The CR immersion into a sphere with the degenerate CR Gauss map

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Abstract

It is a classical problem in algebraic geometry to characterize the algebraic subvariety by using the Gauss map. In this note, we study the analogous phenomenon in CR geometry. In particular, under some assumptions, we show that a CR map between spheres is totally geodesic if and only if the CR Gauss map of the image is degenerate.

1 Introduction

Denote by $\mathbb{CP}^n$ the complex projective space, and denote by $G(k,n)$ the Grassmannian of $\mathbb{CP}^k$'s in $\mathbb{CP}^n$. Let V be a complex analytic subvariety in $\mathbb{CP}^n$ and V_{sm} its smooth points. The Gauss map of $V \subset \mathbb{CP}^n$ is defined by $\gamma : V_{sm} \to G(k,n)$, which sends each smooth point $x \in V_{sm}$ to the projective tangent space $T_x(V)$. γ is said to be *degenerate* if its generic fibers have positive dimensional components. Otherwise, γ is called non-degenerate. In Cartan's moving frame theory, the Gauss map has wide geometric applications in Euclidean and projective geometries. For example, one can obtain rigidity results from the degeneracy of the Gauss maps. In fact, the study of subvarieties of complex projective spaces, tori and hyperbolic space forms with degenerate Gauss maps are classical works due to Griffiths-Harris [GH], Ran [R] and Hwang [Hw]. The interested readers are referred to [IL] for more recent progress on subvarieties of complex projective spaces with degenerate Gauss maps.

The Gauss map is also closely related to the second fundamental form as the latter may be interpreted as the derivative of the Gauss map. In CR geometry, the CR second fundamental form appeared in the fundamental work of Chern-Moser [CM] and Webster [W1], as well as

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the work of Ebenfelt-Huang-Zaitsev [EHZ] in the study of the classification and rigidity of CR submanifolds. One of the central problems in CR geometry is the classification of smooth CR maps between spheres. This problem has been extensively studied and important progresses have been made by many authors in recent years (cf. [W2, Fa86, Hu99, Ha05, HJX06, DL, HJY14, Eb] and references therein). If the CR second fundamental form vanishes, S. Ji and the second author showed that the smooth immersed strongly pseudoconvex real hypersurface in a sphere $\partial\mathbb{B}^n$ must be linear [JY]. Cheng-Ji later relaxed the condition to the vanishing of the difference of the second fundamental form and CR second fundamental form and proved the linearity under some codimension restriction [CJ]. However, in CR geometry, the Gauss map is not fully understood. One can define the Gauss map for any C^1 immersed CR submanifold in $\partial\mathbb{B}^N$ as the sphere $\partial\mathbb{B}^N$ may be embedded into $\mathbb{CP}^N$ (The detailed formulation of the CR Gauss map is given in the last paragraph of the next section). The following interesting question is formulated in [CJL]: Let $V \subset \partial\mathbb{B}^N$ be an immersed spherical CR submanifold. Is it true that the CR Gauss map γ is degenerate if and only if V is the image of a linear embedding $F : \partial\mathbb{B}^n \to \partial\mathbb{B}^N$? In [CJL], Cheng-Ji-Liu answered the question in the following two cases: (1) $\dim_{\mathbb{R}} V = 3$, $N = 3$; (2) $V = F(\partial\mathbb{B}^2)$ and $F : \partial\mathbb{B}^2 \to \partial\mathbb{B}^N$ is the restriction of a rational holomorphic map with $\deg(F) = 2$.

We next state our main results, in which the terminology will be defined in the next section.

Theorem 1.1. *Let $F : \partial\mathbb{B}^n \to \partial\mathbb{B}^N$ be a C^3-smooth CR map with geometric rank $\kappa_0 \leq n-2$. Assume that one of the following conditions hold:*

(1) the degeneracy rank ≤ 2,

(2) the third degeneracy rank ≥ 3, and the third degeneracy dimension

$$d_3 \neq \frac{\kappa_0}{6}\Big(3(\kappa_0+3)n - (\kappa_0+1)(2\kappa_0+1)\Big).$$

Then the CR Gauss map of $F(\partial\mathbb{B}^n)$ in $\partial\mathbb{B}^N$ is degenerate if and only if F is a totally geodesic embedding.

As an immediate consequence, we obtain

Theorem 1.2. *Let $F : \partial\mathbb{B}^n \to \partial\mathbb{B}^N$ be a C^3-smooth CR map with geometric rank $\kappa_0 \leq n-2$. Suppose that $N < \frac{1}{2}(\kappa_0+1)(\kappa_0+2)n - \frac{1}{6}\kappa_0(\kappa_0+1)(2\kappa_0+1)$. Then the CR Gauss map of $F(\partial\mathbb{B}^n)$ in $\partial\mathbb{B}^N$ is degenerate if and only if F is a totally geodesic embedding.*

2 Notations and Preliminaries

In this section, we start by recalling some notations and properties associated to the proper holomorphic maps between balls, which were established in [Hu99][Hu03] and [HJX06]. Next, we define the CR Gauss maps of these maps and reduce the condition on the CR Gauss maps to a proper form, following the lines of [Hw] and [CJL].

Let $\partial\mathbb{B}^n$ be the sphere in $\mathbb{C}^n$ and write $\partial\mathbb{H}_n := \{(z,w) \in \mathbb{C}^{n-1}\times\mathbb{C} : \ \mathrm{Im}(w) = |z|^2\}$ for the Heisenberg group. By the Cayley transformation

$$\rho_n : \mathbb{H}_n \to \mathbb{B}^n, \quad \rho_n(z,w) = \left(\frac{2z}{1-iw}, \frac{1+iw}{1-iw}\right) \tag{2.1}$$

we can identify a CR map F from $\partial\mathbb{B}^n$ into $\partial\mathbb{B}^N$ with $\partial\rho_N^{-1}\circ F\circ\partial\rho_n$, which is a CR map from $\partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$.

Parameterize $\partial\mathbb{H}_n$ by $(z,\overline{z},u)$ through the map $(z,\overline{z},u) \to (z, u+i|z|^2)$. For a non-negative integer m and a function $h(z,\overline{z},u)$ defined over a small ball U of 0 in $\partial\mathbb{H}_n$, we say $h(z,\overline{z},u) = o_{wt}(m)$ if $\frac{h(tz,t\overline{z},t^2u)}{|t|^m} \to 0$ uniformly for (z,u) on any compact subset of U as $t(\in\mathbb{R}) \to 0$. For a holomorphic function (or map) $H(z,w)$, we write

$$H(z,w) = \sum_{k,l=0}^{\infty} H^{(k,l)}(z)w^l = \sum_{i_1\cdots,i_{n-1},l=0}^{\infty} H^{(i_1 I_1+\cdots+i_{n-1}I_{n-1}+lI_n)} z_1^{i_1}\cdots z_{n-1}^{i_{n-1}} w^l.$$

Here $H^{(k,l)}(z)$ is a polynomial of degree k in z.

Let $F = (f,\phi,g) = (\widetilde{f},g) = (f_1,\cdots,f_{n-1},\phi_1,\cdots,\phi_{N-n},g)$ be a non-constant C^2-smooth CR map from $\partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$ with $F(0)=0$. For each $p=(z_0,w_0)\in M$ close to 0, we write $\sigma_p^0 \in \mathrm{Aut}(\mathbb{H}_n)$ for the map sending (z,w) to $(z+z_0, w+w_0+2i\langle z,\overline{z_0}\rangle)$ and $\tau_p^F \in \mathrm{Aut}(\mathbb{H}_N)$ by defining

$$\tau_p^F(z^*,w^*) = (z^* - \widetilde{f}(z_0,w_0), w^* - \overline{g(z_0,w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0,w_0)}\rangle).$$

Then F is equivalent to

$$F_p = \tau_p^F\circ F\circ\sigma_p^0 = (f_p,\phi_p,g_p). \tag{2.2}$$

Notice that $F_0 = F$ and $F_p(0)=0$. Let

$$\begin{aligned} E_l(p) &= (\frac{\partial\widetilde{f}_p}{\partial z_l})\big|_0 = (\frac{\partial f_{p,1}}{\partial z_l},\cdots,\frac{\partial f_{p,n-1}}{\partial z_l},\frac{\partial\phi_{p,1}}{\partial z_l},\cdots,\frac{\partial\phi_{p,N-n}}{\partial z_l})\big|_0 = L_l(\widetilde{f})(p), \\ E_w(p) &= (\frac{\partial\widetilde{f}_p}{\partial w})\big|_0 = (\frac{\partial f_{p,1}}{\partial w},\cdots,\frac{\partial f_{p,n-1}}{\partial w},\frac{\partial\phi_{p,1}}{\partial w},\cdots,\frac{\partial\phi_{p,N-n}}{\partial w})\big|_0 = T(\widetilde{f})(p). \end{aligned} \tag{2.3}$$

Then the rank of $\{E_1(p),\cdots,E_{n-1}(p)\}$ is $n-1$. Write $\lambda(p) = g'_w(p) - 2i\langle\widetilde{f}'_w(p),\overline{\widetilde{f}(p)}\rangle = |L_j(\widetilde{f})|^2$. We can choose vectors $C_l(p)$ for $1\le l\le N-n$ such that

$$A(p) = \Big(\frac{E_1^t(p)}{\sqrt{\lambda(p)}},\cdots,\frac{E_1^t(p)}{\sqrt{\lambda(p)}},C_1^t(p),\cdots,C_{N-n}^t(p)\Big) \tag{2.4}$$

is a unitary matrix. Define

$$F_p^* = (\widetilde{f}_p^*, g_p^*) = \frac{1}{\sqrt{\lambda(p)}} F_p \cdot \begin{pmatrix} \overline{A^t(p)} & 0 \\ 0 & \frac{1}{\sqrt{\lambda(p)}} \end{pmatrix}. \tag{2.5}$$

Then F_p^* has the following form:

$$\begin{aligned} f_j^* &= z_j + a_j w + O(|(z,w)|^2), \\ \phi_j^* &= b_j w + O(|(z,w)|^2), \\ g^* &= w + dw^2 + O(|zw|) + o(|(z,w)|^2). \end{aligned} \tag{2.6}$$

Write $a = (a_1, \cdots, a_{n-1}, b_1, b_{N-n})$, $b = (b_1, \cdots, b_{N-n})$ and define F_p^{**} by

$$\widetilde{f}_p^{**} = \frac{1}{q^*(z,w)} \big(\widetilde{f}_p^*(z,w) - a g_p^*(z,w)\big),\ \widetilde{g}_p^{**} = \frac{1}{q^*(z,w)} g_p^*. \tag{2.7}$$

Here we have set

$$q^*(z,w) = 1 + 2i\overline{a}\widetilde{f}_p^* + (r - i|a|^2) g_p^*(z,w),\ r = \frac{1}{2}\mathrm{Re}\big(\frac{\partial^2 g_p^*}{\partial w^2}(0)\big). \tag{2.8}$$

F_p^{**} has the following normalization, which is fundamentally important for the understanding of the geometric properties of F.

