CR SINGULAR IMAGES OF GENERIC SUBMANIFOLDS UNDER HOLOMORPHIC MAPS

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Abstract. The purpose of this paper is to organize some results on the local geometry of CR singular real-analytic manifolds that are images of CR manifolds via a CR map that is a diffeomorphism onto its image. We find a necessary (sufficient in dimension 2) condition for the diffeomorphism to extend to a finite holomorphic map. The multiplicity of this map is a biholomorphic invariant that is precisely the Moser invariant of the image when it is a Bishop surface with vanishing Bishop invariant. In higher dimensions, we study Levi-flat CR singular images and we prove that the set of CR singular points must be large, and in the case of codimension 2, necessarily Levi-flat or complex. We also show that there exist real-analytic CR functions on such images that satisfy the tangential CR conditions at the singular points, yet fail to extend to holomorphic functions in a neighborhood. We provide many examples to illustrate the phenomena that arise.

1. Introduction

Let M be a smooth real submanifold in $\mathbb{C}^n$, $n \geq 2$. Given $p \in M$, let $T_p^{0,1}M$ denote the CR tangent space to M at p, i.e., the subspace of antiholomorphic vectors in $\mathbb{C}T_p\mathbb{C}^n$ that are also tangent to M. M is called a *CR submanifold* when the function $\phi(p) = \dim_{\mathbb{C}} T_p^{0,1}M$ is constant. In this paper, we focus on submanifolds for which ϕ has jump discontinuities. We call such an M a *CR singular submanifold* and call those points where ϕ is discontinuous *CR singular points* of M. A CR singular submanifold is necessarily of real codimension at least 2. A two-dimensional CR singular submanifold in $\mathbb{C}^2$ already has a rich structure and the biholomorphic equivalence problem in this situation has been extensively studied by many authors, for example [4, 8, 12, 13, 17, 18]. If M is real-analytic,then a local real-analytic parametrization from $\mathbb{R}^2$ onto M gives rise to a holomorphic map from $\mathbb{C}^2$ to $\mathbb{C}^2$ whose restriction to $\mathbb{R}^2$ is a diffeomorphism onto M. This observation motivates us to consider the general situation in which a CR singular submanifold of $\mathbb{C}^n$ is diffeomorphic to a generic submanifold of $\mathbb{C}^n$ of the same codimension via a CR map.

To be specific, let $N \subset \mathbb{C}^n$ be a generic real-analytic CR manifold. That is, N is a locally minimally embedded CR submanifold. Let $f \colon N \to \mathbb{C}^n$ be a real-analytic CR map such that f is a diffeomorphism onto its image $M = f(N)$, which is CR singular at some point $p \in M$, and suppose that M is generic at some point. We call such an M a *CR singular image*. Since N is real-analytic, f extends to a holomorphic map F from a neighborhood of N in $\mathbb{C}^n$ into a neighborhood of M in $\mathbb{C}^n$. If the map F does not have constant rank at a point, the image of that point is a CR singular point of M (see Lemma 4.2). This observation was

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also made in [7], where the images of CR submanifolds under finite holomorphic maps were studied. We consider the following questions regarding a CR singular image M:

(i) What can be said about the holomorphic extension F of the map f? In particular, when is F a finite map?
(ii) What is the structure of the set of CR singular points of M?
(iii) Does every real-analytic CR function (appropriately defined) on M extend to a neighborhood of M in $\mathbb{C}^n$?

In Section 3, we first consider a real-analytic submafold M of real dimension n in $\mathbb{C}^n$ with one-dimensional complex tangent at a point. Any such M is an image of a totally real N under a CR map that is a diffeomorphism onto M. In special coordinates for M we can write the map f in a rather explicit form. We then provide a necessary and sufficient condition for f to extend to a *finite* holomorphic map F, in terms of the defining equation of M. In this case, the multiplicity of F is a biholomorphic invariant. In two dimensions, this invariant is closely related to the Moser invariant (see [17]) of M when it is a Bishop surface with elliptic complex tangent, or the lowest order invariants studied in [10] by Harris. Furthermore, by invoking a theorem of Moser [17], we can show that the only Bishop surface in $\mathbb{C}^2$ that cannot be realized as an image of $\mathbb{R}^2 \subset \mathbb{C}^2$ via a finite holomorphic map is the surface $M_0 := \{w = |z|^2\}$ (Theorem 3.3). When $\dim_{\mathbb{R}} M > n$, we are able to provide an explicit example of a CR singular manifold M which is not an image of a generic submanifold of the same codimension (see Example 5.5). However, when M is a singular image, we prove that an analogous condition is necessary for the extended map to be finite, and that the multiplicity remains a biholomorphic invariant (see Section 2).

We then study a CR singular image M that contains complex subvarieties of positive dimension. The main result along these lines (Theorem 4.1) shows that if M is a CR singular image with a CR singular set S, and M contains a complex subvariety L of complex dimension j that intersects S, then $S \cap L$ is a complex subvariety of (complex) dimension j or $j-1$. Furthermore, if M contains a continuous family of complex varieties L_t of dimension j and $L_0 \cap S$ is of dimension $j-1$, then $L_t \cap S$ is nonempty for all t near 0.

We apply these ideas in Theorem 5.1 to characterize the CR singular set of a Levi-flat CR singular image. We construct several examples illustrating the different possibilities for the CR structure of the set of CR singular points. A corollary to our theorem shows that a Levi-flat CR singular image necessarily has a CR singular set of large dimension, depending on the generic CR dimension of the singular image. When the codimension is 2, we obtain that the CR singular set is necessarily Levi-flat or complex. One of the primary motivations for studying CR singular Levi-flat manifolds is to understand the singularities of non-smooth Levi-flat varieties in general. For example, it has been proved by the first author [16] that the singular locus of a singular Levi-flat hypersurface is Levi-flat or complex. The next step in this program would be to find a Levi-flat stratification, for which we need to understand the CR singular set that may arise in higher codimension flat submanifolds.

We next attempt to find a convenient set of coordinates for a nowhere minimal CR singular image M along the lines of the standard result, Theorem 6.1, for CR manifolds. In particular, a generic submanifold N has coordinates in which some of the equations are of the form $\operatorname{Im} w' = 0$, where $\{w' = s\} \cap N$ give the CR orbits of N. The theorem does not generalize directly, but when M is a CR singular image under a finite holomorphic map, we obtain a partial analogue, namely that M is contained in the intersection of singular Levi-flat hypersurfaces.

In Section 7, we consider the extension of real-analytic CR functions defined on a CR singular image M. When M is generic at every point, then all real-analytic CR functions on M extend to holomorphic functions on a neighborhood of M in $\mathbb{C}^n$. In contrast, we show that if M is a CR singular image, then there exists a real-analytic function satisfying all tangential CR conditions, yet *fails* to extend to a holomorphic function on a neighborhood of M. This result is closely related to an earlier result [9] in which the author provided a necessary and sufficient condition for a CR function on the *generic part* of M to extend past a CR singular point.

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2. Preliminaries

In this section we will recall some basic notations and results that will be needed in the rest of the paper. We refer to [3] for more details. Let M be a smooth real submanifold of real codimension d in $\mathbb{C}^n$. Then at every point $p \in M$, there is a neighborhood U of p and d real-valued smooth functions $r_1, \dots, r_d$ defined on U such that

$$M \cap U = \{z \in \mathbb{C}^n : r_k(z, \bar{z}) = 0,\ k = 1, 2, \dots, d\}, \tag{1}$$

where $dr_1 \wedge dr_2 \wedge \dots \wedge dr_d$ does not vanish on U. For any point $p \in M$, we denote by $T_p^{0,1}M$ the subspace in $T_p^{0,1}\mathbb{C}^n$ that annihilates r_k for $k = 1, 2, \dots, d$. Thus, we see that

$$\dim_{\mathbb{C}} T_p^{0,1}M = n - \operatorname{rank}_{\mathbb{C}} \left(\frac{\partial r_j}{\partial \bar{z}_k}(p, \bar{p}) \right)_{j,k}. \tag{2}$$

If $\dim_{\mathbb{C}} T_q^{0,1}M$ is constant for all q near p and equals $n - d$, then we say that M is *generic* at p. In this paper, we shall assume that M is connected and generic at some point and thus the matrix $\left(\frac{\partial r_j}{\partial \bar{z}_k}\right)_{j,k}$ is of generic full rank. If further we denote by S the set of CR singular points of M, then

$$S = \left\{ z \in M : \operatorname{rank}_{\mathbb{C}} \left(\frac{\partial r_j}{\partial \bar{z}_k} \right)_{j,k} \leq d - 1 \right\}. \tag{3}$$

The set of CR singular points S is a *proper* subvariety of M and $M \setminus S$ is generic at all points.

Remark 2.1. We note that when $M \subset \mathbb{C}^n$ is a CR singular submanifold such that there exists a subbundle $\mathcal{V} \subset \mathbb{C} \otimes TM$ such that $\mathcal{V}_q = T_q^{0,1}M \subset T_q^{0,1}\mathbb{C}^n$ for all $q \in M \setminus S$, then $(M, \mathcal{V})$ becomes an abstract real-analytic CR manifold. Hence, $(M, \mathcal{V})$ is locally integrable (see [3, Theorem 2.1.11]). Therefore, for every $p \in M$ we obtain a generic $N \subset \mathbb{C}^n$ and a real-analytic CR map $f\colon N \to \mathbb{C}^n$ such that f is a diffeomorphism onto an open neighbourhood of p in M. We shall call such pair (N, f) (or simply N) a resolution of CR singularity of M near p. The converse is also true; if there exists a resolution of CR singularity of M near p, then the CR bundle on $M \setminus S$ extends to a subbundle of $\mathbb{C} \otimes_{\mathbb{R}} TM$ in a neighborhood of p on M. The resolution of CR singularity N, if exists, is unique, modulo a biholomorphic equivalence (Proposition 2.3).

