omega \setminus \{0\} : u(z) = 0\}$. Let

$$V := \begin{cases} \frac{\bar\partial u}{u}, & \text{on} \quad (\Omega \setminus \{0\}) \setminus S; \\ 0, & \text{on} \quad S \cup \{0\}. \end{cases}$$

Then

$$\bar\partial u = Vu \quad \text{on} \quad (\Omega \setminus \{0\}) \setminus S \tag{5.5}$$

in the sense of distributions. One can also verify that

$$\bar\partial V = 0 \quad \text{on} \quad (\Omega \setminus \{0\}) \setminus S \tag{5.6}$$

in the sense of distributions. In fact, given a subdomain $U \subset\subset (\Omega \setminus \{0\}) \setminus S$, since $u \in W^{1,q_0}_{loc}(\Omega \setminus \{0\}) \subset C^0(\Omega \setminus \{0\})$, $|u| > c$ on U for some constant $c > 0$. Letting $\{u_j\}_{j=1}^\infty \in C^\infty(U) \to u$ in the $W^{1,q_0}(U)$ norm. In particular, $u_j \to u$ in the $C^0(U)$ norm. Thus by passing to a subsequence, we can assume $|u_j| > \frac{c}{2}$ on U. Let $V_j := \frac{\bar\partial u_j}{u_j}$ on U. Clearly $\bar\partial V_j = 0$ on U. Moreover,

$$\begin{aligned} \|V_j - V\|_{L^1(U)} = \left\| \frac{\bar\partial u_j}{u_j} - \frac{\bar\partial u}{u} \right\|_{L^1(U)} &\le \left\| \frac{1}{u_j}(\bar\partial u_j - \bar\partial u) \right\|_{L^1(U)} + \left\| \left(\frac{1}{u_j} - \frac{1}{u} \right) \bar\partial u \right\|_{L^1(U)} \\ &\le \frac{2}{c} \left\| \bar\partial u_j - \bar\partial u \right\|_{L^1(U)} + \frac{2}{c^2} \|u_j - u\|_{C^0(U)} \left\| \bar\partial u \right\|_{L^1(U)} \to 0 \end{aligned}$$

as $j \to \infty$. In particular, $V_j \to V$ in the sense of distributions. Thus the $\bar\partial$-closedness passes onto V on $(\Omega \setminus \{0\}) \setminus S$.

Since $|V| \le \frac{C}{|z|} \in L^\infty_{loc}(\Omega \setminus \{0\})$, according to Theorem 5.8, S is a complex analytic variety in $\Omega \setminus \{0\}$. If $S = \Omega \setminus \{0\}$, then we are done. Otherwise, S is of complex dimension less than n. Since $n \ge 2$, $V \in L^2_{loc}(\Omega)$. Applying a removable singularity result of Demailly [6, Lemma 6.9] to (5.6),

$$\bar\partial V = 0 \quad \text{on} \quad \Omega \tag{5.7}$$

in the sense of distributions. On the other hand, noting that $u \in L^2_{loc}(\Omega)$ and $Vu \in L^1_{loc}(\Omega)$, we can apply Demailly's result to (5.5) and obtain

$$\bar\partial u = Vu \quad \text{on} \quad \Omega \tag{5.8}$$

in the sense of distributions.

Let f_0, u_0 and v_0 be as in (3.3) with V defined above. Then $\bar\partial f_0 = V$ on B_r in the sense of distributions for some $r > 0$. Restricting on $B_r \setminus \{0\}$, use the ellipticity Lemma 3.1 to further obtain $f_0 \in W^{1,p}_{loc}(B_r \setminus \{0\})$ for all $p < \infty$. Hence we can apply Proposition 2.1 to f_0 on $B_r \setminus \{0\}$ and get

$$\bar\partial e^{-f_0} = -e^{-f_0} V \quad \text{on} \quad B_r \setminus \{0\} \tag{5.9}$$

in the sense of distributions. Let

$$h := u e^{-f_0} \quad \text{on} \quad B_r \setminus \{0\}.$$

Repeating a similar argument as in the proof of Theorem 1.2 to h on $B_r \setminus \{0\}$, and using (5.8) and (5.9), one can further show that

$$\bar\partial h = 0 \quad \text{on} \quad B_r \setminus \{0\}.$$

Namely, h is holomorphic on $B_r \setminus \{0\}$.