Lemma 2.1 (Lemma 5.3, [Hu99])**.** *Let F be a C^2-smooth CR map from $\partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$, $2 \le n \le N$. For each $p \in \partial\mathbb{H}_n$, there is an automorphism $\tau_p^{**} \in Aut_0(\mathbb{H}_N)$ such that $F_p^{**} := \tau_p^{**} \circ F_p$ satisfies the following normalization:*

$$f_p^{**} = z + \frac{i}{2} a_p^{**(1)}(z) w + o_{wt}(3),\ \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2),\ g_p^{**} = w + o_{wt}(4),\ \textit{with}$$

$$\langle \overline{z}, a_p^{**(1)}(z) \rangle |z|^2 = |\phi_p^{**(2)}(z)|^2.$$

Write $\mathcal{A}(p) = -2i(\frac{\partial^2 (f_p^{**})_l}{\partial z_j \partial w}|_0)_{1 \le j,l \le (n-1)}$ in the above lemma. In [Hu03], Huang defined the *geometric rank* of F at p, denoted by $Rk_F(p)$, to be the rank of the $(n-1) \times (n-1)$ matrix $\mathcal{A}(p)$. Now we can define the *geometric rank* of F to be $\kappa_0(F) = max_{p \in \partial\mathbb{H}_n} Rk_F(p)$. For a C^2 smooth CR map F from $\partial\mathbb{B}^n$ into $\partial\mathbb{B}^N$, the *geometric rank* of the map F is defined by the map $\rho_N^{-1} \circ F \circ \rho_n$. By [Hu03], $\kappa_0(F)$ depends only on the equivalence class of F and $\kappa_0(F) \le n-2$ when $N < \frac{n(n+1)}{2}$.

Let F_p^{***} be defined as follows:

$$F_P^{***} = \Big(f_p^{**}(zU, w)U^{-1}, \phi_p^{**}(zU, w)U^*, g_p^{**}(zU, w)\Big). \tag{2.9}$$

When $\kappa_0 \leq n-2$, F_p^{***} satisfies the following normalizations:

$$\begin{cases} f_j = z_j + \frac{i}{2}\mu_j z_j w + o_{wt}(3) \text{ for } j \leq \kappa_0, \\ f_j = z_j + o_{wt}(3) \text{ for } \kappa_0 < j \leq n-1, \\ \phi_{jk} = \mu_{jk} z_j z_k + \sum_{h=1}^{n-1} e_{h,jk} z_h w + d_{jk} w^2 + O(|(z,w)|^3) \text{ for } (j,k) \in \mathcal{S}_0, \\ \phi_{jk} = \sum_{h=1}^{n-1} e_{h,jk} z_h w + d_{jk} w^2 + O(|(z,w)|^3) \text{ for } (j,k) \in \mathcal{S}_1, \\ g = w + o_{wt}(4). \end{cases} \tag{2.10}$$

Here, for $1 \leq \kappa_0 \leq n-2$, we write $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$, the index set for all components of ϕ, where $\mathcal{S}_0 = \{(j,l) : 1 \leq j \leq \kappa_0, 1 \leq l \leq n-1, j \leq l\}$ and $\mathcal{S}_1 = \{(j,l) : j = \kappa_0 + 1, \kappa_0 + 1 \leq l \leq N - n - \frac{(2n-\kappa_0-1)\kappa_0}{2}\}$. Also, $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ *for* $j < l \leq \kappa_0$; and $\mu_{jl} = \sqrt{\mu_j}$ if $j \leq \kappa_0 < l$ or if $j = l \leq \kappa_0$.

Let $\tau \in \mathrm{Aut}_0(\mathbb{H}_n)$ and $\sigma \in \mathrm{Aut}_0(\mathbb{H}_n)$ be given by

$$\sigma(z,w) = \frac{(z - cw, w)}{q(z,w)},\ \tau(z^*, w^*) = \frac{(z^* + (c,0)w^*, w^*)}{q^*(z^*, w^*)} \tag{2.11}$$

with

$$\begin{aligned} q(z,w) &= 1 + 2i\langle \overline{c}, z\rangle - i|c|^2 w, \\ q^*(z^*, w^*) &= 1 - 2i\langle \overline{c}, z^*\rangle - i|c|^2 w^*, \\ c &= (c_1, \cdots, c_{n-1}). \end{aligned} \tag{2.12}$$

Then by suitably choosing c_j for $1 \leq j \leq \kappa_0$, we can make $F_p^{****} = \tau \circ F_P^{***} \circ \sigma$ still have the form (2.10). Furthermore, we can make $\frac{\partial^2 f_j}{\partial w^2}(0) = 0$ for $1 \leq j \leq \kappa_0$. In [HJX06], the authors proved the following normalization theorem for maps with geometric rank bounded by $n-2$, though only part of it is needed later:

Theorem 2.2. *Suppose that F is a rational proper holomorphic map from $\mathbb{H}_n$ into $\mathbb{H}_N$, which has geometric rank $1 \leq \kappa_0 \leq n-2$ with $F(0) = 0$. Then there are $\sigma \in Aut(\mathbb{H}_n)$ and $\tau \in Aut(\mathbb{H}_N)$ such that $\tau \circ F \circ \sigma$ takes the following form, which is still denoted by $F = (f, \phi, g)$ for convenience of notation:*

$$\begin{cases} f_l = \sum_{j=1}^{\kappa_0} z_j f_{lj}^*(z,w), \quad l \le \kappa_0, \\ f_j = z_j, \quad \kappa_0 + 1 \le j \le n-1, \\ \phi_{lk} = \mu_{lk} z_l z_k + \sum_{j=1}^{\kappa_0} z_j \phi_{lkj}^*, \quad (l,k) \in \mathcal{S}_0, \\ \phi_{lk} = \sum_{j=1}^{\kappa_0} z_j \phi_{lkj}^* = O_{wt}(3), \quad (l,k) \in \mathcal{S}_1, \\ g = w, \\ f_{lj}^*(z,w) = \delta_l^j + \frac{i\delta_l^j \mu_l}{2} w + b_{lj}^{(1)}(z) w + O_{wt}(4), \quad 1 \le l \le \kappa_0, \ \mu_l > 0, \\ \phi_{lkj}^*(z,w) = O_{wt}(2), \quad (l,k) \in \mathcal{S}_1. \end{cases} \tag{2.13}$$

Here, for $1 \le \kappa_0 \le n-2$, we write $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$, the index set for all components of ϕ, where $\mathcal{S}_0 = \{(j,l) : 1 \le j \le \kappa_0, 1 \le l \le n-1, j \le l\}$ and $\mathcal{S}_1 = \{(j,l) : j = \kappa_0 + 1, \kappa_0 + 1 \le l \le N - n - \frac{(2n-\kappa_0-1)\kappa_0}{2}\}$. Also, $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ for $j < l \le \kappa_0$; and $\mu_{jl} = \sqrt{\mu_j}$ if $j \le \kappa_0 < l$ or if $j = l \le \kappa_0$.

For later use, we will also set

$$\phi^{(1,1)}(z) = \sum_{j=1}^{\kappa_0} e_j z_j \text{ with } e_j \in \mathbb{C}^{\sharp(\mathcal{S})}, \ \phi_{kl}^{(1,1)}(z) = \sum_{j=1}^{\kappa_0} e_{j,kl} z_j \text{ with } (k,l) \in \mathcal{S}.$$

Next we define the degeneracy rank for any smooth CR map F from $\partial\mathbb{H}_n$ to $\partial\mathbb{H}_N$, which is an invariant integer introduced by Lamel [Lam01](see also [EHZ] and [Eb13]. In fact, the degeneracy rank is defined for more general maps).

For any point $p \in \partial\mathbb{H}_n$, we define an increasing sequence of linear subspaces $E_k(p) \subset \mathbb{C}^N$ for F,

$$E_k(p) = span_{\mathbb{C}}\{L^\alpha \hat{\rho}_{\overline{Z}} \circ F(p) \mid \ |\alpha| \le k\} \tag{2.14}$$

where $L^\alpha = L^{\alpha_1} ... L^{\alpha_{n-1}}$, $\alpha = (\alpha_1, ..., \alpha_{n-1})$, $|\alpha| = |\alpha_1| + ... + |\alpha_{n-1}|$, $L_j = \frac{\partial}{\partial z_j} + 2i\overline{z_j}\frac{\partial}{\partial w}$, $\hat{\rho}(Z, \overline{Z})$ is the defining function of the real hypersurface $\partial\mathbb{H}_N$, and $\hat{\rho}_{\overline{Z}} := \overline{\partial}\hat{\rho}$ is the complex gradient of ρ. Note that $\{L_j\}_{1 \le j \le n-1}$ form a basis of tangent vector fields of $(1,0)$ along $\partial\mathbb{H}_n$.