Let $N \subset \mathbb{C}^n$ be a real-analytic generic submanifold and $f\colon N \to \mathbb{C}^n$ a real-analytic CR map that is a diffeomorphism onto its image $M = f(N)$. By the real-analyticity, the map f extends to a holomorphic map F in a neighborhood of N in $\mathbb{C}^n$. One of the questions we are interested in is whether F a finite holomorphic map. For $p \in \mathbb{C}^n$ we denote by $\mathcal{O}_p$ the ring of germs of holomorphic functions at p.

Definition 2.2. A germ of a holomorphic map $F = (F_1, \ldots, F_n)$ defined in a neighborhood of $p \in \mathbb{C}^n$ is said to be finite at p if the ideal $\mathcal{I}(F)$ generated by $F_1, \ldots, F_n$ in $\mathcal{O}_p$ is of finite codimension, that is, if $\dim_{\mathbb{C}} \mathcal{O}_p/\mathcal{I}(F)$ is finite. This number, denoted by $\operatorname{mult}_p(F)$, is called the multiplicity of F at p.

Equivalently, a holomorphic map defined as above is finite if and only if the germ of the complex analytic variety $F^{-1}\big(F(p)\big)$ is an isolated point (cf. [1]). In this case, for any q close enough to $F(p)$, the number of preimages $\#F^{-1}(q)$ is finite and always less than or equal to $\operatorname{mult}_p(F)$. The equality holds for generic points in a neighborhood of $F(p)$.

Proposition 2.3. *Let $M, \widetilde{M} \subset \mathbb{C}^n$ be connected CR singular real-analytic submanifolds that are generic at some point and φ a biholomorphic map of a neighbourhood of M to a neighbourhood of $\widetilde{M}$ such that $\varphi(M) = \widetilde{M}$. Let $N, \widetilde{N} \subset \mathbb{C}^n$ be generic real-analytic submanifolds, F be a holomorphic map from a neighborhood of N to a neighborhood of M, and $\widetilde{F}$ be holomorphic map from a neighborhood of $\widetilde{N}$ to a neighborhood of $\widetilde{M}$, such that $F|_N$ and $\widetilde{F}|_{\widetilde{N}}$ and are diffeomorphisms onto M and $\widetilde{M}$ respectively. Then N and $\widetilde{N}$ are biholomorphically equivalent.*

Furthermore, for any point $p \in M$, $\operatorname{mult}_p(F)$ is a local biholomorphic invariant of M (i.e., does not depend on N and F).

Proof. Write $f = F|_N$ and $\widetilde{f} = \widetilde{F}|_{\widetilde{N}}$. We have the following commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\ f\ } & M \\ {\scriptstyle \widetilde{f}^{-1}\circ\varphi\circ f}\big\downarrow & & \big\downarrow{\scriptstyle \varphi|_M} \\ \widetilde{N} & \xrightarrow{\ \widetilde{f}\ } & \widetilde{M}. \end{array} \tag{4}$$

Let S be the set of CR singular points of M, and hence $\varphi(S)$ the set of CR singular points of $\widetilde{M}$. As $\widetilde{M}$ is generic outside $\varphi(S)$ and the Jacobian of $\widetilde{F}$ does not vanish on $\widetilde{M} \setminus \varphi(S)$, then $\widetilde{f}^{-1}$ is a CR map on $\widetilde{M} \setminus \varphi(S)$. The map $\widetilde{f}^{-1} \circ \varphi \circ f$ is a diffeomorphism that is a CR map outside of $f^{-1}(S)$, which is nowhere dense in N. It is, therefore, a real-analytic CR diffeomorphism of the generic submanifolds N and $\widetilde{N}$, and so it extends to a biholomorphism of a neighbourhood. By uniqueness of the extension of CR maps from generic submanifolds, the diagram still commutes after we extend. Hence, the extensions F and $\widetilde{F}$ have the same multiplicity. $\square$

If M is a real-analytic submanifold of codimension $d = 2$ in $\mathbb{C}^n$, then $\dim_{\mathbb{C}} T^{0,1}_p M = n - 1$ at a CR singular point $p \in M$. Thus we can find a linear change of coordinates such that the new coordinates $Z = (z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ vanishes at p and M is given by one complex equation:

$$w = \rho(z, \bar{z}), \tag{5}$$

where ρ vanishes to order at least 2 at 0.

Proposition 2.4. *Suppose $M \subset \mathbb{C}^n$ is a CR singular real-analytic submanifold of codimension 2 near zero, defined by $w = \rho(z, \bar{z})$, where ρ vanishes to order at least 2 at 0. Suppose that N is a real-analytic generic submanifold of $\mathbb{C}^n$ and F a holomorphic map of a neighborhood of N into $\mathbb{C}^n$ such that $F|_N : N \to \mathbb{C}^n$ is a diffeomorphism onto M. If $\rho(0, \bar{z}) \equiv 0$, then F cannot be a finite map.*

Proof. Write $F = (F', F_n)$. Let N be given by normal coordinates (see [3, Proposition 4.2.12])

$$\omega = r(\zeta, \bar{\zeta}, \bar{\omega}), \tag{6}$$

where $(\zeta, \omega) \in \mathbb{C}^{n-2} \times \mathbb{C}^2$, and r is a $\mathbb{C}^2$-valued holomorphic function satisfying $r(\zeta, 0, \bar{\omega}) = r(0, \bar{\zeta}, \bar{\omega}) = \bar{\omega}$. Then for $(\zeta, \omega) \in N$ we have

$$F_n(\zeta, \omega) = \rho\big(F'(\zeta, \omega), \bar{F}'(\bar{\zeta}, \bar{\omega})\big). \tag{7}$$

By plugging in the defining equation of N as $\bar{\omega} = \bar{r}(\bar{\zeta}, \zeta, \omega)$, we get

$$F_n(\zeta, \omega) = \rho\big(F'(\zeta, \omega), \bar{F}'(\bar{\zeta}, \bar{r}(\bar{\zeta}, \zeta, \omega))\big) \tag{8}$$

holds near 0. In particular (8) holds when $\bar{\zeta} = 0$. Using the fact that in normal coordinates $\bar{r}(0, \bar{\zeta}, \omega) \equiv \omega$ we get

$$F_n(\zeta, \omega) = \rho\big(F'(\zeta, \omega), \bar{F}'(0, \bar{r}(0, \zeta, \omega))\big) = \rho\big(F'(\zeta, \omega), \bar{F}'(0, \omega)\big). \tag{9}$$

So F cannot be finite if $\rho(0, \bar{z}) \equiv 0$, as in that case F_n is in the ideal generated by components of F'. ☐

Remark 2.5. We conclude this section by a remark that the existence of a finite holomorphic map F and a generic submanifold N such that F restricted to N is a diffeomorphism onto $M = F(N)$ implies that the set of CR singular points on M is contained in a proper complex subvariety of $\mathbb{C}^n$. This fact follows from Lemma 4.2, which shows that the inverse image of the CR singular points is contained in the set where the Jacobian of F vanishes, and from Remmert proper map theorem.

3. Images under finite maps of totally real submanifolds

Let M be a real-analytic submanifold of real dimension n in $\mathbb{C}^n$. Assume that M is totally real at a generic point and has CR singularities along $S \subset M$. Suppose that $p \in S$ and $\dim_{\mathbb{C}} T_p^{0,1} M = 1$. Then, after a change of coordinates, we can assume that $p = 0$ and M can be defined by the following equations

$$\begin{aligned} z_n &= \rho(z_1, \bar{z}_1, x'), \\ y_\alpha &= r_\alpha(z_1, \bar{z}_1, x'), \quad \alpha = 2, \dots, n-1. \end{aligned} \tag{10}$$

Here, $z = (z_1, \dots, z_n)$ are the coordinates in $\mathbb{C}^n$, $z' = (z_2, \dots, z_{n-1})$ (when $n = 2$, z' is omitted) and $z_j = x_j + iy_j$. The functions ρ and r_α have no linear terms, and r_α are real-valued. By making another change of coordinates, we can eliminate the harmonic terms in r_α to obtain

$$r_\alpha(0, \bar{z}, 0) = 0. \tag{11}$$

The case when p is a nondegenerate CR singularity of M (i.e., ρ vanishes to order exactly 2 at 0) has been studied from a different view point (see, e.g., [4, 11, 14]). Here we make no such assumption.

Proposition 3.1. *Let $M \subset \mathbb{C}^n$ be a real-analytic submanifold of real dimension n, with a complex tangent at 0, defined by equations (10). Then the following are equivalent*

(i) There is a totally real submanifold N of dimension n in $\mathbb{C}^n$ and a germ of a finite *holomorphic map $F\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ such that $F|_N$ is diffeomorphic onto $M = F(N)$ (as germs at 0).*

(ii) $\rho(0,\bar{z}_1,0) \not\equiv 0$.

