Rigidity theorems by capacities and kernels

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Abstract

For any open hyperbolic Riemann surface X, the Bergman kernel K, the logarithmic capacity c_β, and the analytic capacity c_B satisfy the inequality chain $\pi K \geq c_\beta^2 \geq c_B^2$; moreover, equality holds at a single point between any two of the three quantities if and only if X is biholomorphic to a disk possibly less a relatively closed polar set. In this paper, we extend the inequality chain by showing that $c_B^2 \geq \pi v^{-1}(X)$ on planar domains, where $v(\cdot)$ is the Euclidean volume, and characterize the extremal cases when equality holds at one point. Similar rigidity theorems concerning the Szegö kernel, the higher-order Bergman kernels, and the sublevel sets of the Green's function are also developed. Additionally, we explore rigidity phenomena related to the multi-dimensional Suita conjecture for domains in $\mathbb{C}^n, n \geq 1$.

1 Introduction

An open Riemann surface X is said to be (potential-theoretically) hyperbolic if it admits a non-constant negative subharmonic function, and parabolic if it does not. Consider the on-diagonal Bergman kernel $K(z)|dz|^2$, the logarithmic capacity $c_\beta(z)|dz|$, and the analytic capacity $c_B(z)|dz|$ on X, where z is some local coordinate. These three quantities are invariant under changes of local coordinates. In 1972, Suita determined a simple relationship between K and c_B.

Theorem 1.1 (Suita's Theorem in [31]). *Suppose X is an open hyperbolic Riemann surface. Then*

$$\pi K(z) \geq c_B^2(z)$$

and equality holds at some $z_0 \in X$ if and only if X is biholomorphically equivalent to the unit disk less a (possibly empty) closed set of inner capacity zero.

A closed set of inner capacity zero is a relatively closed polar set. In the same paper, Suita conjectured that c_β would satisfy a similar inequality with rigidity as follows.

Suita Conjecture [31]: *Suppose X is an open hyperbolic Riemann surface. Then*

$$\pi K(z) \geq c_\beta^2(z)$$

and equality holds at some $z_0 \in X$ if and only if X is biholomorphically equivalent to the unit disk less a (possibly empty) closed set of inner capacity zero.

Towards the resolution of the Suita Conjecture, Ohsawa first demonstrated in [25] that the Suita Conjecture was connected to his Ohsawa-Takegoshi L^2 extension theorem, and he proved

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that $750\pi K \geq c_\beta^2$. By further considering the sharp L^2 extension problem, Błocki [5] established the optimal inequality $\pi K \geq c_\beta^2$ for bounded domains in $\mathbb{C}$. Later, Guan and Zhou [19] proved both the inequality and equality parts of the conjecture for open Riemann surfaces. See also [4] for a variational approach by Berndtsson and Lempert. Suita's theorem and the resolution of the conjecture may be called rigidity theorems as equality at one point between πK and either c_β or c_B determines the surface up to biholomorphism.

It is straightforward to show $c_\beta^2 \geq c_B^2$, so by the works of Błocki, and Guan and Zhou, for any open hyperbolic Riemann surface it holds that

$$\pi K \geq c_\beta^2 \geq c_B^2. \tag{1.1}$$

The characterization of the equality part for the inequality $c_\beta \geq c_B$ can be deduced from a result of Minda [24] on the behavior of $c_\beta(z)|dz|$ under holomorphic mappings. His result used the sharp form of the Lindelöf principle. In Appendix A, we provide a more direct proof of the equality characterization without adopting this principle.

We first restrict our attention to domains in $\mathbb{C}$. Throughout this paper, denote a disk by $\mathbb{D}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ and let $\mathbb{D} := \mathbb{D}(0, 1)$. For the Euclidean volume $v(\cdot)$, we use the convention that $v^{-1}(\Omega) = 0$ when $v(\Omega) = \infty$. Following Ahlfors [2], a compact set $E \subset \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is said to be *a null set of class* $\mathcal{N}_B$ if all bounded holomorphic functions on $\mathbb{C}_\infty \setminus E$ are constants. Compact polar sets are always of class $\mathcal{N}_B$.