We define $d_1(p) := 0$ and

$$d_k(p) := \dim_{\mathbb{C}} \ E_k(p)/E_1(p). \tag{2.15}$$

Then we have a sequence of dimensions $d_1(p) = 0 \le d_2(p) \le d_3(p) \le ... \le d_k(p) \le ....$ Notice that the dimensions $d_j(p)$ is lower semi-continuous. By moving p to a nearby point p_0 if necessary, we may assume that all $d_l(p)$ are locally constant near p_0 and

$$d_2(p) < d_3(p) < ... < d_{l_0}(p) = d_{l_0+1}(p) = ... \tag{2.16}$$

for some l_0 with $1 \leq l_0 \leq N-n+1$. In other words, there exists an open subset U of $\partial\mathbb{H}_n$ on which all $d_l(p)$ are locally constant near p_0 and (2.16) holds. By [EHZ], we may call l_0 the *degeneracy rank* of F, and d_{l_0} the *degeneracy dimension* of F. These definitions depend on the open subset U. By minimizing l_0 among all such open sets, we can define degeneracy rank l_0 of F as an invariant. we also call $d_j(p)$ for $j \in [2, l_0]$ the j-th degeneracy dimension. The dimensions $d_j(p)$ $j = 1, \cdots, l_0$ can be interpreted as ranks of the CR second fundamental form of f and its covariant derivatives. The interested reader is referred to [Eb13, Section 2] for more details.

We end this section by recalling the CR Gauss map formulated in [CJL]. Let $F : \partial\mathbb{H}_n \to \partial\mathbb{H}_N$ be a rational CR map. Write $F(z,w) = (f(z,w), \phi(z,w), g(z,w))$ and set $L_j = \frac{\partial}{\partial z_j} + 2i\overline{z_j}\frac{\partial}{\partial w}$ for $1 \leq j \leq n-1$. Since F is a CR map, we have $\overline{L_j}f = \overline{L_j}\phi = \overline{L_j}g = 0$. Thus the matrix

$$\begin{pmatrix} L_1 f & L_1 g & L_1\phi \\ \vdots & \vdots & \vdots \\ L_{n-1}f & L_{n-1}g & L_{n-1}\phi \\ Tf & Tg & T\phi \end{pmatrix} \tag{2.17}$$

represents an element in the Grassmanian $G(n,N)$, which is Gauss map associate to the map. By an action of a non-singular $n \times n$ matrix, the element is equivalent to the unique matrix representation $(I_{n\times n}\ G)$, where $I_{n\times n}$ is the unit matrix and

$$G(z,w) = \begin{pmatrix} L_1 f & L_1 g \\ \vdots & \vdots \\ L_{n-1}f & L_{n-1}g \\ Tf & Tg \end{pmatrix}^{-1} \cdot \begin{pmatrix} L_1\phi_1 & \cdots & L_1\phi_{N-n} \\ \vdots & \vdots & \vdots \\ L_{n-1}\phi_1 & \cdots & L_{n-1}\phi_{N-n} \\ T\phi_1 & \cdots & T\phi_{N-n} \end{pmatrix}(z,w). \tag{2.18}$$

The CR Gauss map of the image $F(\partial\mathbb{H}_n)$ is defined by $\gamma : p \to G(p)$ for any $p \in \partial\mathbb{H}_n$.

3 Reduction of the degeneracy of the CR Gauss map

In this section, we will reduce the CR Gauss map condition to a proper form, which is crucial to the proof of our main theorems.

Theorem 3.1. *Let $F : \partial\mathbb{H}_n \to \partial\mathbb{H}_N$ be a C^2 CR immersion, the geometric rank of F at 0 is $\kappa_0 \in [1, n-1]$ which is maximal. We further suppose that F has the normal form (2.13). Then for any fixed $p = (z_0, w_0) \in \partial\mathbb{H}_n$ near the origin, the CR Gauss map equation $\gamma(z,w) =$*

$\gamma(z_0, w_0)$, for (z,w) close to (z_0, w_0), expressed in terms of F_p^{****}, takes the following form

$$\frac{\partial \phi_p^{****}}{\partial z_j} = O(|(z,w)|^2), \quad \frac{\partial \phi_p^{****}}{\partial w} = O(|(z,w)|^2). \tag{3.1}$$

For any fixed $p = (z_0, w_0) \in \partial \mathbb{H}_n$ near the origin, set $\widetilde{z} = z + z_0$ and $\widetilde{w} = w + w_0 + 2i\overline{z_0}z$. We also write

$$P(z,w) = \begin{pmatrix} L_1 f & L_1 g \\ \vdots & \vdots \\ L_{n-1} f & L_{n-1} g \\ Tf & Tg \end{pmatrix} (\widetilde{z}, \widetilde{w}), \; Q(z,w) = \begin{pmatrix} L_1 \phi \\ \vdots \\ L_{n-1} \phi \\ T\phi \end{pmatrix} (\widetilde{z}, \widetilde{w}). \tag{3.2}$$

Then CR Gauss map equation $\gamma(z,w) = \gamma(z_0, w_0)$ is equivalent to

$$Q(z,w) = P(z,w) P^{-1}(0) Q(0). \tag{3.3}$$

Next we express the system (3.3) in terms of F_p^{****} introduced in the preceding section through the following 5 steps.

Step I. Express the CR Gauss map equation in terms of F_p

By the construction of Huang in §2 of [Hu03], F_p defined by (2.2) takes the following form:

$$\begin{aligned} &\widetilde{f}_p(z,w) = \widetilde{f}(\widetilde{z}, \widetilde{w}) - \widetilde{f}(z_0, w_0), \\ &g_p(z,w) = g(\widetilde{z}, \widetilde{w}) - \overline{g(z_0, w_0)} - 2i\overline{\widetilde{f}(z_0, w_0)} \widetilde{f}(\widetilde{z}, \widetilde{w}). \end{aligned} \tag{3.4}$$

A direct computation shows that

$$\begin{aligned} &\frac{\partial \widetilde{f}_p}{\partial z_j}(z,w) = \big(\frac{\partial \widetilde{f}}{\partial z_j} + 2i\overline{z_{0j}} \frac{\partial \widetilde{f}}{\partial w}\big)(\widetilde{z}, \widetilde{w}), \\ &T\widetilde{f}_p(z,w) = (T\widetilde{f})(\widetilde{z}, \widetilde{w}), \\ &(Tg_p)(z,w) = (Tg)(\widetilde{z}, \widetilde{w}) - 2i\overline{\widetilde{f}(z_0, w_0)} \big(T\widetilde{f}\big)(\widetilde{z}, \widetilde{w}). \end{aligned} \tag{3.5}$$

Hence we infer

$$\begin{aligned} &\big(L_j \widetilde{f}\big)(\widetilde{z}, \widetilde{w}) = L_j \widetilde{f}_p(z,w), \; \big(T\widetilde{f}\big)(\widetilde{z}, \widetilde{w}) = T\widetilde{f}_p(z,w), \\ &(Tg)(\widetilde{z}, \widetilde{w}) = Tg_p(z,w) + 2i\overline{\widetilde{f}(z_0, w_0)} T\widetilde{f}_p(z,w). \end{aligned} \tag{3.6}$$

Applying T to these equations, we can further get

$$\begin{aligned}&\big(TL_j\widetilde{f}\big)(\widetilde{z},\widetilde{w})=TL_j\widetilde{f}_p(z,w),\ \big(T^2\widetilde{f}\big)(\widetilde{z},\widetilde{w})=T^2\widetilde{f}_p(z,w),\\&(T^2g)(\widetilde{z},\widetilde{w})=T^2g_p(z,w)+2i\overline{\widetilde{f}(z_0,w_0)}T^2\widetilde{f}_p(z,w).\end{aligned}\tag{3.7}$$

By (3.4) and (3.6), we obtain

$$\begin{aligned}\frac{\partial g_p}{\partial z_j}(z,w)=&\Big(\frac{\partial g}{\partial z_j}+2i\overline{z_{0j}}\frac{\partial g}{\partial w}-2i\overline{\widetilde{f}(z_0,w_0)}\big(\frac{\partial\widetilde{f}}{\partial z_j}+2i\overline{z_{0j}}\frac{\partial\widetilde{f}}{\partial w}\big)\Big)(\widetilde{z},\widetilde{w})\\=&(L_jg)(\widetilde{z},\widetilde{w})-2i\overline{z_j}(Tg)(\widetilde{z},\widetilde{w})-2i\overline{\widetilde{f}(z_0,w_0)}\big(L_j\widetilde{f}_p(z,w)-2i\overline{z_j}T\widetilde{f}(\widetilde{z},\widetilde{w})\big)\\=&(L_jg)(\widetilde{z},\widetilde{w})-2i\overline{z_j}Tg_p(z,w)-2i\overline{\widetilde{f}(z_0,w_0)}L_j\widetilde{f}_p(z,w).\end{aligned}\tag{3.8}$$

Thus

$$(L_jg)(\widetilde{z},\widetilde{w})=L_jg_p(z,w)+2i\overline{\widetilde{f}(z_0,w_0)}L_j\widetilde{f}_p(z,w).\tag{3.9}$$

Applying T on this equation, we further get

$$(TL_jg)(\widetilde{z},\widetilde{w})=TL_jg_p(z,w)+2i\overline{\widetilde{f}(z_0,w_0)}TL_j\widetilde{f}_p(z,w).\tag{3.10}$$