Furthermore, if (ii) *holds, then*

$$\rho(0,\bar{z}_1,0) = c\bar{z}_1^k + O(\bar{z}_1^{k+1}), \quad c \neq 0, \tag{12}$$

with $k = \operatorname{mult}_0(F)$.

Proof. We can assume that $N = \mathbb{R}^n \subset \mathbb{C}^n$. Consider the map $f = (f_1, f', f_n)\colon \mathbb{R}^n \to \mathbb{C}^n$ given by

$$f_1(t_1,t',t_n) = t_1 + it_n, \tag{13}$$

$$f_\alpha(t_1,t',t_n) = t_\alpha + ir_\alpha(t_1+it_n, t_1-it_n, t'), \tag{14}$$

$$f_n(t_1,t',t_n) = \rho(t_1+it_n, t_1-it_n, t'). \tag{15}$$

Then clearly, f is a diffeomorphism from a neighborhood of 0 in $\mathbb{R}^n$ onto M. Furthermore, f extends to a holomorphic map F from a neighborhood of 0 in $\mathbb{C}^n$. Thus, by abuse of notation, we also denote the coordinates in $\mathbb{C}^n$ by t. Let V_F be the germ of $F^{-1}(0)$ at 0.

(i) $\Rightarrow$ (ii): Assume that $\rho(0,\bar{z},0) \equiv 0$. By making use of the fact that $r_\alpha(0,\bar{z},0) \equiv 0$, one can check that

$$\{(t_1,t',t_n) \in \mathbb{C}^n : t_1 = -it_n, t' = 0\} \subset V_F. \tag{16}$$

Thus V_F has positive dimension, and so F is not finite at 0.

(ii) $\Rightarrow$ (i): Assume that (ii) holds. Let $(t_1,t',t_n) \in V_F$. From (13) we have $t_1 = -i\,t_n$. Substitute into (14) and (15) we get

$$t_\alpha + i\,r_\alpha(0,2t_1,t') = 0 \tag{17}$$

$$\rho(0,2t_1,t') = 0. \tag{18}$$

As r_α has no linear terms, by implicit function theorem, we can see that (17) has unique solution. Furthermore, since r_α has no harmonic terms, the unique solution must be $t' = 0$.

Substituting $t' = 0$ into (18), we get

$$\rho\big(0,2t_1,0\big) = 0. \tag{19}$$

Using (12) we get

$$2^k c\, t_1^k + O(t^{k+1}) = 0. \tag{20}$$

We then deduce that $t_1 = t_n = 0$. Hence, $V_F = \{0\}$ is isolated and thus F is finite.

Finally, let $\tilde{z} = (\tilde{z}_1, \tilde{z}', \tilde{z}_n')$ be a point close enough to 0. We will show that for generic q, in a small neighborhood of 0 there are k solutions to the equation $F(t) = \tilde{z}$. Indeed, if $F(t) = \tilde{z}$ then

$$t_1 + it_n = \tilde{z}_1, \tag{21}$$

$$t_\alpha + i\,r_\alpha(t_1+it_n, t_1-it_n, t') = \tilde{z}_\alpha, \tag{22}$$

$$\rho(t_1+it_n, t_1-it_n, t') = \tilde{z}_n. \tag{23}$$

From (21) we have

$$t_1 = -i\, t_n + \tilde{z}_1. \tag{24}$$

Substitute (24) into (22) we get

$$t_\alpha + i\, r_\alpha(\tilde{z}_1, 2t_1 - \tilde{z}_1, t') = \tilde{z}_\alpha. \tag{25}$$

For $\tilde{z}$ close enough to 0, the implicit function theorem applied to (25) gives unique solution $t' = \varphi(t_1, \tilde{z}_1)$. Substitute this into (23) we get

$$\rho(\tilde{z}_1, 2t_1 - \tilde{z}_1, \varphi(t_1, \tilde{z}_1)) = \tilde{z}_n. \tag{26}$$

From (11) we have that $\varphi(t_1, 0) \equiv 0$. Thus, (26) is a small pertubation of (20) depending on the size of $|\tilde{z}|$. Thus, there is a neighborhood U of 0 such that for generic $\tilde{z}$ close enough to 0, the equation (26) has exactly k solutions in U for t_1, by Rouché theorem. Therefore $F^{-1}(\tilde{z})$ consists of k distinct points near 0, and hence $\mathrm{mult}_0(F) = k$. □

In two dimensions, we get further results along this line of reasoning. Let M be a real-analytic surface (i.e., a 2-dimensional real submanifold) in $\mathbb{C}^2$ and $p \in M$. If p is a CR singular point of M, then we can find a change of coordinates such that $p = 0$ and M is given by $w = \rho(z, \bar{z})$, where ρ vanishes to order at least 2 at 0. A *Bishop surface* is a surface where ρ vanishes exactly to order 2 at the origin.

As in Theorem 3.1, we see that the condition $\rho(0, \bar{z}) \not\equiv 0$, says precisely when M is the image of $\mathbb{R}^2$ under a finite map. Let us first show that this condition is also an invariant under *formal* invertible transformations.

Lemma 3.2. *Let M and M' be a CR singular real-analytic surfaces in $\mathbb{C}^2$ near 0 given by $w = \rho(z, \bar{z})$ and $w' = \rho'(z', \bar{z}')$ and F a formal invertible transformation that sends M into M'. Then $\rho(0, z) \equiv 0$ if and only if $\rho'(0, \bar{z}') \equiv 0$.*

Proof. Assume that $F = (F_1, F_2)$. Since F preserves the origin and the complex tangent plane at the origin,

$$F_1(z, w) = az + bw + \varphi(z, w), \quad F_2(z, w) = cw + \psi(z, w), \tag{27}$$

where $ac \neq 0$ and $\varphi(z, w) = O(2), \psi(z, w) = O(2)$. Thus

$$cw + \psi(z, w) = \rho'\big(F_1(z, w), \bar{b}\bar{z} + \bar{b}\bar{w} + \bar{\varphi}(\bar{z}, \bar{w})\big), \text{ when } w = \rho(z, \bar{z}). \tag{28}$$

Therefore,

$$c\rho(z, \bar{z}) + \psi\big(z, \rho(z, \bar{z})\big) = \rho'\Big(F_1\big(z, \rho(z, \bar{z})\big), \bar{a}\bar{z} + \bar{b}\bar{\rho}(\bar{z}, z) + \varphi\big(z, \rho(z, \bar{z})\big)\Big). \tag{29}$$

Now assume that $\rho(0, \bar{z}) \equiv 0$. Put $z = 0$

$$0 = \rho'\big(0, \bar{a}\bar{z}\big). \tag{30}$$

Since $a \neq 0$ we deduce that $\rho'(0, \bar{z}) \equiv 0$. □

We conclude this section by the following theorem that completely analyzes the situation in $n = 2$.

Theorem 3.3. *Let $(M, 0)$ be a germ of a CR singular real-analytic surface at 0 in $\mathbb{C}^2$. Assume that M is defined near 0 by $w = \rho(z, \bar{z})$, where ρ vanishes to order at least 2 at 0. Then the following are equivalent:*

(i) There is a germ of a totally real, real-analytic surface $(N,0)$ in $\mathbb{C}^2$ and a finite *holomorphic map $F\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ such that $F|_N$ is diffeomorphic onto $M = F(N)$ (as germs at 0).*
(ii) $\rho(0,\bar{z}) \not\equiv 0$.

In addition, if M is a Bishop surface then (i) *and* (ii) *are equivalent to the following: M is* not *(locally) biholomorphically equivalent to $M_0 := \{w = |z|^2\}$. If the Bishop invariant $\gamma \neq 0$ then* $\operatorname{mult}_0(F) = 2$. *Otherwise, $s =$* $\operatorname{mult}_0(F)$ *is the Moser invariant of M.*

Proof. The equivalence of (i) and (ii) follows from Theorem 3.1.

Now suppose that M is a Bishop surface given by $w = \rho(z,\bar{z})$. After a biholomorphic change of coordinates, we can write

$$\rho(z,\bar{z}) = |z|^2 + \gamma(z^2 + \bar{z}^2) + O(|z|^3), \tag{31}$$

where $0 \leq \gamma \leq \infty$ is the Bishop invariant [4]. When $\gamma = \infty$ the equation (31) is understood as $\rho(z,\bar{z}) = z^2 + \bar{z}^2 + O(|z|^3)$.

Suppose that (i) or (ii) holds, then from Lemma 3.2 we see that M is not equivalent to $M_0 := \{w = |z|^2\}$. Conversely, assume that M is not equivalent to M_0. Then either $\gamma \neq 0$ and hence (ii) holds, or $\gamma = 0$. In the latter case, by the work of Moser [17], after a formal change of coordinates, M can be brought to a "pseudo-normal" form

$$w = |z|^2 + z^s + \bar{z}^s + \sum_{i+j \geq s+1} a_{ij} z^i \bar{z}^j. \tag{32}$$

Here, $0 \leq s \leq \infty$ is the Moser invariant. Assume for contradiction that (ii) does not hold. Then $\rho(0,\bar{z}) \equiv 0$. Notice that the property $\rho(0,\bar{z}) \equiv 0$ is preserved under formal transformations carried in [17], by Lemma 3.2, we deduce that $s = \infty$ and so M is formally equivalent to M_0. By [17], M is biholomorphically equivalent to M_0. We obtain a contradiction.