On the other hand, by the construction of f_0 in Lemma 3.1 and the fact that $|V| \le \frac{C}{|z|}$ on Ω, we have v_0 to be bounded on $B_{\frac{r}{2}}$. Moreover, apply Lemma 5.4 to u_0 and get

$$|f_0(z)| \le C\left(1 + \int_{B_r} \frac{dv_\zeta}{|\zeta||\zeta - z|^{2n-1}}\right) \le M\left(1 + |\ln|z||\right), \quad z \in B_{\frac{r}{2}}$$

for some constant $M > 0$. Hence there exists some $C_1 > 0$ such that

$$\left|e^{-f_0}\right| \le \frac{C_1}{|z|^M} \quad \text{on} \quad B_{\frac{r}{2}}. \tag{5.10}$$

Since u vanishes to infinite order in the L^2 sense at 0, by (5.10) and Lemma 5.2, $h \in L^2_{loc}(B_r)$ and vanishes to infinite order in the L^2 sense at 0 as well. As a consequence of Lemma 5.3, $h = 0$ on B_r and thus $u (= e^{f_0} h) = 0$ on B_r. Since $\frac{C}{|z|} \in L^\infty_{loc}(\Omega \setminus \{0\})$, applying the weak unique continuation property of $|\bar\partial u| \le V|u|$ on $\Omega \setminus \{0\}$ with L^∞_{loc} potentials, one further gets $u \equiv 0$. ☐

Remark 5.9. *It should be pointed out that, although the statement of [8, Proposition A] does not explicitly mention it, the $\bar\partial$-closedness of V as indicated in (5.7) near S has already been established in its proof towards the analyticity of the zero set S. We opt to utilize the statement directly and subsequently employ Demailly's result to demonstrate it for the convenience of readers.*

6 Unique continuation for potentials involving $\frac{1}{|z|}$ for $N \ge 2$

In this section, we study the unique continuation for a $H^1_{loc}(\Omega)$ solution $u : \Omega \to \mathbb{C}^N$ to the inequality

$$|\bar\partial u| \le \frac{C}{|z|}|u| \quad \text{a.e. on} \quad \Omega,$$

when the target dimension $N \ge 2$. As stated in Theorem 5.1, the unique continuation property holds true when $N = 1$ for any constant multiple $C > 0$ in the potential. However, when $N \ge 2$, this property no longer holds in general if C is large, as indicated by an example below of the first author and Wolff in [15]. See also [1] by Alinhac and Baouendi for an alternative example.

Example 5. *Let $v_0 : \mathbb{C} \to \mathbb{C}$ be the nontrivial smooth scalar function constructed in [15] that vanishes to infinite order at 0 and satisfies $|\triangle v_0| \le \frac{C^\sharp}{|z|}|\nabla v_0|$ on $\mathbb{C}$ for some constant $C^\sharp > 0$. Letting $u_0 := (\partial \Re v_0, \partial \Im v_0)$, then $u_0 : \mathbb{C} \to \mathbb{C}^2$ is smooth, vanishes to infinite order at 0, and satisfies $|\bar\partial u_0| \le \frac{C^\sharp}{2|z|}|u_0|$ on $\mathbb{C}$.*

In spite of Example 5, we shall prove that the unique continuation property still holds if the constant multiple C is small enough.

Theorem 6.1. *Let Ω be a domain in $\mathbb{C}$ and $0 \in \Omega$. Let $u : \Omega \to \mathbb{C}^N$ with $u \in H^1_{loc}(\Omega)$, and satisfy $|\bar\partial u| \le \frac{C}{|z|}|u|$ a.e. on Ω for some positive constant $C < \frac{1}{4}$. If u vanishes to infinite order in the L^2 sense at 0, then u vanishes identically.*

In order to prove Theorem 6.1, we need to establish a Carleman inequality for $\bar\partial$ (and its conjugate ∂), making use of a Fourier analysis method, along with the following lemma.

Lemma 6.2. *Let $f : (-\infty, 0) \to \mathbb{C}^N$ with $f \in C_c^\infty((-\infty, 0))$. Then for any $\lambda, k \in \mathbb{R}$,*

$$\int_{-\infty}^0 e^{-2\lambda t} \left|(\partial_t + k)f(t)\right|^2 dt \ge (\lambda + k)^2 \int_{-\infty}^0 e^{-2\lambda t}|f(t)|^2 dt.$$

Proof. Letting $g(t) := e^{-\lambda t} f(t), t \in (-\infty, 0)$, then its derivative $g_t = e^{-\lambda t}(f_t - \lambda f)$, and further

$$e^{-\lambda t}(\partial_t + k)f = e^{-\lambda t}(f_t + kf) = g_t + (\lambda + k)g = (\partial_t + \lambda + k)g.$$

Consequently,

$$\begin{aligned}\int_{-\infty}^0 e^{-2\lambda t} \left|(\partial_t + k)f(t)\right|^2 dt &= \int_{-\infty}^0 \left|(\partial_t + \lambda + k)g(t)\right|^2 dt \\ &= \int_{-\infty}^0 |g_t(t)|^2 dt + (\lambda + k)^2 \int_{-\infty}^0 |g(t)|^2 dt + 2(\lambda + k) Re \int_{-\infty}^0 \langle g_t(t), g(t)\rangle dt \\ &\ge (\lambda + k)^2 \int_{-\infty}^0 e^{-2\lambda t}|f(t)|^2 dt + 2(\lambda + k) Re \int_{-\infty}^0 \langle g_t(t), g(t)\rangle dt.\end{aligned}$$

Note that since $g \in C_c^\infty((-\infty, 0))$,

$$0 = \int_{-\infty}^0 \frac{d}{dt}(|g|^2) dt = 2Re\left(\int_{-\infty}^0 \langle g_t, g\rangle dt\right).$$