Theorem 1.2 (Rigidity theorem of c_B and c_β)**.** *Let $\Omega \subset \mathbb{C}$ be a domain. Then*

$$c_B^2(z) \geq \frac{\pi}{v(\Omega)}, \quad z \in \Omega. \tag{1.2}$$

Moreover, equality holds at some $z_0 \in \Omega$ if and only if either of the following holds true:

1. $v(\Omega) < \infty$ *and* $\Omega = \mathbb{D}(z_0, \sqrt{\pi^{-1} v(\Omega)}) \setminus P$, *where P satisfies*

$$P \cap \overline{\mathbb{D}(z_0, s)} \in \mathcal{N}_B, \quad \text{for all } s < \sqrt{\pi^{-1} v(\Omega)};$$

   if additionally

$$c_\beta^2(z_0) = \frac{\pi}{v(\Omega)},$$

   then P above is a relatively closed polar set;

2. $v(\Omega) = \infty$ *and* $\Omega = \mathbb{C} \setminus P$ *where* $P \in \mathcal{N}_B$.

It is worth pointing out that, as demonstrated in Remark 2.6, the equality condition in (1.2) does not necessarily imply that P is polar. See also Theorem 2.4 and Lemma 2.5 for rigidity theorems with weaker assumptions than (1.2).

Theorem 1.2 extends the chain of inequalities (1.1) to

$$\pi K \geq c_\beta^2 \geq c_B^2 \geq \frac{\pi}{v(\Omega)}, \tag{1.3}$$

and essentially gives the equality conditions between $\pi v^{-1}(\Omega)$ and the other quantities in (1.3) for a hyperbolic domain $\Omega \subset \mathbb{C}$. As a corollary (see Corollary 3.1), we can reprove the main theorem of [14], a rigidity theorem which established the equality conditions of $K(z) \geq v^{-1}(\Omega)$. Moreover, let $K^{(j)}, j \in \mathbb{N}$, denote the higher-order Bergman kernels of a domain $\Omega \subset \mathbb{C}$. Combining Theorem 1.2 with a result of Błocki and Zwonek in [7], we get the inequalities

$$K^{(j)}(z) \geq \frac{j!(j+1)!\pi^j}{v^{j+1}(\Omega)}, \quad z \in \Omega, \quad j \in \mathbb{N},$$

and the following characterization of their equality conditions.

Corollary 1.3 (Rigidity theorem of higher-order Bergman kernels)**.** *Let Ω be a domain in $\mathbb{C}$. Then for each $j \in \mathbb{N}$, there exists a $z_0 \in \Omega$ such that*

$$K^{(j)}(z_0) = \frac{j!(j+1)!\pi^j}{v^{j+1}(\Omega)}$$

if and only if either of the following holds true:

1. $v(\Omega) < \infty$ *and* $\Omega = \mathbb{D}(z_0, \sqrt{\pi^{-1}v(\Omega)}) \setminus P$, *where P is a relatively closed polar set;*
2. $v(\Omega) = \infty$ *and* $\Omega = \mathbb{C} \setminus P$, *where P is a closed polar set.*

A classical question ([30, p. 20]) raised by Stein is *what are the relations between the Bergman kernel K and the Szegö kernel S*? On any bounded domain in $\mathbb{C}$ with Lipschitz boundary (see [3,17] for the C^∞ smooth case and Proposition 3.6 for the Lipschitz case), the Szegö kernel S and the analytic capacity c_B satisfy the identity $c_B(z) = 2\pi S(z)$. Using this and (1.3), we deduce the following inequality chain, which gives a relation between the on-diagonal Bergman and Szegö kernels:

$$\sqrt{\pi K(z)} \geq c_\beta(z) \geq 2\pi S(z) \geq \sqrt{\frac{\pi}{v(\Omega)}} \geq \frac{2\pi}{\sigma(\partial\Omega)}. \tag{1.4}$$

Here $\sigma(\partial\Omega)$ denotes the arc-length of $\partial\Omega$, and the last inequality in (1.4) is the isoperimetric inequality. In particular, the inequalities say that K always dominates $4\pi S^2$ on Lipschitz domains. See also [10] and the references therein for results on the comparison of these two kernels. We characterize the equality conditions in (1.4) as below.