Finally, if $\gamma \neq 0$ then from (31) we see that $\rho(0,\bar{z}) = \gamma\bar{z}^2 + O(\bar{z}^3)$ and hence $\operatorname{mult}_0(F) = 2$. Otherwise, from (32), we have $\rho(0,\bar{z}) = \bar{z}^s + O(\bar{z}^{s+1})$ and thus $\operatorname{mult}_0(F) = s$ is the Moser invariant of M. ∎

4. Images containing a family of discs

The following theorem is one of the main tools to study CR singular submanifolds containing complex subvarieties developed in this paper.

Theorem 4.1. *Let $N \subset \mathbb{C}^n$ be a connected generic real-analytic submanifold and let $f\colon N \to \mathbb{C}^n$ be a real-analytic CR map that is a diffeomorphism onto its image, $M = f(N)$. Let $S \subset M$ be the CR singular set of M and suppose that M is generic at some point.*

(i) If $L \subset M$ is a complex subvariety of dimension j, then $S \cap L$ is either empty or a complex subvariety of L of dimension $j-1$ or j.
(ii) Let $A\colon [0,\epsilon) \times \overline{\Delta} \to M$ be a family of analytic discs such that $A(0,0) = p \in S$. If $A(0,\overline{\Delta}) \setminus S$ is nonempty, then there exists an $\epsilon' > 0$ such that $A(t,\overline{\Delta}) \cap S \neq \emptyset$ for all $0 \leq t < \epsilon'$.
(iii) If M contains a continuous one real dimensional family of complex manifolds of complex dimension j, and if S intersects one of the manifolds, then S must be of real dimension at least $2j-1$.

By a one-dimensional family of analytic discs we mean a map

$$A\colon I\times\overline{\Delta}\to\mathbb{C}^n, \tag{33}$$

where $I\subset\mathbb{R}$ is an interval, $\Delta\subset\mathbb{C}$ is the unit disc, A is a continuous function such that $z\mapsto A(t,z)$ is nonconstant and holomorphic in Δ for every $t\in I$.

In essence, part (i) says that if M contains a complex variety, the intersection of this variety with S must be large (and complex). Part (ii) of the theorem says that if there exists a one-dimensional family of complex varieties in M, and S intersects one of them properly, then it must intersect all of them. Part (iii) puts the two parts together.

In the proof of Theorem 4.1, we need the following lemma.

Lemma 4.2. *Let $N\subset\mathbb{C}^n$ be a connected generic submanifold and let $F\colon\mathbb{C}^n\to\mathbb{C}^n$ be a holomorphic map such that $f=F|_N$ is a diffeomorphism onto its image, $M=f(N)$. If d is the real codimension of N, then for $p\in N$ we have*

$$\dim_{\mathbb{C}} T_p^{1,0}M = 2n-d-\operatorname{rank}_{\mathbb{C}}\left[\frac{\partial F_i}{\partial z_j}(p)\right]. \tag{34}$$

In particular, let $S\subset M$ be the CR singular set of M, and suppose that M is a generic submanifold at some point. Then

$$f^{-1}(S)=\{z\in\mathbb{C}^n : J_F(z)=0\}\cap N. \tag{35}$$

Here $J_F(z)=\det\left[\frac{\partial F_i}{\partial z_j}(z)\right]$ denotes the holomorphic Jacobian of F.

Proof. Let $p\in N$ and $q=F(p)\in M$. We denote by J the complex structure in $\mathbb{C}^n$ as usual. Since N is generic,

$$T_pN+JT_pN=T_p\mathbb{C}^n. \tag{36}$$

On the other hand, since F is holomorphic, we have $F_*\circ J=J'\circ F_*$. Furthermore, $F_*(T_pN)=T_qM$ because $F|_N$ is diffeomorphism. Therefore,

$$F_*(T_p\mathbb{C}^n)=F_*(T_pN+JT_pN)=F_*(T_pN)+F_*(JT_pN)=T_qM+JT_qM. \tag{37}$$

Consequently,

$$\begin{aligned}\operatorname{rank}_{\mathbb{R}} F_*|_p &= \dim_{\mathbb{R}} F_*(T_p\mathbb{C}^n)\\ &= \dim_{\mathbb{R}}(T_qM+JT_qM)\\ &= \dim_{\mathbb{R}} T_qM+\dim_{\mathbb{R}} JT_qM-\dim_{\mathbb{R}}(T_qM\cap JT_qM).\end{aligned} \tag{38}$$

Since F is holomorphic, the real rank $\operatorname{rank}_{\mathbb{R}} F_*|_p$ equals to $2\operatorname{rank}_{\mathbb{C}}\left[\frac{\partial F_i}{\partial z_j}(p)\right]$ (twice the rank of the complex Jacobian matrix of F). Hence

$$2\operatorname{rank}_{\mathbb{C}}\left[\frac{\partial F_i}{\partial z_j}(p)\right]=2(2n-d)-2\dim_{\mathbb{C}} T_q^{1,0}M. \tag{39}$$

In other words,

$$\dim_{\mathbb{C}} T_q^{0,1}M = 2n-d-\operatorname{rank}_{\mathbb{C}}\left[\frac{\partial F_i}{\partial z_j}(p)\right]. \tag{40}$$

The second part of the lemma now follows at once. □

Lemma 4.3. *Let $N \subset \mathbb{C}^n$ be a connected generic real-analytic submanifold and let $f\colon N \to \mathbb{C}^n$ be a CR map that is a real-analytic diffeomorphism onto its image, $M = f(N)$. Let S be the CR singular set of M and suppose that M is generic at some point. If $p \in S$, then there is a neighborhood U of p in M and a real-analytic function u on U that is CR on $U \setminus S$ that does not extend to a holomorphic function past p.*

Proof. Let $q = f^{-1}(p) \in N$ and let F be the unique holomorphic extension of f to some neighborhood of N in $\mathbb{C}^n$. Then by Lemma 4.2, F has degenerate rank at q. Assume for contradiction that for all neighborhoods U of p, every real-analytic function g on U that is CR on $U \setminus S$ extends to a holomorphic function past p. We claim that the homomorphism

$$F^*\colon \mathcal{O}_p \to \mathcal{O}_q, \quad F^*(g) = g \circ F \tag{41}$$

is surjective, where $\mathcal{O}_p$ and $\mathcal{O}_q$ are the rings of germs holomorphic functions at p and q. For if h is a holomorphic function in a neighborhood V of q in $\mathbb{C}^n_t$, we consider the function $u = (h|_N) \circ f^{-1}$. Clearly, u is a real-analytic function on U that is CR on $U \setminus S$, where $U = f(V \cap N)$. By assumption u extends past p to an element $\hat{u} \in \mathcal{O}_p$. It is straightforward to verify that $F^*(\hat{u}) = h$ on $M \setminus S$ near p. By the genericity of M at points in $M \setminus S$, we obtain that $F^*(\hat{u}) = h$ as germs near p and hence, the claim follows. In particular, there are germs of holomorphic functions g_j at p, such that the coordinate functions $t_j = g_j \circ F$. Let $G = (g_1, \ldots, g_j)$, then G is a germ of a holomorphic map satisfying $G \circ F = \mathrm{Id}$. This is impossible since F has degenerate rank at q. ☐

We also need the following result of Diederich and Fornæss [6, claim in section 6].

Lemma 4.4 (Diederich-Fornæss)**.** *Let $U \subset \mathbb{C}^n$ be an open set and let $S \subset U$ be a real-analytic subvariety. For every $p \in S$, there exists a neighborhood U' of p such that for every $q \in U'$ and every germ of a complex variety $(V, q) \subset (S, q)$, there exists a (closed) complex subvariety $W \subset U'$ such that $(V, q) \subset (W, q)$ and such that $W \subset S \cap U'$.*

The lemma has the following useful corollary.

Corollary 4.5. *Let $U \subset \mathbb{C}^n$ be an open set and let $X \subset U$ be a real-analytic subvariety. Suppose that there exists an open dense set $E \subset X$ such that for every $p \in E$ there exists a neighborhood U' of p such that $X \cap U'$ is a complex manifold. Then X is a complex analytic subvariety of U.*

Proof of Theorem 4.1. Let us begin with (i). First look at the inverse image $f^{-1}(L)$. This set is a real subvariety of N, though we cannot immediately conclude that $f^{-1}(L)$ is a complex variety.

Let F be the unique holomorphic extension of f to a neighborhood of N. Near points of $N \setminus f^{-1}(S)$, the map F is locally biholomorphic, by Lemma 4.2. Hence, $f^{-1}(L) \setminus f^{-1}(S)$ is a complex analytic variety.