We obtain the desired inequality. $\square$

Proposition 6.3. *For any $u : \mathbb{C} \to \mathbb{C}^N$ with $u \in H^1(\mathbb{C})$ and supported outside a neighborhood of 0, and for any $\lambda \in \mathbb{Z} + \left\{\frac{1}{2}\right\}$,*

$$\int_{\mathbb{C}} \frac{|u(z)|^2}{|z|^{2\lambda+2}} dv_z \le 16 \int_{\mathbb{C}} \frac{|\partial u(z)|^2}{|z|^{2\lambda}} dv_z \tag{6.1}$$

and

$$\int_{\mathbb{C}} \frac{|u(z)|^2}{|z|^{2\lambda+2}} dv_z \le 16 \int_{\mathbb{C}} \frac{|\bar\partial u(z)|^2}{|z|^{2\lambda}} dv_z.$$

Proof. We shall only prove (6.1) in terms of ∂u, as $\bar{\partial} u = \overline{\partial \bar{u}}$. Since the proof involves derivatives on other variables as well, we use u_z instead of ∂u to emphasize its derivative with respect to z.

First, we consider $u \in C_c^\infty(\mathbb{C} \setminus \{0\})$. Since the inequality is scaling-invariant, without loss of generality we assume u is supported inside the unit disc D_1. Let $v(t,\theta) := u(e^{t+i\theta})$, $t \in (-\infty, 0), \theta \in (0, 2\pi)$. Write the Fourier series of v as

$$v(t,\theta) = \sum_{k\in\mathbb{Z}} v_k(t) e^{ik\theta},$$

where

$$v_k(t) := \frac{1}{2\pi} \int_0^{2\pi} v(t,\theta) e^{-ik\theta} d\theta \in C_c^\infty((-\infty, 0)).$$

According to the Parseval's identity,

$$\int_0^{2\pi} |v(t,\theta)|^2 d\theta = 2\pi \sum_{k\in\mathbb{Z}} |v_k(t)|^2. \tag{6.2}$$

Then under the coordinate change $r = e^t$, we have

$$\begin{aligned}\int_{D_1} \frac{|u(z)|^2}{|z|^{2\lambda+2}} dv_z &= \int_0^{2\pi} \int_0^1 r^{-2\lambda-1} |u(r,\theta)|^2 dr d\theta = \int_0^{2\pi} \int_{-\infty}^0 e^{(-2\lambda-1)t} |v(t,\theta)|^2 e^t dt d\theta \\ &= \int_{-\infty}^0 \int_0^{2\pi} e^{-2\lambda t} |v(t,\theta)|^2 d\theta dt.\end{aligned} \tag{6.3}$$

On the other hand, note that for $z = e^t e^{i\theta}$, one has $z\partial_z = \frac{1}{2}(\partial_t - i\partial_\theta)$. Thus $e^{t+i\theta} v_z = z v_z = \frac{1}{2}(\partial_t - i\partial_\theta) v = \frac{1}{2} \sum_{k\in\mathbb{Z}} (\partial_t + k) v_k(t) e^{ik\theta}$ and

$$\int_0^{2\pi} |e^t v_z(t,\theta)|^2 d\theta = \frac{\pi}{2} \sum_{k\in\mathbb{Z}} |(\partial_t + k) v_k(t)|^2.$$

Hence

$$\begin{aligned}\int_{D_1} \frac{|u_z(z)|^2}{|z|^{2\lambda}} dv_z &= \int_0^{2\pi} \int_0^1 r^{-2\lambda+1} |u_z(re^{i\theta})|^2 dr d\theta = \int_0^{2\pi} \int_{-\infty}^0 e^{-2\lambda t + 2t} |v_z(t,\theta)|^2 dt d\theta \\ &= \int_{-\infty}^0 e^{-2\lambda t} \int_0^{2\pi} |e^t v_z(t,\theta)|^2 d\theta dt = \frac{\pi}{2} \sum_{k\in\mathbb{Z}} \int_{-\infty}^0 e^{-2\lambda t} |(\partial_t + k) v_k(t)|^2 dt.\end{aligned}$$

Applying Lemma 6.2 to v_k and making use of the fact that $(\lambda + k)^2 \geq \frac{1}{4}$ whenever $\lambda \in \mathbb{Z} + \left\{\frac{1}{2}\right\}$ and $k \in \mathbb{Z}$,

$$\begin{aligned}\int_{D_1} \frac{|u_z(z)|^2}{|z|^{2\lambda}} dv_z \geq& \frac{\pi}{2} \sum_{k\in\mathbb{Z}} (\lambda + k)^2 \int_{-\infty}^0 e^{-2\lambda t} |v_k(t)|^2 dt \geq \frac{\pi}{8} \sum_{k\in\mathbb{Z}} \int_{-\infty}^0 e^{-2\lambda t} |v_k(t)|^2 dt \\ =& \frac{1}{16} \int_{-\infty}^0 \int_0^{2\pi} e^{-2\lambda t} |v(t,\theta)|^2 d\theta dt.\end{aligned}$$

Here in the last line we also used (6.2). The inequality (6.1) for $u \in C_c^\infty(\mathbb{C} \setminus \{0\})$ is proved by combining the above inequality with (6.3).