Write

$$\begin{aligned}P_p(z,w)&=\begin{pmatrix}L_1f_p(z,w) & L_1g_p(z,w)+2i\overline{\widetilde{f}(z_0,w_0)}L_1\widetilde{f}_p(z,w)\\ \vdots & \vdots\\ L_{n-1}f_p(z,w) & L_{n-1}g_p(z,w)+2i\overline{\widetilde{f}(z_0,w_0)}L_{n-1}\widetilde{f}_p(z,w)\\ Tf_p(z,w) & Tg_p(z,w)+2i\overline{\widetilde{f}(z_0,w_0)}T\widetilde{f}_p(z,w)\end{pmatrix},\\ Q_p(z,w)&=\begin{pmatrix}L_1\phi_p(z,w)\\ \vdots\\ L_{n-1}\phi_p(z,w)\\ T\phi_p(z,w)\end{pmatrix}.\end{aligned}\tag{3.11}$$

By (3.6), (3.9) and (3.11), (3.3) has the form

$$Q_p(z,w)=P_p(z,w)P_p^{-1}(0)Q_p(0).\tag{3.12}$$

Step II. Express the CR Gauss map equation in terms of F_p^*

Recall that F_p^* is defined by

$$F_p^*(z,w) = \frac{1}{\sqrt{\lambda(p)}} F_p(z,w) \begin{pmatrix} \overline{A^t(p)} & 0 \\ 0 & \frac{1}{\sqrt{\lambda(p)}} \end{pmatrix}. \tag{3.13}$$

Rewrite it as $F_p^*(z,w) = (f_p \ g_p \ \phi_p)(z,w) \cdot M(p)$, then $M(p)$ takes the following form:

$$\begin{pmatrix} M_1 & 0 & M_2 \\ 0 & \frac{1}{\lambda(p)} & 0 \\ M_3 & 0 & M_4 \end{pmatrix}. \tag{3.14}$$

Write

$$\hat{M}(p) = \begin{pmatrix} M_1 & -\frac{2i}{\lambda}\overline{f^t(z_0,w_0)} & M_2 \\ 0 & \frac{1}{\lambda} & 0 \\ M_3 & -\frac{2i}{\lambda}\overline{\phi^t(z_0,w_0)} & M_4 \end{pmatrix}. \tag{3.15}$$

Since $M(p)$ is independent of (z,w), we have

$$\begin{aligned} &\big(L_j f_p^* \ \ L_j g_p^* \ \ L_j \phi_p^*\big)(z,w) \\ =&\big(L_j f_p \ \ L_j g_p \ \ L_j \phi_p\big)(z,w) \cdot M(p) \\ =&\big(L_j f_p \ \ L_j g_p + 2i\overline{\widetilde{f}(z_0,w_0)} L_j \widetilde{f}_p \ \ L_j \phi_p\big)(z,w) \cdot \hat{M}(p). \end{aligned} \tag{3.16}$$

Similarly, we get

$$\big(T f_p^* \ \ T g_p^* \ \ T \phi_p^*\big)(z,w) = \big(T f_p \ \ T g_p + 2i\overline{\widetilde{f}(z_0,w_0)} T \widetilde{f}_p \ \ T \phi_p\big)(z,w) \cdot \hat{M}(p). \tag{3.17}$$

Set

$$P_p^*(z,w) = \begin{pmatrix} L_1 f_p^*(z,w) & L_1 g_p^*(z,w) \\ \vdots & \vdots \\ L_{n-1} f_p^*(z,w) & L_{n-1} g_p^*(z,w) \\ T{f^*}_p(z,w) & (T g_p^*)(z,w) \end{pmatrix}, Q_p^*(z,w) = \begin{pmatrix} L_1 \phi_p^*(z,w) \\ \vdots \\ L_{n-1} \phi_p^*(z,w) \\ T \phi_p^*(z,w) \end{pmatrix}. \tag{3.18}$$

Consider the $n \times N$ matrix $(P_p^* \ Q_p^*)(z,w)$. From (3.11), (3.12) and (3.18), we yield

$$\begin{aligned} (P_p^* \ Q_p^*)(z,w) =&(P_p \ Q_p)(z,w) \cdot \hat{M}(p) \\ =&\big(P_p(z,w) \ P_p(z,w) P_p^{-1}(0) Q_p(0)\big) \cdot \hat{M}(p) \\ =&P_p(z,w) P_p^{-1}(0) \cdot \big(P_p(0) \ Q_p(0)\big) \cdot \hat{M}(p) \\ =&P_p(z,w) P_p^{-1}(0) \cdot \big(P_p^*(0) \ Q_p^*(0)\big). \end{aligned} \tag{3.19}$$

Hence we know

$$P_p^*(z,w) = P_p(z,w)P_p^{-1}(0)P_p^*(0),\ Q_p^*(z,w) = P_p(z,w)P_p^{-1}(0)Q_p^*(0), \tag{3.20}$$

from which we deduce

$$Q_p^*(z,w) = P_p^*(z,w)\big(P_p^*(0)\big)^{-1}Q_p^*(0). \tag{3.21}$$

By the normalization properties of F_p^*, we know

$$P_p^*(0) = \begin{pmatrix} I & 0 \\ a & 1 \end{pmatrix},\ Q_p^*(0) = \begin{pmatrix} 0 \\ b \end{pmatrix}.$$

Here we have set $a = (a_1, \cdots, a_{n-1})$ and $b = (b_1, \cdots, b_{N-n})$, where a_j and b_k are defined by (2.6). Hence (3.21) takes the form

$$\begin{pmatrix} L_1\phi_p^*(z,w) \\ \vdots \\ L_{n-1}\phi_p^*(z,w) \\ T\phi_p^*(z,w) \end{pmatrix} = P_p^*(z,w)\begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} L_1 g_p^*(z,w) \\ \vdots \\ L_{n-1}g_p^*(z,w) \\ Tg_p^*(z,w) \end{pmatrix} b. \tag{3.22}$$

Step III. Express the CR Gauss map equation in terms of F_p^{**}

We further express this system in terms of F_p^{**}, which is defined as follows:

$$\widetilde{f}_p^{**} = \frac{1}{q^*(z,w)}\big(\widetilde{f}_p^*(z,w) - ag_p^*(z,w)\big),\ \widetilde{g}_p^{**} = \frac{1}{q^*(z,w)}g_p^*. \tag{3.23}$$

Here we have set

$$q^*(z,w) = 1 + 2i\overline{c}\widetilde{f}_p^* + (r - i|c|^2)g_p^*(z,w),\ c = \frac{\partial \widetilde{f}_p^*}{\partial w}(0) = (a,b),\ r = \frac{1}{2}\mathrm{Re}\big(\frac{\partial^2 g_p^*}{\partial w^2}(0)\big). \tag{3.24}$$

Write $q^{**}(z,w) = \frac{1}{q^*(z,w)}$, then (3.23) and (3.24) give

$$\begin{aligned} q^{**}(z,w) &= 1 - 2i\overline{c}\frac{\widetilde{f}_p^*}{q^*}(z,w) - (r - i|c|^2)\frac{g_p^*}{q^*}(z,w) \\ &= 1 - 2i\overline{c}f_p^{**}(z,w) - (r + i|c|^2)g_p^{**}(z,w). \end{aligned}$$

Together with (3.23), we know

$$\widetilde{f}_p^* = \frac{1}{q^{**}(z,w)}\big(\widetilde{f_p^{**}}(z,w) + cg_p^{**}(z,w)\big),\ g_p^* = \frac{1}{q^{**}(z,w)}g_p^{**}. \tag{3.25}$$

Applying L_j and T to (3.25), we yield

$$\begin{aligned}
L_j\phi_p^* =& \frac{1}{(q^{**}(z,w))^2}\big(L_j(\phi_p^{**}+bg_p^{**})q^{**}-(\phi_p^{**}+bg_p^{**})L_jq^{**}\big),\\
T\phi_p^* =& \frac{1}{(q^{**}(z,w))^2}\big(T(\phi_p^{**}+bg_p^{**})q^{**}-(\phi_p^{**}+bg_p^{**})Tq^{**}\big),\\
L_jg_p^* =& \frac{1}{(q^{**}(z,w))^2}\big(L_jg_p^{**}q^{**}-g_p^{**}L_jq^{**}\big),\\
Tg_p^* =& \frac{1}{(q^{**}(z,w))^2}\big(Tg_p^{**}q^{**}-g_p^{**}Tq^{**}\big).
\end{aligned}$$

Substituting these equations into (3.22), we get

$$\begin{aligned}
&L_j(\phi_p^{**}+bg_p^{**})q^{**}-(\phi_p^{**}+bg_p^{**})L_jq^{**} = b\big(L_jg_p^{**}q^{**}-g_p^{**}L_jq^{**}\big),\\
&T(\phi_p^{**}+bg_p^{**})q^{**}-(\phi_p^{**}+bg_p^{**})Tq^{**} = b\big(Tg_p^{**}q^{**}-g_p^{**}Tq^{**}\big).
\end{aligned}$$

A quick simplification gives

$$L_j\phi_p^{**}q^{**}-\phi_p^{**}L_jq^{**}=0,\ T\phi_p^{**}q^{**}-\phi_p^{**}Tq^{**}=0. \tag{3.26}$$

Notice that $\phi_p^{**}=O(|(z,w)|^2)$ and $q_p^{**}=1+O(|(z,w)|)$. Hence (3.26) takes the form

$$\frac{\partial}{\partial z_j}\phi_p^{**}=O(|(z,w)|^2),\ \frac{\partial}{\partial w}\phi_p^{**}=O(|(z,w)|^2). \tag{3.27}$$

Step IV. Express the CR Gauss map equation in terms of F_p^{***}

We express the system (3.27) in terms of the map F_P^{***} defined by (2.9). (3.27) takes the following form:

$$\Big(\frac{\partial\phi_p^{***}}{\partial z_j}\Big)\big(zU,w\big)=O(|(z,w)|^2),\ \Big(\frac{\partial\phi_p^{***}}{\partial w}\Big)\big(zU,w\big)=O(|(z,w)|^2). \tag{3.28}$$

Step V. Express the CR Gauss map equation in terms of F_p^{****}

Let $\tau\in\mathrm{Aut}_0(\mathbb{H}_n)$ and $\sigma\in\mathrm{Aut}_0(\mathbb{H}_N)$ be given by (2.11) so that $F_p^{****}=\tau\circ F_p^{***}\circ\sigma$. Then

$$\begin{aligned}
&\phi_p^{****}(z,w)=\phi_p^{***}\big(\sigma(z,w)\big)+O(|(z,w)|^3),\\
&\sigma(z,w)=(z-cw,w)+O(|(z,w)|^2).
\end{aligned} \tag{3.29}$$

Hence (3.28) takes the following form:

$$\frac{\partial\phi_p^{****}}{\partial z_j}=O(|(z,w)|^2),\quad \frac{\partial\phi_p^{****}}{\partial w}=O(|(z,w)|^2). \tag{3.30}$$

This finishes the proof of the theorem.