If $f^{-1}(L) \setminus f^{-1}(S)$ is empty, then we are finished as $L \subset S$. Let us assume that L is irreducible. As f is a diffeomorphism, $f^{-1}(L)$ is also an irreducible subvariety. So suppose that $f^{-1}(L) \setminus f^{-1}(S)$ is nonempty, and therefore an open dense subset of $f^{-1}(L)$. By applying Corollary 4.5 we obtain that $f^{-1}(L)$ must be a complex variety. As $F^{-1}(S)$ is defined by a single holomorphic function J_F and furthermore, it follows from (35) that

$$f^{-1}(L \cap S) = f^{-1}(L) \cap f^{-1}(S) = f^{-1}(L) \cap F^{-1}(S), \tag{42}$$

then $E := f^{-1}(L \cap S)$ must be a complex subvariety of dimension $j-1$ (or empty). As $L \cap S$ is a real-analytic subvariety, we can invoke Corollary 4.5 again to conclude that $L \cap S$ is complex variety as follows. If $p \in L \cap S$ is a regular point and $q = f^{-1}(p) \in E$ then q is a regular point of E as f is a diffeomorphism. Furthermore, E is a complex manifold near q, i.e., $T_q E = T_q^c E$. Here, $T_q^c E := T_q E \cap J(T_q E)$ is the complex tangent space at q of E. Thus, as f is a CR map and diffeomorphism,

$$T_p(L \cap S) = f_*(T_q E) = f_*(T_q^c E) \subset T_p^c(L \cap S). \tag{43}$$

Consequently, $T_p(L \cap S) = T_p^c(L \cap S)$. Therefore, $L \cap S$ is a complex manifold near p. Since the regular part of $L \cap S$ is dense, $L \cap S$ is a complex manifold near all points on an open dense subset of $L \cap S$ and hence $L \cap S$ is complex variety.

Let us now move to (ii). As $S \cap A(0, \Delta)$ is a nonempty proper subset, we know it is a proper complex subvariety by (i). Without loss of generality we can rescale A such that $S \cap A(0, \Delta) = \{p\}$. Suppose for contradiction that $S \cap A(t, \overline{\Delta})$ is empty for all $0 < t \leq \epsilon'$, in other words, $A(t, \overline{\Delta}) \subset M \setminus S$ for all $0 < t \leq \epsilon'$ for some $\epsilon' > 0$. Then by the Kontinuitätssatz (see, e.g., [19, page 190]) any holomorphic function defined on a neighborhood of $M \setminus S$ extends to a neighborhood of $A(0, \overline{\Delta})$ (and hence past p). In view of Lemma 4.3, we obtain a contradiction.

(iii) follows as a consequence of (i) and (ii). □

5. Levi-flats

In this section we study CR singular, Levi-flat submanifolds in $\mathbb{C}^n$. Unlike in two dimensions, a codimension 2, CR submanifold in $\mathbb{C}^3$ or larger must have nontrivial CR geometry. The simplest case is the Levi-flat. A CR manifold is said to be *Levi-flat* if the Levi-form vanishes identically. Equivalently, for each $p \in M$, there is a neighborhood U of p such that $M \cap U$ is foliated by complex manifolds whose leaves L_c satisfy $T_q L_c = T_q M \cap J(T_q M)$ for all $q \in M \cap V$ and all c. This foliation is unique, it is simply the foliation by CR orbits.

In the real-analytic case, a generic Levi-flat submanifold of codimension d is locally biholomorphic to the submanifold defined by

$$\operatorname{Im} z_1 = 0, \operatorname{Im} z_2 = 0, \dots, \operatorname{Im} z_d = 0, \tag{44}$$

that is, a submanifold locally equivalent to $\mathbb{R}^d \times \mathbb{C}^{n-d}$. The situation is different if we allow a CR singularity. The fact that there are infinitely many different CR singular Levi-flat submanifolds not locally biholomorphically equivalent is already evident from the theory of Bishop surfaces in $\mathbb{C}^2$ (see Section 3).

We apply the result in the previous section to study CR singular Levi-flats that are images of a neighborhood of $\mathbb{R}^d \times \mathbb{C}^{n-d}$ under a CR diffeomorphism. We show that in dimension 3 and higher, unlike in 2 dimensions, there exist Levi-flats that are not images of a CR submanifold. First, let us study the CR singular set of an image of $N = \mathbb{R}^d \times \mathbb{C}^{n-d}$. In this case, the Levi foliation on N gives rise to a real-analytic foliation $\mathcal{L}$ on M that coincides with the Levi-foliation on $M \setminus S$. Moreover, it is readily seen that the leaves of $\mathcal{L}$ are complex submanifolds (even near points in S) and thus M is also foliated by complex manifolds. We call a "leaf" of M a leaf of this foliation.

Theorem 5.1. *Let $n \geq 3$, $n > d \geq 2$. Let $U \subset \mathbb{R}^d \times \mathbb{C}^{n-d}$ be a connected open set and let $f : U \to \mathbb{C}^n$ be a real-analytic CR map that is a diffeomorphism onto its image, $M = f(U)$. Let $S \subset M$ be the CR singular set of M and suppose that M is generic at some point.*

(i) If (x, ξ) are the coordinates in $\mathbb{R}^d \times \mathbb{C}^{n-d}$, then the set $f^{-1}(S)$ is locally the zero set of a real-analytic function that is holomorphic in ξ.
(ii) If L is a leaf of M, then $S \cap L$ is either empty or a complex analytic variety of dimension $n-d$ or $n-d-1$.
(iii) If $S \cap L$ is of dimension $n-d-1$, then S must intersect all the leaves in some neighborhood of L.

In particular, the theorem says that the CR singularity cannot be isolated if M is an image of $\mathbb{R}^d \times \mathbb{C}^{n-d}$. In fact, the CR singularity cannot be a real one-dimensional curve either. It cannot be a curve inside a leaf L as it is a complex variety when intersected with L. When it is a point, S must intersect all leaves nearby, and there is at least a 2-dimensional family of leaves of M. That is, the singular set is always 2 or more dimensional.

For example when $n = 3$, $d = 2$, it is possible that the singular set is either 2 or 3 dimensional. We show below that where it is CR it must be Levi-flat in the following sense. If we include complex manifolds among Levi-flat manifolds we can say that a CR submanifold K (not necessarily a generic submanifold) is *Levi-flat* if near every $p \in K$ there exist local coordinates z such that K is defined by

$$\operatorname{Im} z_1 = \operatorname{Im} z_2 = \cdots = \operatorname{Im} z_j = 0 \tag{45}$$

$$z_{j+1} = z_{j+2} = \cdots = z_{j+k} = 0. \tag{46}$$

for some j and k, where we interpret $j = 0$ and $k = 0$ appropriately. This definition includes complex manifolds ($j = 0$) and generic Levi-flats ($k = 0$), although we generally call complex submanifolds complex rather than Levi-flat.

Corollary 5.2. *Let $n \geq 3$, $n > d \geq 2$. Let $U \subset \mathbb{R}^d \times \mathbb{C}^{n-d}$ be a connected open set and let $f : U \to \mathbb{C}^n$ be a real-analytic CR map that is a diffeomorphism onto its image, $M = f(U)$. Let $S \subset M$ be the CR singular set of M and suppose that M is generic at some point. If S is nonempty, then it is of real dimension at least $2(n-d)$. Furthermore, near points where $f^{-1}(S)$ is a CR submanifold, it is Levi-flat or complex.*

Proof. Suppose $p \in S$ and L is the leaf passing through p. From Theorem 5.1, $L \cap S$ is a complex variety of complex dimension $n-d$ or $n-d-1$ near p. If $L \cap S$ is of dimension $(n-d)$ near p, then we are done. Otherwise, using Theorem 5.1 again, we have that S intersects a d-parameter family of leaves near p. Therefore, the real dimension of S near p is $2(n-d-1) + d \geq 2(n-d)$.

Let (x, ξ) be our parameters in U as in the theorem, then $f^{-1}(S)$ is given by a real-analytic function that is holomorphic in ξ. In other words, $f^{-1}(S)$ is a subvariety that is a Levi-flat submanifold at all regular points where it is CR. To see this fact, it is enough to look at a generic point of $f^{-1}(S)$. ☐

The fact that $f^{-1}(S)$ is Levi-flat does not imply that S must be Levi-flat. This is precisely because the complex Jacobian of the holomorphic extension F of f vanishes on $f^{-1}(S)$. In fact, as the examples in Section 8 suggest, the CR structure of $f^{-1}(S)$ and S may be quite different. In particular CR dimension of S may be strictly greater than the CR dimension of $f^{-1}(S)$. What we can say for sure is that through each point, S must contain complex varieties of complex dimension at least $n-d-1$. Using Theorem 4.1, we can obtain the following information on the CR structure of S.

Corollary 5.3. *Let M be as above and let p be a generic point of S (in particular, S is CR near p). Suppose L is the leaf on M through p, that is, an image of the leaf of the Levi-foliation through $f^{-1}(p)$ and let $E = S \cap L$.*

(i) The following table lists all the possibilities for the CR structure of S near p when the codimension is $d = 2$ (DNO means 'Does not occur').

$\dim_{\mathbb{R}}(S)$ \ $\mathrm{CRdim}(S)$	$n-3$	$n-2$
$2n-4$	*S is Levi-flat* $\dim_{\mathbb{C}} E = n-3$	*S is complex* $\dim_{\mathbb{C}} E = n-2$ *or* $\dim_{\mathbb{C}} E = n-3$
$2n-3$	*DNO*	*S is Levi-flat* $\dim_{\mathbb{C}} E = n-2$

(ii) The following table lists all the possibilities for the CR structure of S near p when the codimension is $d = 3$ († marks the case when it is not known if S must necessarily be Levi-flat).