For general $u \in H^1(\mathbb{C})$ in the proposition, let $r > 0$ be small such that the support of u is outside D_r. Pick a family $u_j \in C_c^\infty(\mathbb{C} \setminus D_r) \to u$ in $H^1(\mathbb{C})$ norm. Then applying (6.1) to $u_j \in C_c^\infty(\mathbb{C} \setminus \{0\})$, we get

$$\begin{aligned}\left(\int_{\mathbb{C}} \frac{|u(z)|^2}{|z|^{2\lambda+2}} dv_z\right)^{\frac{1}{2}} \leq& \left(\int_{\mathbb{C}\setminus D_r} \frac{|u(z) - u_j(z)|^2}{|z|^{2\lambda+2}} dv_z\right)^{\frac{1}{2}} + \left(\int_{\mathbb{C}} \frac{|u_j(z)|^2}{|z|^{2\lambda+2}} dv_z\right)^{\frac{1}{2}}\\ \leq& r^{-\lambda-1} \left(\int_{\mathbb{C}} |u(z) - u_j(z)|^2 dv_z\right)^{\frac{1}{2}} + 4 \left(\int_{\mathbb{C}} \frac{|(u_j)_z(z)|^2}{|z|^{2\lambda}} dv_z\right)^{\frac{1}{2}}.\end{aligned}$$

Since

$$\begin{aligned}\left(\int_{\mathbb{C}} \frac{|(u_j)_z(z)|^2}{|z|^{2\lambda}} dv_z\right)^{\frac{1}{2}} \leq& \left(\int_{\mathbb{C}\setminus D_r} \frac{|u_z(z) - (u_j)_z(z)|^2}{|z|^{2\lambda}} dv_z\right)^{\frac{1}{2}} + \left(\int_{\mathbb{C}} \frac{|u_z(z)|^2}{|z|^{2\lambda}} dv_z\right)^{\frac{1}{2}}\\ \leq& r^{-\lambda} \left(\int_{\mathbb{C}} |u_z(z) - (u_j)_z(z)|^2 dv_z\right)^{\frac{1}{2}} + \left(\int_{\mathbb{C}} \frac{|u_z(z)|^2}{|z|^{2\lambda}} dv_z\right)^{\frac{1}{2}},\end{aligned}$$

one thus has

$$\left(\int_{\mathbb{C}} \frac{|u(z)|^2}{|z|^{2\lambda+2}} dv_z\right)^{\frac{1}{2}} \leq \left(r^{-\lambda-1} + 4r^{-\lambda}\right) \|u - u_j\|_{H^1(\mathbb{C})} + 4 \left(\int_{\mathbb{C}} \frac{|u_z(z)|^2}{|z|^{2\lambda}} dv_z\right)^{\frac{1}{2}}.$$

Letting $j \to \infty$, we have the desired inequality (6.1) for $u \in H^1(\mathbb{C})$ with support away from 0. $\square$

By employing an induction process along with a similar argument as in the proof to Proposition 6.3, one can further get the following higher order edition.

Corollary 6.4. *Let $k, l \in \mathbb{Z}^+$ with $k \leq l$, and $\lambda \in \mathbb{Z} + \left\{\frac{1}{2}\right\}$. For any $u : \mathbb{C} \to \mathbb{C}^N$ with $u \in H^l_{loc}(\mathbb{C})$ and supported outside a neighborhood of 0, and any 2-tuples $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2)$ with $|\alpha| = k, |\beta| = l$ and $\alpha_j \leq \beta_j, j = 1, 2$, there exists a constant C dependent only on l such that*

$$\int_{\mathbb{C}} \frac{|\partial^{\alpha_1} \bar{\partial}^{\alpha_2} u(z)|^2}{|z|^{2\lambda+2(l-k)}} dv_z \leq C \int_{\mathbb{C}} \frac{|\partial^{\beta_1} \bar{\partial}^{\beta_2} u(z)|^2}{|z|^{2\lambda}} dv_z.$$

Proof of Theorem 6.1: Let $r > 0$ be small such that $D_{2r} \subset\subset \Omega$. Choose $\eta \in C_c^\infty(\mathbb{C})$ with $\eta = 1$ on D_r, $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq \frac{2}{r}$ on $D_{2r} \setminus D_r$, and $\eta = 0$ outside D_{2r}. Let $\psi \in C^\infty(\mathbb{C})$ be such that $\psi = 0$ in D_1, $0 \leq \psi \leq 1$ and $|\nabla \psi| \leq 2$ on $D_2 \setminus D_1$, and $\psi = 1$ outside D_2. For each $k \geq \frac{4}{r}$ (thus $\frac{2}{k} \leq \frac{r}{2}$), let $\psi_k = \psi(k\cdot)$ and $u_k = \psi_k \eta u$. Then $u_k \in H^1(\mathbb{C})$ with support outside $D_{\frac{1}{k}}$.