4 Some normalization properties

In this section, we will derive some properties for proper holomorphic maps between balls that will be essential for the proof of our main theorems.

Let $F : \partial\mathbb{H}_n \to \partial\mathbb{H}_N$ be a rational CR immersion. Assume the geometric rank κ_0 of F at 0 is maximal with $\kappa_0 \in [1, n-2]$. We also suppose that F has the following expansion near 0:

$$\begin{cases} f_j = z_j + \frac{i}{2}\mu_j z_j w + d_j w^2 + O(|(z,w)|^3) \text{ for } j \le \kappa_0, \\ f_j = z_j + d_j w^2 + O(|(z,w)|^3) \text{ for } \kappa_0 < j \le n-1, \\ \phi_{jk} = \mu_{jk} z_j z_k + \sum_{h=1}^{n-1} e_{h,jk} z_h w + d_{jk} w^2 + O(|(z,w)|^3) \text{ for } (j,k) \in \mathcal{S}_0, \\ \phi_{jk} = \sum_{h=1}^{n-1} e_{h,jk} z_h w + d_{jk} w^2 + O(|(z,w)|^3) \text{ for } (j,k) \in \mathcal{S}_1, \\ g = w + O(|(z,w)|^3). \end{cases} \tag{4.1}$$

Let $\tau \in \mathrm{Aut}_0(\mathbb{H}_n)$ and $\sigma \in \mathrm{Aut}_0(\mathbb{H}_n)$ be given by (2.11).

Lemma 4.1. *Let $\hat{F} = (\hat{f}, \hat{\phi}, g) := \tau \circ F \circ \sigma$, where τ and σ are given in (2.11), then $\hat{F}$ has the following expansion:*

$$\begin{cases} \hat{f}_j = z_j + \frac{i}{2}\mu_j z_j w + (d_j - \frac{i}{2}\mu_j c_j) w^2 + O(|(z,w)|^3) \text{ for } j \le \kappa_0, \\ \hat{f}_j = z_j + d_j w^2 + O(|(z,w)|^3) \text{ for } \kappa_0 < j \le n-1, \\ \hat{\phi}_{jk} = \mu_{jk}(z_j - c_j w)(z_k - c_k w) + \sum_{h=1}^{n-1} e_{h,jk}(z_h - c_h w) w + d_{jk} w^2 \\ \qquad + O(|(z,w)|^3) \text{ for } (j,k) \in \mathcal{S}_0, \\ \hat{\phi}_{jk} = \sum_{h=1}^{n-1} e_{h,jk}(z_h - c_h w) w + d_{jk} w^2 + O(|(z,w)|^3) \text{ for } (j,k) \in \mathcal{S}_1, \\ \hat{g} = w + O(|(z,w)|^3). \end{cases} \tag{4.2}$$

Proof. A direct computation from (2.11) shows that

$$\begin{aligned} \sigma(z,w) =& \Big(\big((z_j - c_j w) \cdot \big(1 - 2i\langle \bar{c}, z\rangle + i|c|^2 w\big) \big)_{1 \le j \le n-1}, w\big(1 - 2i\langle \bar{c}, z\rangle + i|c|^2 w\big) \Big) \\ & + O(|(z,w)|^3). \end{aligned} \tag{4.3}$$

For $1 \le j \le \kappa_0$ and $\kappa_0 + 1 \le k \le n-1$, we have

$$\begin{aligned} f_j \circ \sigma(z,w) =& \big(z_j - c_j w\big) \cdot \big(1 - 2i\langle \bar{c}, z\rangle + i|c|^2 w\big) + \frac{i}{2}\mu_j (z_j - c_j w) w \\ & + d_j w^2 + O(|(z,w)|^3), \\ f_k \circ \sigma(z,w) =& \big(z_k - c_k w\big) \cdot \big(1 - 2i\langle \bar{c}, z\rangle + i|c|^2 w\big) + d_k w^2 + O(|(z,w)|^3). \end{aligned} \tag{4.4}$$

Similarly, for $(j,l)\in\mathcal{S}_0$ and $(j',l')\in\mathcal{S}_1$, we get

$$\begin{aligned}\phi_{jl}\circ\sigma(z,w)&=\mu_{jl}(z_j-c_jw)(z_l-c_lw)+\sum_{h=1}^{n-1}e_{h,jl}(z_h-c_hw)w+d_{jl}w^2+O(|(z,w)|^3),\\ \phi_{j'l'}\circ\sigma(z,w)&=\sum_{h=1}^{n-1}e_{h,j'l'}(z_h-c_hw)w+d_{j'l'}w^2+O(|(z,w)|^3),\\ g\circ\sigma(z,w)&=w\big(1-2i\langle\overline{c},z\rangle+i|c|^2w\big)+O(|(z,w)|^3).\end{aligned}\tag{4.5}$$

Hence for $1\le j\le\kappa_0$ and $\kappa_0+1\le k\le n-1$, we get

$$\begin{aligned}f_j\circ\sigma(z,w)+c_jg\circ\sigma(z,w)=&z_j\cdot\big(1-2i\langle\overline{c},z\rangle+i|c|^2w\big)+\frac{i}{2}\mu_j(z_j-c_jw)w\\ &+d_jw^2+O(|(z,w)|^3),\\ f_k\circ\sigma(z,w)+c_kg\circ\sigma(z,w)=&z_k\cdot\big(1-2i\langle\overline{c},z\rangle+i|c|^2w\big)+d_kw^2+O(|(z,w)|^3).\end{aligned}\tag{4.6}$$

Substituting the formulas above into (2.11), we yield for $1\le j\le\kappa_0$ that

$$\begin{aligned}\hat{f}_j=&\Big(z_j\cdot\big(1-2i\langle\overline{c},z\rangle+i|c|^2w\big)+\frac{i}{2}\mu_j(z_j-c_jw)w+d_jw^2\Big)\\ &\cdot\big(1+2i\langle\overline{c},z-cw\rangle+i|c|^2w\big)+O(|(z,w)|^3)\\ =&z_j+\frac{i}{2}\mu_jz_jw+(d_j-\frac{i}{2}\mu_jc_j)w^2+O(|(z,w)|^3).\end{aligned}\tag{4.7}$$

Similarly, we obtain for $\kappa_0+1\le k\le n-1$ that

$$\hat{f}_k=z_k+d_kw^2+O(|(z,w)|^3).\tag{4.8}$$

For ϕ, we have

$$\begin{aligned}\hat{\phi}_{jl}=&\mu_{jl}(z_j-c_jw)(z_l-c_lw)+\sum_{h=1}^{\kappa_0}e_{h,jl}(z_h-c_hw)w\\ &+d_{jl}w^2+O(|(z,w)|^3)\text{ for }(j,l)\in\mathcal{S}_0,\\ \hat{\phi}_{jl}=&\sum_{h=1}^{\kappa_0}e_{h,jl}(z_h-c_hw)w+d_{jl}w^2+O(|(z,w)|^3)\text{ for }(j,l)\in\mathcal{S}_1.\end{aligned}\tag{4.9}$$

For g, we have

$$\begin{aligned}\hat{g}=&w\big(1-2i\langle\overline{c},z\rangle+i|c|^2w\big)\cdot(1+2i\langle\overline{c},z-cw\rangle+i|c|^2w\big)+O(|(z,w)|^3)\\ =&w+O(|(z,w)|^3).\end{aligned}\tag{4.10}$$

The proof of Lemma 4.1 is complete. □

As a consequence of the lemma, we can further normalize the map (2.13) such that

$$e_{1,1\alpha} = 0 \text{ for } \alpha > \kappa_0. \tag{4.11}$$

Indeed, if we choose $c_j = 0$ for $1 \le j \le \kappa_0$, then

$$\hat{\phi}_{j\alpha}^{(I_j+I_n)} = e_{j,j\alpha} - \sqrt{\mu_j} c_\alpha \text{for } 1 \le j \le \kappa_0 < \alpha \le n-1.$$

Thus we can choose c_α for $\kappa_0 + 1 \le \alpha \le n-1$ such that $\hat{\phi}_{1\alpha}^{(I_1+I_n)} = 0$ with $\kappa_0 < \alpha \le n-1$. By [HJX06], the map has the following form:

$$\begin{cases} f_{p,j}^{(****)} = z_j + \frac{i}{2}\mu_{p,j} z_j w + O(|(z,w)|^3) \text{ for } j \le \kappa_0, \\ f_{p,j}^{(****)} = z_j \text{ for } \kappa_0 < j \le n-1, \\ \phi_{p,jk}^{(****)} = \mu_{p,jk} z_j z_k + \sum\limits_{h=1}^{\kappa_0} e_{ph,jk} z_h w + O(|(z,w)|^3) \text{ for } (j,k) \in \mathcal{S}_0, \\ \phi_{p,jk}^{(****)} = \sum\limits_{h=1}^{n-1} e_{ph,jk} z_h w + O(|(z,w)|^3) \text{ for } (j,k) \in \mathcal{S}_1, \\ g_p^{(****)} = w. \end{cases} \tag{4.12}$$

Here $e_{p1,1\alpha} = 0$ for $\kappa_0 + 1 \le \alpha \le n-1$. Write

$$\Phi_p^{****(1,1)} = \big(\sum_{h=1}^{\kappa_0} e_{ph,jk} \big)_{(j,k)\in\mathcal{S}}.$$

Next, we recall some relations derived by analyzing the Chern-Moser equation. The following relations are obtained in [HJY14] for the geometric rank equal to two case, which in fact holds true independent of the geometric rank and the codimension of the maps.