$\dim_{\mathbb{R}}(S)$ \ $\mathrm{CRdim}(S)$	$n-4$	$n-3$	$n-2$
$2n-6$	*DNO*	*S is complex* $\dim_{\mathbb{C}} E = n-3$	*DNO*
$2n-5$	*S is Levi-flat* $\dim_{\mathbb{C}} E = n-4$	*S is Levi-flat* $\dim_{\mathbb{C}} E = n-3$ $\dim_{\mathbb{C}} E = n-4$ †	*DNO*
$2n-4$	*DNO*	*S is Levi-flat* $\dim_{\mathbb{C}} E = n-3$	*S is complex* $\dim_{\mathbb{C}} E = n-3$

We see that when codimension $d = 2$, then S must be Levi-flat or complex, and in fact we understand precisely the CR structure of S at a generic point. Also given the examples in Section 8, all the possibilities for $d = 2$ actually occur.

When $d = 3$ there is one case where we cannot decide if S is Levi-flat or not using Theorem 4.1.

Proof. Note that as p is generic, we can assume that S is a CR submanifold near p and p can be chosen such that $\dim_{\mathbb{C}} S \cap L_q$ is constant on $q \in S$. Here, L_q is the leaf through q.

Let us start with $d = 2$. Let us first note that the dimension of S must be greater than or equal to $2(n-d) = 2n-4$, so the possibilities are $2n-4$ and $2n-3$. Since $S \cap L$ is contained in S, the CR dimension of S must be greater than or equal to that of $S \cap L$. Hence it is either $n-3$ or $n-2$. If $\mathrm{CRdim}\, S = n-3$, then $\dim_{\mathbb{C}} S \cap L = n-3$ of course and so S must be Levi-flat. By Theorem 4.1, S intersects all leaves near L and hence must have real dimension $2(n-3)+2 = 2n-4$ as there is a 2-dimensional family of leaves and the intersection with each of them is of real dimension $2n-6$.

So now consider $\text{CRdim}(S) = n - 2$. If $\dim S = 2n - 4$, then S must be complex. Both $\dim_{\mathbb{C}} S \cap L = n - 3$ and $n - 2$ are possible. When $\dim S = 2n - 3$, then necessarily $\dim_{\mathbb{C}} S \cap L = n - 2$ by the above argument, and so S is Levi-flat.

The case $d = 3$ follows similarly. In this case we have $2(n-d) = 2n-6 \leq \dim S \leq 2n-4$, and $\text{CRdim}\, S \geq n - 4$ as $\dim_{\mathbb{C}} S \cap L \geq n - 4$. Similarly, when $\dim_{\mathbb{C}} S \cap L = n - 4$, then $\dim S = 2(n-4) + 3 = 2n - 5$. The table fills in similarly as above.

The only case when we do not know if S is flat or not is when $\dim_{\mathbb{C}} S \cap L = n - 4$ and the CR dimension of S is $n - 3$. □

Proof of Theorem 5.1. Let us suppose that $f(0) = 0$ and $0 \in S$ for simplicity in the following arguments.

Let us begin with (i). Suppose we have a real-analytic CR map $f(x, \xi)$ from U onto M, where $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^{n-d}$. It follows that f is holomorphic in ξ. As in the proof of Lemma 4.2, the map f extends to a holomorphic maps $F(z, \xi)$ defined in an open set $\hat{U} \subset \mathbb{C}^d \times \mathbb{C}^{n-d}$ by simply replacing the real variable x by a complex variable z. F sends U, as a generic submanifold of $\mathbb{C}^n$ defined by $z = \bar{z}$ in $\hat{U}$, diffeomorphically onto M. By Lemma 4.2,

$$f^{-1}(S) = \{(z, \xi) \in \hat{U} : J_F(z, \xi) = 0\} \cap \{z = \bar{z}\}. \tag{47}$$

Thus, in (x, ξ)-coordinates, $f^{-1}(S)$ is given by the vanishing of the function $J_F(x, \xi)$ on U, where $J_F(x, \xi)$ is real-analytic in x and holomorphic in ξ.

For the proof of (ii), let L be a leaf on M. Since the leaves of M is parametrized by $f(x, \xi)$, where x is regarded as a parameter, L is a complex manifold of dimension $n - d$. The conclusion of (ii) then follows from Theorem 4.1.

Let us now prove (iii). Fix $p \in S$ and take a leaf L of M through p, and suppose $f(0) = p$. Suppose that $S \cap L$ is $(n-d-1)$-dimensional. Take a one-dimensional curve $x \colon [0, \epsilon) \to \mathbb{R}^d$ such that the leaves of M given by $L_t = f(\{x(t)\} \times \mathbb{C}^{n-d} \cap U)$ do not intersect S for all $t > 0$. We find the family of disks $A \colon [0, \epsilon) \to \overline{\Delta}$ such that $A(0, \overline{\Delta}) \setminus S \neq \emptyset$ (as $S \cap L$ is $n - d - 1$ dimensional), and $A(t, \overline{\Delta}) \subset L_t$. We can apply Theorem 4.1 to show that $S \cap L_t$ is nonempty. As the curve x was arbitrary, we are done. □

Let us prove a general proposition about identifying the CR singular set for codimension two submanifolds. It is particularly useful for computing examples.

Proposition 5.4. *Let $w = \rho(z, \bar{z})$ define a CR singular manifold M of in coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, where ρ is real-analytic such that $\rho = 0$ and $d\rho = 0$ at the origin. Then, the CR singularity S is defined precisely by*

$$S = \{(z, w) \in \mathbb{C}^n : \rho_{\bar{z}_k}(z, \bar{z}) = 0 \text{ for } k = 1, \ldots, n-1\}. \tag{48}$$

Proof. Suppose $M \subset \mathbb{C}^{n-1} \times \mathbb{C} = \{Z = (z, w)\}$ is a real-analytic CR singular submanifold of codimension 2 given by

$$M = \{Z = (z, w) \in \mathbb{C}^n : w = \rho(z, \bar{z})\}. \tag{49}$$

We can take $r_1 = \text{Re}(w - \rho)$ and $r_2 = \text{Im}(w - \rho)$ to be real-valued defining equation for M. As explained in Section 1, the CR singular set S is then determined by those points in M where the matrix $\left(\frac{\partial r_j}{\partial \bar{Z}_k}\right)_{j,k}$ is of rank at most one. By a straightforward computation, we can see that

$$S = \{(z,w) \in \mathbb{C}^n : \rho_{\bar{z}_k}(z,\bar{z}) = 0 \text{ for } k = 1,\dots,n-1\}. \tag{50}$$

The proof is complete. ☐

We conclude this section by the following example of a Levi-flat codimension two submanifold M of $\mathbb{C}^{n-1}_z \times \mathbb{C}_w$ ($n \geq 3$) whose CR singular set is isolated. Such a manifold is then not an diffeomorphic image of a codimension two generic submanifold on $\mathbb{C}^n$ under CR map. This conclusion follows by Theorem 5.1.

Example 5.5. Let M be given by

$$w = \operatorname{Re}(z_1^2 + z_2^2 + \cdots + z_{n-1}^2). \tag{51}$$

The CR singular set of M is the origin, by Proposition 5.4. Furthermore, $M \setminus \{0\}$ is Levi-flat. Assume that there exist a generic codimension two submanifold $N \subset \mathbb{C}^n$ and an analytic CR map $f\colon N \to M$ that is diffeomorphism onto M. Then $N \setminus f^{-1}(S)$ is Levi-flat and so is N. From Theorem 5.1 we obtain that S is of dimension at least $n-2 \geq 1$. This is a contradiction.

The manifold M is the intersection of two nonsingular Levi-flat hypersurfaces, One defined by $\operatorname{Im} w = 0$, and the other by $\operatorname{Re} w = \operatorname{Re}(z_1^2 + \cdots + z_{n-1}^2)$. Further, the manifold M contains the singular complex analytic set $\{z_1^2 + z_2^2 + \cdots + z_{n-1}^2 = 0, w = 0\}$ through the origin.

6. Singular coordinates for nowhere minimal finite images

We would like to find at least a partial analogue for CR singular manifolds of the following standard result for CR manifolds. If M is a CR submanifold, and $p \in M$ then the CR orbit Orb_p is the germ of the smallest CR submanifold of M of the same CR dimension as M through p. For a real-analytic M, the CR orbit exists and is unique by a theorem of Nagano (see [3]). Near a generic point where the orbit is of maximal possible dimension in M, the CR orbits give a real-analytic foliation of M. The following theorem gives a way to describe this foliation.

Theorem 6.1 (see [2])**.** *Let $M \subset \mathbb{C}^n$ be a generic real-analytic nowhere minimal submanifold of real codimension d, and let $p \in M$. Suppose that all the CR orbits are of real codimension j in M. Then there are local holomorphic coordinates $(z, w', w'') \in \mathbb{C}^k \times \mathbb{C}^{d-j} \times \mathbb{C}^j = \mathbb{C}^n$, vanishing at p, such that near p, M is defined by*

$$\operatorname{Im} w' = \varphi(z, \bar{z}, \operatorname{Re} w', \operatorname{Re} w''), \tag{52}$$

$$\operatorname{Im} w'' = 0, \tag{53}$$

where φ is a real valued real-analytic function with $\varphi(z, 0, s', s'') \equiv 0$. Moreover, the local CR orbit of the point $(z, w', w'') = (0, 0, s'')$, for $s'' \in \mathbb{R}^j$, is given by

$$\operatorname{Im} w' = \varphi(z, \bar{z}, \operatorname{Re} w', s''), \tag{54}$$

$$w'' = s''. \tag{55}$$

For convenience, we will call a subvariety of codimension one a *hypervariety*. For a real hypervariety $H \subset \mathbb{C}^n$ let H^* denote the set of points near which H is a real-analytic nonsingular hypersurface. We say H is a Levi-flat hypervariety if H^* is Levi-flat. The subvariety defined by

$$\operatorname{Im} w''_j = 0 \tag{56}$$

is a Levi-flat hypervariety (in this case, it is nonsingular). We cannot find coordinates as in Theorem 6.1 for a CR singular manifold, but we can at least find Levi-flat hypervarieties that play the role of $\{\operatorname{Im} w''_j = 0\}$. We should note that *not* every Levi-flat hypervariety is of the form $\operatorname{Im} h = 0$ for some holomorphic function h (see [5]).