Since $C < \frac{1}{4}$, one can choose $\epsilon_0 > 0$ with

$$16(1 + 2\epsilon_0)C^2 < 1. \tag{6.4}$$

Making use of the following elementary inequality

$$(a + b + c)^2 \leq (1 + 2\epsilon)a^2 + (2 + \epsilon^{-1})b^2 + (2 + \epsilon^{-1})c^2, \quad \text{for all } \ a, b, c \in \mathbb{R}, \epsilon > 0,$$

together with Proposition 6.3 and the inequality (5.1), we have for each $\lambda \in \mathbb{Z} + \{\frac{1}{2}\}$,

$$\begin{aligned}
\int_{D_{2r}} \frac{|u_k(z)|^2}{|z|^{2\lambda}} dv_z \leq & 16 \int_{D_{2r}} \frac{|\bar\partial u_k(z)|^2}{|z|^{2\lambda-2}} dv_z \\
\leq & 16(1+2\epsilon_0) \int_{D_{2r}} \frac{|\psi_k(z)\eta(z)|^2|\bar\partial u(z)|^2}{|z|^{2\lambda-2}} dv_z + 16(2+\epsilon_0^{-1}) \int_{D_r} \frac{|\bar\partial \psi_k(z)|^2|u(z)|^2}{|z|^{2\lambda-2}} dv_z \\
& + 16(2+\epsilon_0^{-1}) \int_{D_{2r}\setminus D_r} \frac{|\bar\partial \eta(z)|^2|u(z)|^2}{|z|^{2\lambda-2}} dv_z \\
\leq & 16(1+2\epsilon_0)C^2 \int_{D_{2r}} \frac{|u_k(z)|^2}{|z|^{2\lambda}} dv_z + 16(2+\epsilon_0^{-1}) \int_{D_r} \frac{|\bar\partial \psi_k(z)|^2|u(z)|^2}{|z|^{2\lambda-2}} dv_z \\
& + 16(2+\epsilon_0^{-1}) \int_{D_{2r}\setminus D_r} \frac{|\bar\partial \eta(z)|^2|u(z)|^2}{|z|^{2\lambda-2}} dv_z.
\end{aligned}$$

Noting that (6.4) holds, one can subtract $16(1+2\epsilon_0)C^2 \int_{D_{2r}} \frac{|u_k(z)|^2}{|z|^{2\lambda}} dv_z$ from both sides and get

$$\int_{D_{2r}} \frac{|u_k(z)|^2}{|z|^{2\lambda}} dv_z \leq C_0 \left(\int_{D_r} \frac{|\nabla \psi_k(z)|^2|u(z)|^2}{|z|^{2\lambda-2}} dv_z + \int_{D_{2r}\setminus D_r} \frac{|\nabla \eta(z)|^2|u(z)|^2}{|z|^{2\lambda-2}} dv_z \right), \tag{6.5}$$

where

$$C_0 := \frac{16(2+\epsilon_0^{-1})}{1-16(1+2\epsilon_0)C^2} > 0.$$

Next, we show that

$$\lim_{k\to\infty} \int_{D_r} \frac{|\nabla \psi_k(z)|^2|u(z)|^2}{|z|^{2\lambda-2}} dv_z = 0. \tag{6.6}$$

Indeed, since $\nabla\psi_k$ is only supported on $D_{\frac{2}{k}} \setminus D_{\frac{1}{k}}$,

$$\int_{D_r} \frac{|\nabla \psi_k(z)|^2|u(z)|^2}{|z|^{2\lambda-2}} dv_z \leq \int_{\frac{1}{k}<|z|<\frac{2}{k}} \frac{|\nabla \psi_k(z)|^2|u(z)|^2}{|z|^{2\lambda-2}} dv_z \leq k^{2\lambda} \int_{|z|<\frac{2}{k}} |u(z)|^2 dv_z \to 0$$

as $k \to \infty$, as a consequence of the flatness of u at 0 in the L^2 sense.

Letting $k \to \infty$ in (6.5), and making use of (6.6) and Fatou's Lemma, we obtain that

$$\int_{D_{2r}} \frac{|\eta(z)u(z)|^2}{|z|^{2\lambda}} dv_z \leq C_0 \int_{D_{2r}\setminus D_r} \frac{|\nabla \eta(z)|^2|u(z)|^2}{|z|^{2\lambda-2}} dv_z.$$

Since

$$\int_{D_{2r}} \frac{|\eta(z)u(z)|^2}{|z|^{2\lambda}} dv_z \geq \int_{D_{\frac{r}{2}}} \frac{|u(z)|^2}{|z|^{2\lambda}} dv_z \geq \left(\frac{2}{r}\right)^{2\lambda} \int_{D_{\frac{r}{2}}} |u(z)|^2 dv_z$$

and

$$\int_{D_{2r}\setminus D_r} \frac{|\nabla \eta(z)|^2|u(z)|^2}{|z|^{2\lambda-2}} dv_z \leq \frac{1}{r^{2\lambda-2}} \int_{D_{2r}\setminus D_r} |\nabla \eta(z)|^2|u(z)|^2 dv_z,$$

we have

$$\int_{D_{\frac{r}{2}}} |u(z)|^2 dv_z \leq \frac{C_0 r^2}{2^{2\lambda}} \int_{D_{2r}\setminus D_r} |\nabla \eta(z)|^2|u(z)|^2 dv_z.$$