As in [HJY14], write $\xi_j(z) = \overline{e_j} \cdot \Phi_0^{(2,0)}(z)$. From [HJY14, (3.5)], we know

$$\overline{z} f^{(2,1)}(z) = -\overline{(z_1, ..., z_{\kappa_0})} \cdot \xi(z). \tag{4.13}$$

By a similar argument as in [HJY14, (4.3)], we obtain

$$\phi^{(1,2)} \in Span\{e_1, \cdots, e_{\kappa_0}\}. \tag{4.14}$$

From [HJY14, (4.10)], we know

$$2\mathrm{Re}\big(\overline{z} f^{(1,2)}(z)\big) + |f^{(1,1)}(z)|^2 + |\phi^{(1,1)}(z)|^2 = 0. \tag{4.15}$$

With these preparations, we arrive at the following main theorem of this section.

Theorem 4.2. *Let $F : \partial\mathbb{H}_n \to \partial\mathbb{H}_N$ be a CR immersion as in Theorem 1.1. For generic p around 0, the associated vector $\Phi_p^{****(1,1)}(z) \not\equiv 0$.*

Proof. By Theorem 2.2, we can suppose that near 0, the expansion has the form (2.13). Since $\Phi_p^{****(1,1)}(z)$ is a smooth function with respect to p and $\Phi_0^{****(1,1)}(z) = \Phi^{(1,1)}(z)$ by notation, it suffices to prove $\Phi^{(1,1)}(z) \neq 0$. Assume by contradiction that $\Phi^{(1,1)}(z) \equiv 0$. By our notation, this implies $e = \xi = 0$. From (4.13) and (4.14), we know

$$f^{(2,1)} = \phi^{(1,2)} = 0. \tag{4.16}$$

Next we would like to give some asymptotic properties of the coefficients F_p^{****}. For $1 \leq j \leq \kappa_0$, we have

$$\begin{aligned} E_j(p) =& \big(\frac{\partial \widetilde{f}_p}{\partial z_j}\big)\big|_0 = L_j(\widetilde{f})(p) = \big((\frac{\partial}{\partial z_j} + 2i\overline{z_{0j}}\frac{\partial}{\partial w})\widetilde{f}\big)(p) \\ :=& \big(E_j^{[0]}(p), E_j^{[1]}(p), \cdots, E_j^{[k_0+1]}(p)\big). \end{aligned} \tag{4.17}$$

Here we have set

$$\begin{aligned} &E_j^{[0]}(p) = L_j(f)(p) \in \mathbb{C}^{n-1}, \\ &E_j^{[k]}(p) = \big(L_j(\phi_{kk})(p), L_j(\phi_{k(k+1)})(p), \cdots, L_j(\phi_{k(n-1)})(p)\big) \in \mathbb{C}^{n-k}. \end{aligned}$$

Then

$$\begin{aligned} E_j^{[0]}(p) &= \big((\frac{\partial}{\partial z_j} + 2i\overline{z_{0j}}\frac{\partial}{\partial w})f\big)(p) \\ &= \Big(0, \cdots, 0, 1 + \frac{i}{2}\mu_j u_0 \ (j\text{-th position}), 0, \cdots, 0\Big) + O(2). \end{aligned} \tag{4.18}$$

and

$$E_j^{[k]}(p) = \begin{cases} (0, \cdots, 0, \mu_{kj} z_{0k}, 0, \cdots, 0) + O(2), & k < j, \\ (2\mu_{kk} z_{0k}, \mu_{k(k+1)} z_{0(k+1)}, \cdots, \mu_{k(n-1)} z_{0(n-1)}) + O(2), & k = j, \\ O(2), & k > j. \end{cases} \tag{4.19}$$

By the definition of C_{ij} (see [Hu99, p.17]), the asymptotic expansion of $E_j(p)$ given in (4.18) and (4.19), we can suppose that C_{jk} has the following form

$$C_{jk} = \big(o(1), \cdots, o(1), 1 + o(1), 0, \cdots, 0\big),$$

where $1+o(1)$ is in the position corresponding to that of ϕ_{jk} in $\widetilde{f}$. Since A defined by (2.4) is unitary, we can use the implicit function theory to get that

$$C_{jk}=\begin{cases}(0,\cdots,0,-\mu_{jk}\overline{z_{0k}},0,\cdots,0,-\mu_{jk}\overline{z_{0j}},0,\cdots,0,1,0,\cdots,0)+O(2) & j<k, j\le\kappa_0,\\ (0,\cdots,0,-2\mu_{jj}\overline{z_{0j}},0,\cdots,0,1,0,\cdots,0)+O(2) & j=k,\\ (0,\cdots,0,1,0,\cdots,0)+O(2) & j=\kappa_0+1,\end{cases} \tag{4.20}$$

From (3.7) and $f^{(2,1)}(z)\equiv 0$, we know for $1\le j\le\kappa_0$ the following

$$f_{p,j}^{(1,1)}=\sum_{k=1}^{n-1}L_kTf_j(p)z_k=\frac{i}{2}\mu_jz_j+\sum_{k=1}^{\kappa_0}2f_j^{(I_k+2I_n)}u_0z_k+O(|(z_0,u_0)|^2)z. \tag{4.21}$$

For $\kappa_0+1\le j\le n-1$, we have

$$f_{p,j}^{(1,1)}=\sum_{k=1}^{n-1}L_kTf_j(p)z_k=0. \tag{4.22}$$

For $1\le j\le n-1$, we have

$$f_{p,j}^{(0,2)}=\frac{1}{2}T^2f_{p,j}(0)=\frac{1}{2}T^2f_j(p)=\sum_{k=1}^{\kappa_0}f_j^{(I_k+2I_n)}z_{0k}+O(|(z_0,u_0)|^2). \tag{4.23}$$

From (3.7), we know

$$L_jT\phi_p(0)=\frac{\partial^2}{\partial z_j\partial w}\phi(p)+2i\overline{z_{0j}}\frac{\partial^2}{\partial w^2}\phi(p),\ T^2\phi_p(0)=\frac{1}{2}\frac{\partial^2}{\partial w^2}\phi(p). \tag{4.24}$$

Thus we get

$$\phi_{p,jl}^{(1,1)}=\sum_{k=1}^{\kappa_0}2\phi_{jl}^{(2I_k+I_n)}z_{0k}z_k+\sum_{k\ne h,k,h=1}^{n-1}\phi_{jl}^{(I_h+I_k+I_n)}z_{0h}z_k+O(|(z_0,u_0)|^2)z \tag{4.25}$$

and

$$\phi_{p,jl}^{(0,2)}=\phi_{jl}^{(1,2)}(z_0)+O(|(z_0,u_0)|^2)=O(|(z_0,u_0)|^2). \tag{4.26}$$

Here the last equality follows from (4.16). From (3.10), we know

$$\begin{aligned}g_p^{(1,1)}&=\sum\left(L_jTg(p)-2i\overline{\widetilde{f}(p)}\cdot L_jT\widetilde{f}(p)\right)z_j=O(|(z_0,u_0)|)z,\\ g_p^{(0,2)}&=O(|(z_0,u_0)|).\end{aligned} \tag{4.27}$$

Notice that $\lambda(p) = 1 + O(2)$, thus for $1 \le j \le \kappa_0$,

$$\begin{aligned}\phi^{*(1,1)}_{p,jj} =&\big(\frac{1}{\sqrt{\lambda(p)}}\widetilde{f}_p \cdot \overline{C^t_{jj}}\big)^{(1,1)}\\ =&f^{(1,1)}_{p,j}\cdot(-2\mu_{jj}z_{0j}) + \phi^{(1,1)}_{p,jj} + O(|(z_0,u_0)|^2)z\\ =&-i\mu_j\mu_{jj}z_{0j}z_j + \sum_{k=1}^{\kappa_0} 2\phi^{(2I_k+I_n)}_{jj} z_{0k}z_k + \sum_{k\neq h,k,h=1}^{n-1} \phi^{(I_h+I_k+I_n)}_{jj} z_{0h}z_k\\ &+ O(|(z_0,u_0)|^2)z.\end{aligned} \tag{4.28}$$

Similarly, we have

$$\begin{aligned}f^{*(0,2)}_{p,j} =&\big(\frac{1}{\lambda(p)}\widetilde{f}_p \cdot \overline{E^t_j}\big)^{(0,2)} = \sum_{k=1}^{\kappa_0} f^{(I_k+2I_n)}_j z_{0k} + O(|(z_0,u_0)|^2),\\ \phi^{*(0,2)}_{p,jj} =&O(|(z_0,u_0)|^2).\end{aligned} \tag{4.29}$$

Note that

$$a, r = o(1),\ U = I + o(1),\ U^* = I + o(1),\ \phi^{*(1,1)}_{p,jk} = o(1),\ f^{*(0,2)}_{p,j} = o(1),\ \phi^{*(0,2)}_{p,jk} = o(1).$$

A straightforward computation shows that

$$\begin{aligned}\phi^{***(1,1)}_{p,jj} =&-i\mu_j\mu_{jj}z_{0j}z_j + \sum_{k=1}^{\kappa_0} 2\phi^{(2I_k+I_n)}_{jj} z_{0k}z_k + \sum_{k\neq h,k,h=1}^{n-1} \phi^{(I_h+I_k+I_n)}_{jj} z_{0h}z_k\\ &+ O(|(z_0,u_0)|^2)z,\\ f^{***(0,2)}_{p,j} =&\sum_{k=1}^{\kappa_0} f^{(I_k+2I_n)}_j z_{0k} + O(|(z_0,u_0)|^2),\\ \phi^{***(0,2)}_{p,jj} =&O(|(z_0,u_0)|^2).\end{aligned} \tag{4.30}$$