Theorem 1.4 (Rigidity theorem of the Szegö kernel)**.** *Suppose $\Omega \subset \mathbb{C}$ is a bounded domain with Lipschitz boundary. Then,*

1. *there exists some $z_0 \in \Omega$ such that*

$$2\pi S(z_0) = \quad c_\beta(z_0) \quad \text{or} \quad \sqrt{\pi K(z_0)}$$

if and only if Ω is simply connected;

2. *there exists some $z_0 \in \Omega$ such that*

$$2\pi S(z_0) = \quad \sqrt{\frac{\pi}{v(\Omega)}} \quad \text{or} \quad \frac{2\pi}{\sigma(\partial\Omega)} \quad \text{or} \quad \frac{1}{\delta(z_0)}$$

if and only if $\Omega = \mathbb{D}(z_0, \sqrt{\pi^{-1}v(\Omega)})$.

Next, by a monotonicity result of Błocki and Zwonek in [6] concerning the Euclidean volume of the sublevel sets of the Green's function, for $z \in \Omega, -\infty < t_1 < t_2 < 0$,

$$c_\beta^2(z) \geq \frac{\pi e^{2t_1}}{v(\{G(\cdot, z) < t_1\})} \geq \frac{\pi e^{2t_2}}{v(\{G(\cdot, z) < t_2\})} \geq \frac{\pi}{v(\Omega)}. \tag{1.5}$$

See Remark 4.2. Using the isoperimetric inequality and the classical PDE theory on the unique continuation property for harmonic functions, we obtain the following rigidity theorem which examines the extremal cases and characterizes domains on which equalities in (1.5) hold.

Theorem 1.5 (Rigidity theorem of sublevel sets of Green's function)**.** *Let Ω be a bounded domain in $\mathbb{C}$. Then Ω is a disk centered at z_0 possibly less a relatively closed polar subset if and only if either of the following holds true:*

1. $$c_\beta^2(z_0) = \frac{\pi e^{2t_0}}{v(\{G(\cdot, z_0) < t_0\})}$$
   for some $z_0 \in \Omega$, and $t_0 \in (-\infty, 0)$;

2. $$\frac{\pi e^{2t_1}}{v(\{G(\cdot, z_0) < t_1\})} = \frac{\pi e^{2t_2}}{v(\{G(\cdot, z_0) < t_2\})}$$
   for some $z_0 \in \Omega$, and $t_1 \neq t_2$ in $(-\infty, 0)$;

3. $$\frac{\pi e^{2t_0}}{v(\{G(\cdot, z_0) < t_0\})} = \frac{\pi}{v(\Omega)}$$
   for some $z_0 \in \Omega$, and $t_0 \in (-\infty, 0)$.

At last, we study rigidity properties for bounded domains in $\mathbb{C}^n, n \geq 1$. Błocki and Zwonek [6] proved the following **multi-dimensional Suita Conjecture** concerning the Bergman kernel and the Azukawa indicatrix: *for any pseudoconvex domain $\Omega \subset \mathbb{C}^n$,*

$$K(z) \geq v^{-1}(I^A(z)).$$

Here I^A denotes the Azukawa indicatrix (see (5.2) for its definition). As the proof relied on an approximation of the domain by hyperconvex sub-domains, the pseudoconvexity was needed in [6]. For our final result, by considering the connection between the two involved quantities in the multi-dimensional Suita Conjecture and the Euclidean distance function $\delta(z)$, we derive the following rigidity theorem without requiring the domains in $\mathbb{C}^n$ to be pseudoconvex when $n > 1$.

Theorem 1.6. *Let Ω be a bounded domain in $\mathbb{C}^n$, $n \geq 1$. Then for all $z \in \Omega$,*

$$v(I^A(z)) \geq \frac{\pi^n}{n!}\delta^{2n}(z), \tag{1.6}$$

and

$$K(z) \leq \frac{n!}{\pi^n \delta^{2n}(z)}. \tag{1.7}$$

Moreover, either equality holds at some $z_0 \in \Omega$ if and only if Ω is a ball centered at z_0.