Theorem 6.2. *Let $N \subset \mathbb{C}^n$ be a real-analytic generic connected submanifold and let $f \colon N \to \mathbb{C}^n$ be a real-analytic CR map that is a diffeomorphism onto its image, $M = f(N)$. Suppose that f extends to a finite holomorphic map F from a neighborhood of N to a neighborhood of M. Suppose that all the CR orbits of N are of real codimension j in N, and $p \in M$ is such that M is CR singular at p. Then there exists a neighborhood U of p and j distinct Levi-flat hypervarieties $H_1, H_2, \ldots, H_j$ such that $\dim_{\mathbb{R}} H_1 \cap \cdots \cap H_j = 2n - j$ and*

$$M \subset H_1 \cap \cdots \cap H_j. \tag{57}$$

Furthermore, if Orb_q is a germ of a CR orbit of N at $q \in N$, Then there exists an $n - j$ dimensional germ of a complex variety $\big(L, f(q)\big)$ with $\big(L, f(q)\big) \subset \big(H_k, f(q)\big)$ for all $k = 1, \ldots, j$ and as germs

$$f(\operatorname{Orb}_q) \subset \big(L, f(q)\big). \tag{58}$$

In particular, if N is Levi-flat then we can find Levi-flat hypervarieties $H_1, \ldots, H_j$ such that M is one of the components of $H_1 \cap \cdots \cap H_j$.

Proof. Suppose that $q \in N$ is such that $F(q) = p$. Let V, U be connected open subsets of $\mathbb{C}^n$, $q \in V$ and $p = F(q) \in U$ and $F(V) = U$, and as F is finite we can assume that F is proper onto U. Let us assume that V is the domain of F, that N is closed in V and furthermore that the N is defined in V using coordinates of Theorem 6.1 (the coordinates are defined in all of V).

Fix a nonzero vector $v \in \mathbb{R}^j$. Take the variety $\{\operatorname{Im}\langle w'', v\rangle = 0\}$ and let us push it forward by F. The image need not necessarily be a real-analytic subvariety. We claim, however, that as F is finite, then $F(\{\operatorname{Im}\langle w'', v\rangle = 0\})$ is contained in a real-analytic subvariety of codimension one. To show the claim we complexify F and $\{\operatorname{Im}\langle w'', v\rangle = 0\}$, push the set forward using the Remmert proper map theorem, and then restrict back to the diagonal.

As F is proper, then the function $\mathcal{F}(\zeta, \xi) = \big(F(\zeta), \bar{F}(\xi)\big)$ is a proper map of $V \times V^*$ to $U \times U^*$ where $V^* = \{\xi : \bar{\xi} \in V\}$. So as $\{\operatorname{Im}\langle w'', v\rangle = 0\}$ complexifies to a complex submanifold $\mathcal{H} \subset V \times V^*$, then as $\mathcal{F}$ is proper, $\mathcal{F}(\mathcal{H})$ is an irreducible complex subvariety of $U \times U^*$. Let (ζ', ξ') denote the coordinates in $U \times U^*$ and $\pi_{\zeta'}$ the projection onto the ζ' coordinates. Let H denote the set $\pi_{\zeta'}\big(\mathcal{F}(\mathcal{H}) \cap \{\overline{\zeta'} = \xi'\}\big)$. The defining equation for $\mathcal{F}(\mathcal{H})$ defines H once we plug in $\overline{\zeta'}$ for ξ'. Therefore H is an irreducible real subvariety of U and $M \subset H$. A holomorphic function that is not identically zero cannot vanish identically on the maximally totally real set $\{\overline{\zeta'} = \xi'\}$, and hence H must be a proper subvariety of U. By construction, $F(\{\operatorname{Im}\langle w'', v\rangle = 0\}) \subset H$. Therefore, the subvariety H must be of real codimension one as F is finite.

Since $\{\operatorname{Im}\langle w'', v\rangle = 0\}$ is a real Levi-flat hypersurface and F is a local biholomorphism outside of a complex subvariety, we see that H must be Levi-flat at some point. By a lemma of Burns and Gong (see [5] or [15]) then as H is irreducible, it is Levi-flat at all smooth points of top dimension, and hence Levi-flat by definition.

Suppose we have taken k linearly independent vectors $v_1, \ldots, v_k$ such that the corresponding $H_1, \ldots, H_k$ have an intersection that is of real codimension k. Suppose that $k < j$. We have

$$H_1 \cap \cdots \cap H_k = \pi_{\zeta'}\big(\mathcal{F}(\mathcal{H}_1) \cap \cdots \cap \mathcal{F}(\mathcal{H}_k) \cap \{\overline{\zeta'} = \xi'\}\big). \tag{59}$$

Let

$$\mathcal{V} = \mathcal{F}^{-1}\big(\mathcal{F}(\mathcal{H}_1) \cap \cdots \cap \mathcal{F}(\mathcal{H}_k)\big). \tag{60}$$

The variety $\mathcal{V}$ has codimension k. Let us treat (z, w', w'') and $(\bar{z}, \bar{w}', \bar{w}'')$ as different variables

If $\mathcal{V} = \mathcal{H}_1 \cap \cdots \cap \mathcal{H}_k$, then pick any vector vector $v \in \mathbb{R}^j$ linearly independent from $v_1, \ldots, v_k$, and let $\mathcal{H}$ be defined by $\{\langle w'', v\rangle - \overline{\langle w'', v\rangle} = 0\}$. Then the intersection $\mathcal{V} \cap \mathcal{F}^{-1}\big(\mathcal{F}(\mathcal{H})\big)$ is of codimension $k+1$. It now follows that $\mathcal{F}(\mathcal{H}_1) \cap \cdots \cap \mathcal{F}(\mathcal{H}_k) \cap \mathcal{F}(\mathcal{H})$ is of codimension $k+1$. And hence if $H = \pi_{\zeta'}\big(\mathcal{F}(\mathcal{H}) \cap \{\overline{\zeta'} = \xi'\}\big)$, then $H_1 \cap \cdots \cap H_k \cap H$ has real codimension $k+1$. This claim follows because if $H_1 \cap \cdots \cap H_k \cap H$ has codimension k, it would have some point where it is a smooth real codimension k manifold. The complexification at that point would have to be a complex codimension k manifold, and we know that is not true.

In case $\mathcal{V}$ has other components, then let $\mathcal{C}$ be any irreducible component of $\mathcal{V}$ that is not contained in $\mathcal{H}_1 \cap \cdots \cap \mathcal{H}_k$. Let us treat (z, w', w'') and $(\bar{z}, \bar{w}', \bar{w}'')$ as different variables as usual. Note that $\mathcal{X} = \mathcal{F}^{-1}\big(\mathcal{F}(\{w'' = \bar{w}''\})\big)$ is of dimension $2n - j$ as $\mathcal{F}$ is finite. Furthermore $\mathcal{X} \subset \mathcal{V}$. There must exist a point $x = (z_0, w_0', w_0'', \bar{z}_0, \bar{w}_0', \bar{w}_0'') \in \mathcal{C}$ where $x \notin \mathcal{X}$. Hence $\mathcal{F}^{-1}\big(\mathcal{F}(x)\big) \notin \mathcal{X}$. In particular, $\mathcal{F}^{-1}\big(\mathcal{F}(x)\big) \cap \{w'' = \bar{w}''\}$ is the empty set. We can pick a vector $v \in \mathbb{R}^j$ linearly independent from $v_1, \ldots, v_k$, such that the set $\mathcal{H}$ defined by $\{\langle w'', v\rangle - \overline{\langle w'', v\rangle} = 0\}$ does not contain any point of $\mathcal{F}^{-1}\big(\mathcal{F}(x)\big)$ and so the intersection $\mathcal{V} \cap \mathcal{F}^{-1}\big(\mathcal{F}(\mathcal{H})\big)$ is of codimension $k+1$. We then proceed as above.

Hence we can find j distinct Levi-flat hypervarieties $H_1, H_2, \ldots, H_j$ such that $\dim_{\mathbb{R}}(H_1 \cap \cdots \cap H_j) = 2n - j$ and $M \subset H_1 \cap \cdots \cap H_j$.

To find L, we push forward the complex variety $\{w'' = s''\}$ by F, which is finite. ☐

Remark 6.3. Theorem 6.1 also generalizes to some extent to certain CR manifolds at points where the dimension of the CR orbits is not constant and hence where the CR orbits do not form a foliation. At such singular points it is not always true that such N lie inside Levi-flat hypervarieties, despite all CR orbits being of positive codimension. See [15] for more on these matters.

7. Failure of extensions of real-analytic CR functions

In this section, we focus our attention on functions satisfying the pointwise Cauchy-Riemann conditions on a CR singular image with a nonempty CR singular set S. We shall show for each $p \in S$, there exists a real-analytic function on a neighborhood of p in M satisfying all the pointwise Cauchy-Riemann conditions that does not extend to a holomorphic function at p. This result generalizes Lemma 4.3.