By Lemma 4.1, to make $f^{****(0,2)}_p = 0$, we need to choose c_j for $1 \le j \le \kappa_0$ such that

$$c_j = -\frac{2i}{\mu_j} f^{***(0,2)}_{p,j} = -\frac{2i}{\mu_j}\sum_{k=1}^{\kappa_0} f^{(I_k+2I_n)}_j z_{0k} + O(|(z_0,u_0)|^2). \tag{4.31}$$

From Lemma 4.1 and (4.30), we know

$$\begin{aligned}\phi^{****(1,1)}_{p,jj}(z) =&-i\mu_j\mu_{jj}z_{0j}z_j + \sum_{k=1}^{\kappa_0} 2\phi^{(2I_k+I_n)}_{jj} z_{0k}z_k + \sum_{k\neq h,k,h=1}^{n-1} \phi^{(I_h+I_k+I_n)}_{jj} z_{0h}z_k\\ &-2\sqrt{\mu_j}c_jz_j + O(|(z_0,u_0)|^2)z.\end{aligned} \tag{4.32}$$

Thus the coefficients of z_k term in $\phi^{****(1,1)}_{p,jj}(z)$ is the following:

$$\begin{aligned} I_j(z_0,u_0) :=& -i\mu_j\mu_{jj}z_{0j} + 2\phi^{(2I_j+I_n)}_{jj}z_{0j} + \sum_{k\neq j,k=1}^{n-1}\phi^{(I_k+I_j+I_n)}_{jj}z_{0k} + \frac{4i}{\sqrt{\mu_j}}\sum_{k=1}^{\kappa_0} f^{(I_k+2I_n)}_j z_{0k} \\ &+ O(|(z_0,u_0)|^2), \\ I_k(z_0,u_0) :=& 2\phi^{(2I_k+I_n)}_{jj}z_{0k} + \sum_{h\neq k}\phi^{(I_h+I_k+I_n)}_{jj}z_{0h} + O(|(z_0,u_0)|^2) \text{ for } k\neq j. \end{aligned} \tag{4.33}$$

If $\phi^{****(1,1)}_{p,jj}(z)\equiv 0$ in a neighborhood of 0, then $I_h(z_0,u_0)\equiv 0$ in a a neighborhood of 0 for $1\leq h\leq n-1$. From $I_k(z_0,u_0)\equiv 0$ for $k\neq j$, we know $\phi^{(I_k+I_h+I_n)}_{jj}=0$ for $k\neq j$ and $1\leq j\leq n-1$. From $I_j(z_0,u_0)\equiv 0$, we get

$$\begin{aligned} &f^{(I_k+2I_n)}_j = 0 \text{ for } k\neq j, \\ &-i\mu_j\mu_{jj} + 2\phi^{(2I_j+I_n)}_{jj} + \frac{4i}{\sqrt{\mu_j}}f^{(I_j+2I_n)}_j = 0. \end{aligned} \tag{4.34}$$

Thus

$$\begin{aligned} \phi^{***(1,1)}_{p,jj} &= \Big(-i\mu_j\mu_{jj} + 2\phi^{(2I_j+I_n)}_{jj}\Big)z_{0j}z_j, \\ c_j &= -\frac{2i}{\mu_j}f^{(I_j+2I_n)}_j z_{0j} + O(|(z_0,u_0)|^2). \end{aligned} \tag{4.35}$$

For the pair (j,l) satisfying $1\leq j\leq l\leq\kappa_0$, by the asymptotic expansion of C_{jl}, we get

$$\begin{aligned} \phi^{*(1,1)}_{p,jl} =& \Big(\frac{1}{\sqrt{\lambda(p)}}\widetilde{f}_p\cdot\overline{C^t_{jl}}\Big)^{(1,1)} \\ =& f^{(1,1)}_{p,j}\cdot(-\mu_{jl}z_{0l}) + f^{(1,1)}_{p,l}\cdot(-\mu_{jl}z_{0j}) + \phi^{(1,1)}_{p,jl} + O(|(z_0,u_0)|^2)z \\ =& -\frac{i}{2}\mu_j\mu_{jl}z_{0l}z_j - \frac{i}{2}\mu_l\mu_{jl}z_{0j}z_l + \sum_{k=1}^{\kappa_0}2\phi^{(2I_k+I_n)}_{jl}z_{0k}z_k \\ &+ \sum_{k\neq h,k,h=1}^{n-1}\phi^{(I_h+I_k+I_n)}_{jl}z_{0h}z_k + O(|(z_0,u_0)|^2)z. \end{aligned} \tag{4.36}$$

We further know

$$\begin{aligned}\phi_{p,jl}^{****(1,1)}(z) =& -\frac{i}{2}\mu_j\mu_{jl}z_{0l}z_j - \frac{i}{2}\mu_l\mu_{jl}z_{0j}z_l + \sum_{k=1}^{\kappa_0} 2\phi_{jl}^{(2I_k+I_n)}z_{0k}z_k \\ &+ \sum_{k\neq h,k,h=1}^{n-1}\phi_{jl}^{(I_h+I_k+I_n)}z_{0h}z_k - \mu_{jl}c_l z_j - \mu_{jl}c_j z_l + O(|(z_0,u_0)|^2)z.\end{aligned} \tag{4.37}$$

By comparing the coefficients of z_k term in $\phi_{p,jl}^{****(1,1)}(z)$, we obtain

$$\begin{aligned}&\phi_{jl}^{(I_h+I_k+I_n)} = 0 \text{ for } h<l,\ (h,k)\neq(j,j),(j,l),(l,j),(l,l). \\ &-\frac{i}{2}\mu_j\mu_{jl}z_{0l} + 2\phi_{jl}^{(2I_j+I_n)}z_{0j} + \sum_{h\neq j,h=1}^{n-1}\phi_{jl}^{(I_j+I_h+I_n)}z_{0h} - \mu_{jl}c_l = 0, \\ &-\frac{i}{2}\mu_l\mu_{jl}z_{0j} + 2\phi_{jl}^{(2I_l+I_n)}z_{0l} + \sum_{h\neq l,h=1}^{n-1}\phi_{jl}^{(I_l+I_h+I_n)}z_{0h} - \mu_{jl}c_j = 0.\end{aligned} \tag{4.38}$$

By the formula for c_j in (4.35), we know $\phi_{jl}^{(I_k+I_h+I_n)} = 0$ for $(k,h)\neq(j,l),(l,j)$ and

$$\begin{aligned}&-\frac{i}{2}\mu_j\mu_{jl} + \phi_{jl}^{(I_j+I_l+I_n)} + \frac{2i\mu_{jl}}{\mu_l}f_l^{(I_l+2I_n)} = 0, \\ &-\frac{i}{2}\mu_l\mu_{jl} + \phi_{jl}^{(I_j+I_l+I_n)} + \frac{2i\mu_{jl}}{\mu_j}f_j^{(I_j+2I_n)} = 0.\end{aligned} \tag{4.39}$$

Eliminating $\phi_{jl}^{(I_j+I_l+I_n)}$ in the above system, we get

$$\frac{i}{2}\mu_j + \frac{2i}{\mu_j}f_j^{(I_j+2I_n)} = \frac{i}{2}\mu_l + \frac{2i}{\mu_l}f_l^{(I_l+2I_n)}. \tag{4.40}$$

Write

$$A_i = f_i^{(I_i+2I_n)},\ B_i = \phi_{ii}^{(2I_i+I_n)}. \tag{4.41}$$

Then by (4.34) and (4.40),

$$\begin{aligned}&-i\mu_j\mu_{jj} + 2B_j + \frac{4i}{\sqrt{\mu_j}}A_j = 0. \\ &\frac{i}{2}\mu_j + \frac{2i}{\mu_j}A_j = \frac{i}{2}\mu_l + \frac{2i}{\mu_l}A_l.\end{aligned} \tag{4.42}$$

From (4.39), we get

$$\phi_{jl}^{(I_j+I_l+I_n)} = \frac{i}{2}\mu_j\mu_{jl} - \frac{2i\mu_{jl}}{\mu_l}A_l. \tag{4.43}$$

By the asymptotic expansion of $C_{j\alpha}$ for $1 \le j \le \kappa_0 < \alpha \le n-1$, we get

$$\begin{aligned}\phi_{p,j\alpha}^{*(1,1)} =& (\frac{1}{\sqrt{\lambda(p)}}\widetilde{f}_p \cdot \overline{C_{j\alpha}^t})^{(1,1)} \\ =& f_{p,j}^{(1,1)} \cdot (-\mu_{j\alpha}z_{0\alpha}) + \phi_{p,j\alpha}^{(1,1)} + O(|(z_0,u_0)|^2)z \\ =& -\frac{i}{2}\mu_j\mu_{j\alpha}z_{0\alpha}z_j + \sum_{k=1}^{\kappa_0} 2\phi_{j\alpha}^{(2I_k+I_n)}z_{0k}z_k \\ &+ \sum_{k\neq h,k,h=1}^{n-1} \phi_{j\alpha}^{(I_h+I_k+I_n)}z_{0h}z_k + O(|(z_0,u_0)|^2)z.\end{aligned} \tag{4.44}$$

We further know

$$\begin{aligned}\phi_{p,j\alpha}^{****(1,1)}(z) =& -\frac{i}{2}\mu_j\mu_{j\alpha}z_{0\alpha}z_j + \sum_{k=1}^{\kappa_0} 2\phi_{j\alpha}^{(2I_k+I_n)}z_{0k}z_k \\ &+ \sum_{k\neq h,k,h=1}^{n-1} \phi_{j\alpha}^{(I_h+I_k+I_n)}z_{0h}z_k - \mu_{j\alpha}c_\alpha z_j - \mu_{j\alpha}c_j z_\alpha + O(|(z_0,u_0)|^2)z.\end{aligned} \tag{4.45}$$