As a consequence of (1.7), we may add the distance function δ to the inequality chains (1.3), (1.4), and (1.5) on bounded domains in $\mathbb{C}$. See Remark 5.2 for more details.

The organization of the paper is as follows. In Sec. 2 we prove a rigidity theorem of c_B and c_β, and as applications, in Sec. 3 we prove rigidity theorems of the Bergman kernels, and new results on the Szegö kernel on Lipschitz domains. In Sec. 4, we investigate rigidity phenomena for the logarithmic capacity and the sublevel sets of the Green's function. In Sec. 5, for bounded domains in $\mathbb{C}^n$ we prove Theorem 1.6. In the appendices, we prove a rigidity phenomenon between c_B and c_β based on a rigidity lemma of the Green's function, and a stability result of the Szegö kernel on Lipschitz domains.

2 Rigidity theorem of c_B and c_β

Let $SH^-(X)$ denote the set of negative subharmonic functions on an open Riemann surface X. Given $z_0 \in X$, let w be a fixed local coordinate in a neighbourhood of z_0 such that $w(z_0) = 0$. The (negative) Green's function is

$$G(z, z_0) = \sup\{u(z) : u \in SH^-(X), \limsup_{z \to z_0} u(z) - \log |w(z)| < \infty\}. \tag{2.1}$$

An open Riemann surface admits a Green's function if and only if there is a non-constant, negative subharmonic function defined on it. The Green's function is strictly negative on the surface and harmonic except on the diagonal.

We say that a Borel set P is polar if there is a subharmonic function $u \not\equiv -\infty$ defined in a neighbourhood of P so that $P \subset \{z : u(z) = -\infty\}$. For a domain $\Omega \subset \mathbb{C}$, a point $\zeta_0 \in \partial\Omega$ is said to be irregular if there is a $\zeta \in \Omega$ such that $\lim_{z \to \zeta_0} G(z, \zeta) \neq 0$ and regular otherwise. By Kellogg's Theorem, cf. [27, Theorem 4.2.5], the set of irregular boundary points of a domain is a polar set. The Green's function on a Riemann surface induces the logarithmic capacity, which is used prominently (see [27] for its applications) in potential theory. Ahlfors introduced the analytic capacity for domains [1, 2] in order to study Painlevé's question: *which compact sets E in the complex plane are removable for the bounded holomorphic functions*?

Definition 2.1. *Let X be an open hyperbolic Riemann surface. The logarithmic capacity c_β is defined as*

$$c_\beta(z_0) = \lim_{z \to z_0} \exp(G(z, z_0) - \log |w(z)|).$$

The analytic capacity of a Riemann surface X is defined as

$$c_B(z_0) = \sup\left\{\left|\frac{\partial f}{\partial w}(z_0)\right| : f \in Hol(X, \mathbb{D}), f(z_0) = 0\right\}. \tag{2.2}$$

When we wish to emphasize the surface X, we will use the notations G_X and $c_{m;X}(\cdot)$, $m = \beta$ or B. If $h : X_1 \to X_2$ is a biholomorphism, then

$$c_{m;X_1}(z) = |h'(z)| c_{m;X_2}(h(z)), \quad m = \beta \text{ or } B.$$

Thus, the logarithmic and analytic capacity are independent of the choice of the local coordinates and define conformally-invariant metrics $c_\beta(z)|dz|$ and $c_B(z)|dz|$. For each $z_0 \in X$, there exists an extremal function f_0 for the analytic capacity, unique up to rotation; that is, f_0 belongs to the family described in (2.2) and $\left|\frac{df_0}{dw}(z_0)\right| = c_B(z_0)$.

We restrict our attention to domains in $\mathbb{C}$. With this restriction we will be able to examine the relationship between the domain functions c_β, c_B and the Euclidean volume of the domain. In the literature, the analytic capacity is often defined in terms of compact sets.