Theorem 7.1. *Let $M \subset \mathbb{C}^n$ be a connected real-analytic CR singular submanifold such that there are a real-analytic generic submanifold $N \subset \mathbb{C}^n$ and a real-analytic CR map $f \colon N \to \mathbb{C}^n$ that is a diffeomorphism onto $M = f(N)$. Let S be the nonempty CR singular set of M and suppose that M is generic at some point. For any $p \in S$, then there exists a neighborhood U of p and a real-analytic function u on $U \cap M$ such that it satisfies $Lu|_q = 0$ for any $q \in U \cap M$ and any $L \in T_q^{(0,1)}M$, but does not extend to a holomorphic function on any neighborhood of p in $\mathbb{C}^n$.*

Proof. Let F be the unique holomorphic extension of f near $f^{-1}(p)$ in $\mathbb{C}^n$. Let $\theta = J_F$ and $\varphi = \theta^2 \circ f^{-1}$. Then φ is a real-analytic function on $f(V \cap N)$. We claim that φ satisfies all CR condition on M. Indeed, for any point $q \in M$ near p, let $L \in T_q^{(0,1)}M$. If $q \in S$, then $q' := f^{-1}(q) \in f^{-1}(S) \subset \{\theta = 0\}$ and hence $\theta(q') = 0$. If $q \in M \setminus S$, then $X_{q'} := (f^{-1})_* L \in T_{q'}^{(0,1)}N$ and thus $X_{q'}(\theta)(q') = 0$. Therefore for all $q \in U \cap M$, we have

$$(L\varphi)(q) = L(\theta^2 \circ f^{-1})(q) = ((f^{-1})_* L)\theta^2\big(f^{-1}(q)\big) = X_{q'}\theta^2(q') = 2\theta(q')\left(X_{q'}\theta\right)(q') = 0. \quad (61)$$

Therefore, the claim follows.

If φ does not extend to a holomorphic function in a neighborhood of p in $\mathbb{C}^n$, then we are done. Otherwise, suppose that φ extends to a neighborhood. Notice that $\varphi \equiv 0$ on S. Without loss of generality we can assume further that φ is radical.

On the other hand, by Lemma 4.3, we can find a real-analytic function u on a neighborhood U of p in M such that u is CR on $U \setminus S$ and u does not extend holomorphically near p. By construction, $u\varphi$ and $u^2\varphi$ restricted to U are both CR functions on $M \cap U$. We claim that at least one of the two functions $u\varphi$ and $u^2\varphi$ does not extend holomorphically past p. Indeed, assume for a contradiction that v_1 and v_2 are holomorphic functions on a neighborhood of p in $\mathbb{C}^n$ whose restrictions to M are $u\varphi$ and $u^2\varphi$, respectively. Observe that the following equalities hold on U.

$$v_1^2 = u^2\varphi^2 = v_2\varphi. \quad (62)$$

Since M is generic at all points on $M \setminus S$, v_1^2 and $v_2\varphi$ are holomorphic, we deduce from (62) that $v_1^2 = v_2\varphi$ in a neighborhood of p in $\mathbb{C}^n$. In other words, we have the following equality in the ring $\mathcal{O}_p$.

$$v_1^2 = v_2\varphi. \quad (63)$$

Note that the ring $\mathcal{O}_p$ is a unique factorization domain and φ is radical. From (63) we obtain that v_1 divides v_2 and hence $\frac{v_2}{v_1}$ is holomorphic near p. Consequently, u extends to the holomorphic function $\frac{v_2}{v_1}$ on a neighborhood of p. We obtain a contradiction. ☐

8. Examples

We start this section by the following proposition, which is helpful in constructing examples.

Proposition 8.1. *Let $w = \rho(z, \bar{z})$ define a connected CR singular manifold M near the origin in coordinates $(z, w) \in \mathbb{C}^2 \times \mathbb{C}$, where ρ is real-analytic and such that $\rho = 0$ and $d\rho = 0$ at the origin. If $\rho_{\bar{z}_1} \equiv 0$, then M is Levi-flat at CR points, and furthermore, the set S of CR singularities is given by $M \cap \{(z, w) : \rho_{\bar{z}_2}(z) = 0\}$.*

Furthermore, for each point $p \in M$, there exists a neighborhood U such that $U \cap M$ is the image under a real-analytic CR diffeomorphism of an open subset of $\mathbb{R}^2 \times \mathbb{C}$.

Note that if $w = \rho(z, \bar{z})$ and $\rho_{\bar{z}_1} \equiv 0$, we could also get that M is a complex manifold, but in this case M is not CR singular.

Proof. If $\rho_{\bar{z}_2} \equiv 0$ then ρ is holomorphic and hence M is complex analytic. Otherwise, $\rho_{\bar{z}_2} \not\equiv 0$ and therefore, from Proposition 5.4, we see that

$$S = M \cap \{(z, w) : \rho_{\bar{z}_2}(z, \bar{z}) = 0\}. \quad (64)$$

Hence, $S \subset M$ is a proper real subvarieties and $M \setminus S$ is generic. To see that $M \setminus S$ is Levi-flat, observe that in a neighborhood of $p \notin S$, $M \setminus S$ is foliated by family of one-dimensional complex submanifolds defined by $L_t = \{(z, w) : w = \rho(z_1, t, 0, \bar{t})\}$ with complex parameter t. Let a local map $f \colon \mathbb{R}^2 \times \mathbb{C} \to M$ be given by

$$f \colon (x, y, \xi) \mapsto \big(\xi, x + iy, \rho(\xi, x + iy, 0, x - iy)\big). \tag{65}$$

The CR structure on $\mathbb{R}^2 \times \mathbb{C}$ is given by $\partial/\partial\bar{\xi}$ and so clearly f is CR map. Since ρ does not depend on $\bar{z}_1$, it follows that f sends $\mathbb{R}^2 \times \mathbb{C}$ into M. The fact that f is local diffeomorphism is immediate. □

Using the proposition we can easily create many examples showing that the CR singular set of a Levi-flat manifold that is an image of a CR diffeomorphisms can have any possible CR structure allowed by Corollary 5.2.

Example 8.2. We can obtain a 3-dimensional CR singularity by simply taking a parabolic CR singular Bishop surface in 2 dimensions and considering it in 3 dimensions. For example,

$$w = |z_2|^2 + \frac{\bar{z}_2^2}{2}. \tag{66}$$

The manifold is the image of $\mathbb{R}^2 \times \mathbb{C}$ by the construction of Proposition 8.1. The CR singular set is the set $\{\operatorname{Re} z_2 = 0\} \cap M$, hence 3 real dimensional.

The submanifold is contained in the nonsingular Levi-flat hypersurface defined by $\operatorname{Im} w = -\operatorname{Im} \frac{z_2^2}{2}$.

Example 8.3. Next, let us consider

$$w = z_1 \bar{z}_2^2. \tag{67}$$

The manifold is the image of $\mathbb{R}^2 \times \mathbb{C}$ by the construction of Proposition 8.1. The CR singular set is the set $(\{z_1 = 0\} \cup \{z_2 = 0\}) \cap M$, that is a union of two 2-dimensional sets, both of which are complex analytic. Note that the set $\{z_1 = 0\} \cap M$ is a complex analytic set that is an image of a totally real submanifold of $\mathbb{R}^2 \times \mathbb{C}$ under the map of Proposition 8.1. We therefore have a complex analytic set that is a subset of M while not being an image of one of the leaves of the Levi-foliation of $\mathbb{R}^2 \times \mathbb{C}$.

Example 8.4. Consider

$$w = z_1 \bar{z}_2 - \frac{\bar{z}_2^2}{2}. \tag{68}$$

Again the manifold is the image of $\mathbb{R}^2 \times \mathbb{C}$. The CR singular set S is the set $\{z_1 = \bar{z}_2\} \cap M$, which is a totally real set; to see this fact simply substitute $\bar{z}_2 = z_1$ in the defining equation for M to find that S is the intersection of $\{z_1 = \bar{z}_2\}$ with a complex manifold.

Example 8.5. Consider

$$w = z_1 \bar{z}_2 - \frac{z_2 \bar{z}_2^2}{2}. \tag{69}$$

The CR singular set S is the set $\{z_1 = |z_2|^2\} \cap M$, which is a CR singular submanifold.

Example 8.6. While we have mostly concerned ourselves with flat manifolds, there is nothing particularly special about flat manifolds. Even a finite-type manifold can map to a CR singular manifold. The following example was given in [7, Example 1.6]. Let $M \subset \mathbb{C}^3$ be given by

$$M = \{(z, w_1, w_2) \in \mathbb{C}^3 : \operatorname{Im} w_1 = \frac{|z|^2}{2}, \operatorname{Im} w_2 = \frac{|z|^4}{2}\} \tag{70}$$

is taken to the CR singular

$$\{(z_1, z_2, w) \in \mathbb{C}^3 : w = \left(\bar{z}_2 + i|z_1|^2 + |z_1|^4\right)^2\} \tag{71}$$

via the finite holomorphic map

$$(z, w_1, w_2) \mapsto \left(z, w_1 + iw_2, (w_1 - iw_2)^2\right). \tag{72}$$

The map is a diffeomorphism onto its image when restricted to M.

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