By comparing the coefficients of z_k term in $\phi_{p,j\alpha}^{****(1,1)}(z)$, we obtain

$$\begin{aligned}&\phi_{j\alpha}^{(I_h+I_k+I_n)} = 0 \text{ for } (h,k) \neq (j,j),(j,\alpha),(\alpha,j). \\ &-\frac{i}{2}\mu_j\mu_{j\alpha}z_{0\alpha} + 2\phi_{j\alpha}^{(2I_j+I_n)}z_{0j} + \sum_{h\neq j,h=1}^{n-1}\phi_{j\alpha}^{(I_j+I_h+I_n)}z_{0h} - \mu_{j\alpha}c_\alpha = 0, \\ &\sum_{h\neq\alpha,h=1}^{n-1}\phi_{j\alpha}^{(I_\alpha+I_h+I_n)}z_{0h} - \mu_{j\alpha}c_j = 0.\end{aligned} \tag{4.46}$$

By the formula for c_j, we know $\phi_{j\alpha}^{(I_h+I_k+I_n)} = 0$ for $(h,k) \neq (j,\alpha),(\alpha,j)$ and

$$\begin{aligned}&-\frac{i}{2}\mu_j\mu_{j\alpha}z_{0\alpha} + \phi_{j\alpha}^{(I_j+I_\alpha+I_n)}z_{0\alpha} = \mu_{j\alpha}c_\alpha, \\ &\phi_{j\alpha}^{(I_j+I_\alpha+I_n)} - \mu_{j\alpha}c_j = 0.\end{aligned} \tag{4.47}$$

Hence

$$\phi_{j\alpha}^{(I_j+I_\alpha+I_n)} = \mu_{j\alpha}(-\frac{2i}{\mu_1})f_j^{(I_j+2I_n)} = -\frac{2i}{\sqrt{\mu_j}}A_j. \tag{4.48}$$

By considering $\phi_{p,\alpha\beta}^{****(1,1)}(z)$ for $(\alpha,\beta)\in\mathcal{S}_1$, we know $\phi_{\alpha\beta}^{(2,1)}=0$.

Furthermore, by [HJY,(4.10)], we have

$$2\mathrm{Re}\Big(\overline{z}f^{(1,2)}(z)\Big) + |f^{(1,1)}(z)|^2 + |\phi^{(1,1)}(z)|^2 = 0.$$

Making use of (2.13), (4.34) and (4.41), we get

$$\begin{aligned} &f_j^{(1,1)} = -\frac{i}{2}\mu_j z_j,\ \ f_j^{(1,2)} = A_j z_j \text{ for } 1\le j\le\kappa_0,\\ &f_k^{(1,1)} = f_k^{(1,2)} = \phi^{(1,1)} = 0 \text{ for } \kappa_0+1\le j\le n-1. \end{aligned}$$

Substituting these relations back to (4), we obtain $\mathrm{Re}(A_j) = -\frac{\mu_j^2}{8}$. Combining this with the latter equation of (4.42), and collecting the imaginary part of its both sizes, we further obtain $\mu_j=\mu_k$. Together with (4.42), we yield $A_j=A_k$ and $B_j=B_k$.

On the other hand, by [HJY,(4.17)], we know

$$2\Big(-2\overline{z}f^{(1,2)}(z)|z|^2 + i\overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(2,1)}(z)\Big)|z|^2 + |\phi^{(3,0)}(z)|^2 = 0. \tag{4.49}$$

Substituting into this equation, we get

$$\begin{aligned} &-4\sum_{j=1}^{\kappa_0}A_j|z_j|^2|z|^4 + 2i\Big\{\sum_{1\le j\le\kappa_0}\overline{\mu_{jj}z_j^2}\cdot B_j z_j^2 + \sum_{1\le j<k\le\kappa_0}\overline{\mu_{jk}z_jz_k}\cdot(\frac{i}{2}\mu_j\mu_{jk} - \frac{2i\mu_{jk}}{\mu_k}A_k)z_jz_k\\ &+\sum_{1\le j\le\kappa_0<\alpha\le n-1}\overline{\mu_{j\alpha}z_jz_\alpha}\cdot(-\frac{2i}{\sqrt{\mu_j}}A_j)z_jz_\alpha\Big\}|z|^2 + |\Phi_1^{(3,0)}(z)|^2 = 0. \end{aligned} \tag{4.50}$$

After a direct simplification, we obtain:

$$|\Phi_1^{(3,0)}(z)|^2 = (\sum_{1\le j\le\kappa_0}\mu_j|z_j|^2)^2|z|^2. \tag{4.51}$$

This means that $(\Phi_1^{(3,0)}(z))$ is a vector of dimension $\frac{1}{2}\kappa_0(\kappa_0+1)(n-\kappa_0)+\frac{1}{6}\kappa_0(\kappa_0+1)(\kappa_0+2)$. Hence the third degeneracy dimension is

$$\begin{aligned} &n+(n-1)+\cdots+(n-\kappa_0)+\frac{1}{2}\kappa_0(\kappa_0+1)(n-\kappa_0)+\frac{1}{6}\kappa_0(\kappa_0+1)(\kappa_0+2)\\ =&\frac{\kappa_0}{6}\Big(3(\kappa_0+3)n-(\kappa_0+1)(2\kappa_0+1)\Big). \end{aligned} \tag{4.52}$$

□

5 Proof of the main theorems

With (3.1) and Theorem 4.2 at our disposal, we are now in a position to prove our main theorems.

Proof of Theorem 1.1. We prove Theorem 1.1 by absurdity. Suppose that the geometric rank of the map is $\kappa_0 \in [1, n-2]$ and the map satisfies the degeneracy rank or the degeneracy dimension conditions. We will prove that the CR Gauss map of the map F must be non-degenerate. Write $\mathcal{M} = \{p \in \partial\mathbb{B}^n : Rk_F(p) = \kappa_0\}$. It will suffice to show the non-degeneracy of the Gauss map for $p_0 \in \mathcal{M}$. This is because, if so, $dim_{\mathbb{R}}\gamma(\mathcal{M}) = 2n-1$ while $dim_{\mathbb{R}}\gamma(\partial\mathbb{B}^n \setminus \mathcal{M}) \leq dim_{\mathbb{R}}\partial\mathbb{B}^n \setminus \mathcal{M} \leq 2n-2$, which would be non-generic. Due to Theorem 2.2 and Theorem 4.2, we also suppose at $p_0 \in \mathcal{M}$, F has the form (4.12) and has the additional condition $\phi_{p_0}^{****(1,1)} \not\equiv 0$. Without loss of generality, we suppose that $p_0 = 0$.

For every p close to 0, write

$$\begin{aligned} \frac{\partial \phi_{p,kl}^{****(2)}}{\partial z_j} &= \sum_{h=1}^{n-1} \Gamma_{j,kl}^{[h]}(p) z_h + \Gamma_{j,kl}^{[n]}(p) w + O(2), \\ T\phi_p^{****(2)} &= \sum_{h=1}^{n-1} \Gamma_{n,kl}^{[h]}(p) z_h + \Gamma_{j,kl}^{[n]}(p) w + O(2). \end{aligned} \tag{5.1}$$

Denote by $\Upsilon(p)$ the following $n(N-n) \times n$ matrix

$$\Upsilon(p) = \left(\Gamma_{j,kl}^{[1]}(p) \ \Gamma_{j,kl}^{[2]}(p) \ \cdots \ \Gamma_{j,kl}^{[n]}(p)\right)_{1 \leq j \leq n, (k,l) \in \mathcal{S}}. \tag{5.2}$$

By our normalization properties, we know, for $1 \leq j \leq \kappa_0$ and $\kappa_0 + 1 \leq \alpha \leq n-1$, the following

$$\frac{\partial}{\partial z_1}\phi_{1\alpha}^{(2)} = \mu_{1\alpha} z_\alpha, \ \frac{\partial}{\partial z_\alpha}\phi_{j\alpha}^{(2)} = \mu_{j\alpha} z_j. \tag{5.3}$$

If $e_{j,kl} \neq 0$, then $\frac{\partial}{\partial z_j}\phi_{kl}^{(2)} = \sum_{h=1}^{n-1} \lambda_h z_h + \tau w$ for some $\tau \neq 0$. Hence we must have $\Upsilon(0)$ is of rank n. Notice that $\Upsilon(p) = \Upsilon(0) + o(1)$, hence when p is sufficiently close to 0, $\Upsilon(p)$ is also of rank n.

By the implicit function theorem, there is a small neighborhood U of 0, such that for every point $p \in U$, (3.3) has only one solution in some neighborhood V. Choose $U_1 \subset U$ sufficiently small such that the solutions are contained in V_1 which is also very small and $(z + \hat{z}, w + \hat{w} + 2i\bar{\hat{z}}z) \in U \cap V$ for $(\hat{z}, \hat{w}) \in U_1$ and $(z, w) \in V_1$. Then for generic point $(z_0, w_0) \in U_1$, (3.3) has only one solution (z, w) with $(\widetilde{z}, \widetilde{w}) \in U$, which means that the CR Gauss map is non-degenerate. □

Proof of Theorem 1.2. Since there are n functions in the mapping containing linear terms, we have

$$
\begin{aligned}
d_3 \leq N - n &< \frac{1}{2}(\kappa_0+1)(\kappa_0+2)n - \frac{1}{6}\kappa_0(\kappa_0+1)(2\kappa_0+1) - n \\
&= \frac{\kappa_0}{6}\Big(3(\kappa_0+3)n - (\kappa_0+1)(2\kappa_0+1)\Big).
\end{aligned}
$$

Namely, the condition in (2) of Theorem 1.1 holds true and the proof of Theorem 1.2 follows directly.

□

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