Letting $\lambda \to \infty$, we see $u = 0$ on $D_{\frac{r}{2}}$. Finally, apply the unique continuation property Theorem 4.1 part 1) to get $u \equiv 0$. $\square$

Proof of Theorem 1.5: Let r be small such that $B_r \subset\subset \Omega$. For each fixed $\zeta \in S^{2n-1}$, let $v(w) := u(w\zeta), w \in D_r$. Then v vanishes to infinite order in the L^2 sense at 0 and satisfies

$$|\bar{\partial} v(w)| = |\zeta \cdot \bar{\partial} u(w\zeta)| \leq \frac{C}{|w|}|u(w\zeta)| = \frac{C}{|w|}|v(w)|, \quad w \in D_r.$$

For a.e. $\zeta \in S^{2n-1}$, we apply Theorem 5.1 when $N = 1$, or Theorem 6.1 when $N \geq 2$ and $C < \frac{1}{4}$, to get $v = 0$ on D_r. Hence $u = 0$ on B_r in either case. The weak unique continuation property further applies to give $u \equiv 0$. $\square$

7 Proof of Theorem 1.4

In this section, we prove Theorem 1.4 – the unique continuation property for $|\bar{\partial} u| \leq V|u|$ on $\Omega \subset \mathbb{C}^2$, with the target dimension $N \geq 1$, and $V \in L^4_{loc}$. As already seen in the proof of Theorem 1.3 in Section 4, its proof can be reduced to that of the following theorem on the complex plane.

Theorem 7.1. *Let Ω be a domain in $\mathbb{C}$ and $0 \in \Omega$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in H^1_{loc}(\Omega)$, and satisfies*

$$|\bar{\partial} u| \leq |z|^{-\frac{1}{2}} V|u| \quad \text{a.e. on} \quad \Omega \tag{7.1}$$

for some $V \in L^4_{loc}(\Omega)$. If u vanishes to infinite order in the L^2 sense at 0, then u vanishes identically.

Note that the $N = 1$ case in Theorem 7.1 is a special case that has been proved in Theorem 5.5. On the other hand, since $|z|^{-\frac{1}{2}} V \notin L^2_{loc}(\Omega)$ given a general $V \in L^4_{loc}(\Omega)$, Theorem 4.1 does not apply.

The key element in proving Theorem 7.1 involves an idea in [16] that utilizes the Cauchy integral, coupled with the technique employed in establishing Theorem 6.1. To begin with, let us recall a representation formula for $u \in H^1(\mathbb{C})$ with compact support in terms of the Cauchy kernel:

$$u(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\bar{\partial} u(\zeta)}{z - \zeta} dv_\zeta, \quad \text{a.e. } z \in \mathbb{C}. \tag{7.2}$$

See, for instance, [16, Lemma 3.1]. Denote by $\|f\|_{L^2_V(\Omega)}$ the weighted $L^2(\Omega)$ norm of a function f on $\Omega \subset \mathbb{C}$ with respect to a weight $V > 0$, with

$$\|f\|_{L^2_V(\Omega)} := \left(\int_\Omega |f(z)|^2 V(z) dv_z \right)^{\frac{1}{2}}.$$

It was proved in [16, Theorem 2.2] that, given a positive function $V \in L^2(\mathbb{C})$, the Riesz potential

$$I_1 f = \int_{\mathbb{C}} \frac{f(\zeta)}{|\zeta - \cdot|} dv_\zeta$$

is a bounded operator from $L^2_{V^{-1}}(\mathbb{C})$ to $L^2_V(\mathbb{C})$. More precisely, there exists a universal constant C_0 such that for any $f \in L^2_{V^{-1}}(\mathbb{C})$,

$$\|I_1 f\|_{L^2_V(\mathbb{C})} \leq C_0 \|V\|_{L^2(\mathbb{C})} \|f\|_{L^2_{V^{-1}}(\mathbb{C})}. \tag{7.3}$$

Proof of Theorem 7.1: Fix an $r > 0$ small such that $D_{2r} \subset\subset \Omega$, and

$$\left\| V\chi_{D_r} + \frac{r}{1+|z|^2} \right\|^4_{L^4(\mathbb{C})} \le \frac{\pi^2}{128C_0^2},$$

where C_0 is the universal constant in (7.3), and χ_{D_r} is the characteristic function for D_r. Replacing V by $V\chi_{D_r} + \frac{r}{1+|z|^2}$, we have (7.1) holds on D_{2r} with $V \in L^4(\mathbb{C})$,

$$V > 0 \quad \text{on} \quad \mathbb{C}; \quad V \ge C_r \quad \text{on} \quad D_{2r} \tag{7.4}$$

for some $C_r > 0$ dependent only on r, and

$$\|V\|^4_{L^4(\mathbb{C})} \le \frac{\pi^2}{128C_0^2}. \tag{7.5}$$

We shall show that $u = 0$ on $D_{\frac{r}{2}}$.