Definition 2.2. [17] *The analytic capacity $\gamma(E)$ of a compact subset $E \subset \mathbb{C}$ is*

$$\gamma(E) = \sup\{|g'(\infty)| : g \in Hol(\mathbb{C}_\infty \setminus E, \mathbb{D}), g(\infty) = 0\}$$

where

$$g'(\infty) = \lim_{z\to\infty} z(g(z) - g(\infty)).$$

Notation. *In this section, $\Omega - \{z_0\}$ will denote the translation of the domain Ω by z_0 and*

$$j_{z_0}(z) := \frac{1}{z - z_0}.$$

For ease of notation, when $z_0 = 0$, let $j = j_{z_0}$.

The two definitions of the analytic capacities presented in this paper are related as follows.

Lemma 2.3. *For any domain $\Omega \subset \mathbb{C}$, $c_B(z_0) = \gamma(\mathbb{C}_\infty \setminus j_{z_0}(\Omega))$.*

Proof. Since $j(\Omega - \{z_0\}) = j_{z_0}(\Omega)$ and $c_{B;\Omega}(z_0) = c_{B;\Omega-\{z_0\}}(0)$, it suffices to prove the lemma when $z_0 = 0 \in \Omega$. Let $E = \mathbb{C}_\infty \setminus j(\Omega)$. Notice that there is a one-to-one correspondence between the two function sets $\mathcal{A} := \{g \in Hol(\mathbb{C}_\infty \setminus E, \mathbb{D}) : g(\infty) = 0\}$ and $\mathcal{B} := \{h \in Hol(\Omega, \mathbb{D}) : h(0) = 0\}$, as $g \circ j \in \mathcal{B}$ whenever $g \in \mathcal{A}$, and vice versa. Moreover, $(g \circ j)'(0) = g'(\infty)$. The lemma then follows directly from the definitions of the two analytic capacities.

□

The analytic capacity is difficult to compute in general for most domains. It does however satisfy a lower bound referred to as the **Ahlfors-Beurling Inequality** (see [2], [17, Theorem 4.6, Chapter III]). Namely, for any compact set $E \subset \mathbb{C}$,

$$\gamma^2(E) \geq \frac{v(E)}{\pi}. \tag{2.3}$$

For any $z \in \Omega$, letting E in (2.3) be $\mathbb{C}_\infty \setminus j_z(\Omega)$, and combining with Lemma 2.3, we obtain

$$c_B^2(z) \geq \frac{v(\mathbb{C}_\infty \setminus j_z(\Omega))}{\pi}. \tag{2.4}$$

The following serves as a rigidity theorem concerning (2.4).

Theorem 2.4. *Let Ω be a domain in $\mathbb{C}$ with $v(\Omega) < \infty$. There exists a $z_0 \in \Omega$ such that*

$$c_B^2(z_0) = \frac{v(\mathbb{C}_\infty \setminus j_{z_0}(\Omega))}{\pi} \tag{2.5}$$

if and only if $\Omega = \mathbb{D}(z_1, r) \setminus P$ for some $z_1 \in \mathbb{C}, r > 0$, where P satisfies

$$P \cap \overline{\mathbb{D}(z_1, s)} \in \mathcal{N}_B, \quad \text{for all } s < r. \tag{2.6}$$

If additionally

$$c_\beta^2(z_0) = \frac{v(\mathbb{C}_\infty \setminus j_{z_0}(\Omega))}{\pi}, \tag{2.7}$$

then P above is a relatively closed polar set.

Proof. If $\Omega = \mathbb{D}(z_1, r) \setminus Q$, where

$$Q \cap \overline{\mathbb{D}(z_1, s)} \in \mathcal{N}_B, \quad \text{for all } s < r,$$

then by the classical Schwarz lemma,

$$c_B(z_0) = \frac{r}{r^2 - |z_0 - z_1|^2}. \tag{2.8}$$

On the other hand, a direct computation gives

$$v(\mathbb{C}_\infty \setminus j_{z_0}(\mathbb{D}(z_1, r))) = \frac{\pi r^2}{(r^2 - |z_0 - z_1|^2)^2}.$$

Equation (2.5) is proved.