Let η and ψ_k be as defined in the proof of Theorem 6.1. Then $u_k := \psi_k \eta u \in H^1(\mathbb{C})$ and is supported inside $D_{2r} \setminus D_{\frac{1}{k}}$. So is $\frac{u_k}{z^m}$ for each $m \in \mathbb{Z}^+$. Applying (7.2) to $\frac{u_k}{z^m}$, we obtain

$$\frac{|u_k(z)|^2}{|z|^{2m}} = \frac{1}{\pi^2}\left|\int_{\mathbb{C}} \frac{\bar\partial u_k(\zeta)}{(z-\zeta)\zeta^m} dv_\zeta\right|^2, \quad z \in D_{2r},$$

and with $\tilde V := V^2$, one has

$$\int_{D_{2r}} \frac{|u_k(z)|^2}{|z|^{2m}} \tilde V(z) dv_z \le \frac{1}{\pi^2}\int_{D_{2r}} \left(\int_{\mathbb{C}} \frac{1}{|z-\zeta|}\frac{|\bar\partial u_k(\zeta)|}{|\zeta|^m} dv_\zeta\right)^2 \tilde V(z) dv_z \le \frac{1}{\pi^2}\left\| I_1\left(\frac{|\bar\partial u_k|}{|\cdot|^m}\right)\right\|^2_{L^2_{\tilde V}(\mathbb{C})}.$$

Make use of (7.3) with respect to the weight $\tilde V$ to further infer

$$\begin{aligned}
&\int_{D_{2r}} \frac{|u_k(z)|^2}{|z|^{2m}} \tilde V(z) dv_z \\
\le & \frac{C_0^2}{\pi^2}\|\tilde V\|^2_{L^2(\mathbb{C})} \int_{\mathbb{C}} \frac{|\bar\partial\left(\psi_k(z)\eta(z)u(z)\right)|^2}{|z|^{2m}\tilde V(z)} dv_z \\
\le & \frac{2C_0^2}{\pi^2}\|V\|^4_{L^4(\mathbb{C})}\left(\int_{D_{2r}} \frac{|\bar\partial\psi_k(z)|^2|u(z)|^2}{|z|^{2m}\tilde V(z)} dv_z + \int_{D_r} \frac{|\psi_k(z)|^2|\bar\partial u(z)|^2}{|z|^{2m}\tilde V(z)} dv_z + \int_{D_{2r}\setminus D_r} \frac{|\bar\partial\left(\eta(z)u(z)\right)|^2}{|z|^{2m}\tilde V(z)} dv_z\right) \\
= : & A + B + C.
\end{aligned} \tag{7.6}$$

Note that for B, by the inequalities (7.1) and (7.5)

$$B \le \frac{2C_0^2}{\pi^2}\|V\|^4_{L^4(\mathbb{C})}\int_{D_r} \frac{|\psi_k(z)|^2|u(z)|^2}{|z|^{2m+1}} dv_z \le \frac{1}{64}\int_{D_{2r}} \frac{|u_k(z)|^2}{|z|^{2m+1}} dv_z.$$

Thus we apply Theorem 6.3 with $\lambda = m - \frac{1}{2}$ to have

$$\begin{aligned}
B \le & \frac{1}{4}\int_{D_{2r}} \frac{|\bar\partial u_k(z)|^2}{|z|^{2m-1}} dv_z \\
\le & \frac{1}{2}\int_{D_{2r}} \frac{|\psi_k(z)\eta(z)|^2|\bar\partial u(z)|^2}{|z|^{2m-1}} dv_z + \frac{1}{2}\int_{D_r} \frac{|\bar\partial\psi_k(z)|^2|u(z)|^2}{|z|^{2m-1}} dv_z + \frac{1}{2}\int_{D_{2r}\setminus D_r} \frac{|\bar\partial\eta(z)|^2|u(z)|^2}{|z|^{2m-1}} dv_z \\
\le & \frac{1}{2}\int_{D_{2r}} \frac{|u_k(z)|^2}{|z|^{2m}} \tilde V(z) dv_z + \frac{1}{2}\int_{D_r} \frac{|\bar\partial\psi_k(z)|^2|u(z)|^2}{|z|^{2m-1}} dv_z + \frac{1}{2}\int_{D_{2r}\setminus D_r} \frac{|\bar\partial\eta(z)|^2|u(z)|^2}{|z|^{2m-1}} dv_z \\
= : & E + F + G.
\end{aligned}$$

Here we used (7.1) in the third inequality. Combining (7.6) with the above,

$$\int_{D_{2r}} \frac{|u_k(z)|^2}{|z|^{2m}} \tilde{V}(z) dv_z \le A + C + E + F + G.$$

One further subtracts E from both sides to get

$$\int_{D_{2r}} \frac{|u_k(z)|^2}{|z|^{2m}} \tilde{V}(z) dv_z \le 2A + 2C + 2F + 2G. \tag{7.7}$$

Similarly as in the proof to (6.6) along with the fact that $\tilde{V} \ge C_r^2$ on D_{2r}, one has

$$\lim_{k\to\infty} A = \lim_{k\to\infty} F = 0.$$

Together, after passing $k \to \infty$ and using Fatou's Lemma in (7.7), we obtain

$$\int_{D_{2r}} \frac{|\eta(z)|^2|u(z)|^2}{|z|^{2m}} \tilde{V}(z) dv_z \le \int_{D_{2r}\setminus D_r} \frac{|\bar{\partial}\left(\eta(z)u(z)\right)|^2}{|z|^{2m}\tilde{V}(z)} dv_z + \int_{D_{2r}\setminus D_r} \frac{|\bar{\partial}\eta(z)|^2|u(z)|^2}{|z|^{2m-1}} dv_z. \tag{7.8}$$

Now multiply two sides of (7.8) by r^{2m}. On the left hand side,

$$\int_{D_{2r}} \frac{r^{2m}|\eta(z)|^2|u(z)|^2}{|z|^{2m}} \tilde{V}(z) dv_z \ge \int_{D_{\frac{r}{2}}} \frac{r^{2m}}{|z|^{2m}} |u(z)|^2 \tilde{V}(z) dv_z \ge 2^{2m} \int_{D_{\frac{r}{2}}} |u(z)|^2 \tilde{V}(z) dv_z.$$

On the right hand side, using the fact that $\tilde{V} \ge C_r^2$ on D_{2r} again,

$$\begin{aligned}&\int_{D_{2r}\setminus D_r} \frac{r^{2m}|\bar{\partial}\left(\eta(z)u(z)\right)|^2}{|z|^{2m}\tilde{V}(z)} dv_z + \int_{D_{2r}\setminus D_r} \frac{r^{2m}|\bar{\partial}\eta(z)|^2|u(z)|^2}{|z|^{2m-1}} dv_z \\ \le& \frac{1}{C_r^2} \int_{D_{2r}\setminus D_r} |\bar{\partial}\left(\eta(z)u(z)\right)|^2 dv_z + r \int_{D_{2r}\setminus D_r} |\nabla\eta(z)|^2|u(z)|^2 dv_z \le \tilde{C}_r \|u\|^2_{H^1(D_{2r})},\end{aligned}$$

for some $\tilde{C}_r$ dependent only on r. Thus

$$2^{2m} \int_{D_{\frac{r}{2}}} |u(z)|^2 \tilde{V}(z) dv_z \le \tilde{C}_r \|u\|^2_{H^1(D_{2r})}.$$

Letting $m \to \infty$ and making use of the positivity of $\tilde{V}$ on $D_{\frac{r}{2}}$, we have $u = 0$ on $D_{\frac{r}{2}}$. The proof is thus complete as a consequence of the weak unique continuation property. □

Proof of Theorem 1.4: As in the proof to Theorem 1.3 yet with $n = 2$ and $p = 4$, let $z_0 = 0$ and $r > 0$ be small such that $V \in L^4(B_r)$. For each fixed $\zeta \in S^3$, let $\tilde{V}(w) := |w|^{\frac{1}{2}} V(w\zeta)$ and $v(w) := u(w\zeta), w \in D_r$. Then v vanishes to infinite order at 0 in the L^2 sense. Moreover, v satisfies

$$|\bar{\partial} v(w)| \le |w|^{-\frac{1}{2}} \tilde{V}(w)|v(w)|, \quad w \in D_r.$$

Note that for a.e. $\zeta \in S^3$, $\tilde{V} \in L^4(D_r)$ by Lemma 4.2. According to Theorem 7.1, $v = 0$ on D_r for a.e. $\zeta \in S^3$. Hence $u = 0$ on B_r. Apply the weak unique continuation property to get $u \equiv 0$. □

Remark 7.2. *The following two questions still remain open. In particular, with an approach similar as in the proof to Theorem 1.4, the resolution of Question 1 can be converted to that of Question 2.*

1. *Let Ω be a domain in $\mathbb{C}^n, n \geq 3$ and $N \geq 2$. Suppose $u : \Omega \to \mathbb{C}^N$ is smooth on Ω and satisfies $|\bar{\partial} u| \leq V|u|$ a.e. on Ω for some $V \in L^{2n}_{loc}(\Omega)$. If u vanishes to infinite order at some $z_0 \in \Omega$, does u vanish identically?*

2. *Let Ω be a domain in $\mathbb{C}$ containing 0, and $n, N \in \mathbb{Z}^+$ with $n \geq 3, N \geq 2$. Suppose $u : \Omega \to \mathbb{C}^N$ is smooth on Ω and satisfies $|\bar{\partial} u| \leq |z|^{-\frac{n-1}{n}} V|u|$ a.e. on Ω for some $V \in L^{2n}_{loc}(\Omega)$. If u vanishes to infinite order at $0 \in \Omega$, does u vanish identically?*

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