For the other direction, let $E = \mathbb{C}_\infty \setminus j_{z_0}(\Omega)$. Then E is compact, and since $v(\Omega) < \infty$, $v(E) > 0$. Define $f \in \mathcal{O}(\Omega) \cap C(\overline{\Omega})$ by

$$f(z) = \frac{1}{\sqrt{\pi v(E)}} \int_E \frac{1}{w - \frac{1}{z - z_0}} dv(w).$$

Then $f(z_0) = 0$ and $f'(z_0) = \sqrt{\pi^{-1} v(E)}$. The Ahlfors-Beurling inequality states that $|f(z)| \leq 1$ for all $z \in \mathbb{C}_\infty$. By the maximum modulus theorem, $|f(z)| < 1$ on Ω. Thus, f is an extremal function for c_B, which implies $\sup_{z \in \Omega} |f(z)| = 1$. By continuity there exists $z_2 \in \partial\Omega$ such that $|f(z_2)| = 1$. We observe, after a careful inspection of the proof of the Ahlfors-Beurling inequality, as given in [27, Lemma 5.3.6], that E must be a union of a closed disk with a closed measure-zero set. For completeness we resupply the proof since this observation is not stated in the literature, cf. [2, 17, 27]. Indeed, after a rotation and translation of E we may assume that

$$1 = |f(z_2)| = \frac{1}{\sqrt{\pi v(E)}} \int_E \frac{1}{w} dv(w). \tag{2.9}$$

Let $D := \{w \in \mathbb{C} : Re w^{-1} > (2a)^{-1}\}$ be a disk such that $v(D) = v(E)$. Then $v(E \setminus D) = v(D \setminus E)$ and so

$$\int_{E \setminus D} Re \frac{1}{w} dv(w) \leq \int_{E \setminus D} \frac{1}{2a} dv(w) = \int_{D \setminus E} \frac{1}{2a} dv(w) \leq \int_{D \setminus E} Re \frac{1}{w} dv(w). \tag{2.10}$$

This implies

$$\begin{aligned}
\int_E \frac{1}{w} dv(w) = \int_E Re \frac{1}{w} dv(w) &= \int_{E \cap D} Re \frac{1}{w} dv(w) + \int_{E \setminus D} Re \frac{1}{w} dv(w) \\
&\leq \int_{D \cap E} Re \frac{1}{w} dv(w) + \int_{D \setminus E} Re \frac{1}{w} dv(w) \\
&= \int_D Re \frac{1}{w} dv(w) \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a \cos\theta} \cos\theta \, dr d\theta = \pi a = \sqrt{\pi v(E)}.
\end{aligned}$$

Comparing with (2.9), both inequalities in (2.10) become equalities and thus $v(E \setminus D) = v(D \setminus E) = 0$. Since E is compact, $\overline{D} \subset E$. Consequently, $\Omega = \mathbb{D}(z_1, r) \setminus P$ for some $z_1 \in \mathbb{C}$ and $r = \sqrt{v(\Omega)\pi^{-1}}$, where P is a relatively closed set of measure 0.

To further show that P satisfies (2.6), consider

$$h(z) = \frac{r(z-z_0)}{r^2 + \bar{z}_0 z_1 - |z_1|^2 - (\bar{z}_0 - \bar{z}_1)z}, \quad z \in \Omega.$$

It is not hard to verify that $h \in Hol(\Omega, \mathbb{D})$, $h(z_0) = 0$ and

$$h'(z_0) = \frac{r}{r^2 - |z_0 - z_1|^2}.$$

Thus h is an extremal function for c_B at z_0, by (2.8). By [20, Theorem 28], the image of the extremal function satisfies $h(\Omega) = \mathbb{D} \setminus Q$ where

$$Q = h(P), \quad Q \cap \overline{\mathbb{D}(0,r)} \in \mathcal{N}_B, \quad \text{for all } 0 \le r < 1.$$

Thus (2.6) is proved.

If additionally (2.7) holds, then by Theorem A.1, h is a biholomorphism from Ω to $\mathbb{D}(0,1) \setminus Q$, where Q is a relatively closed polar set. This implies that P is also a relatively closed polar set. Conversely if $\Omega = \mathbb{D}(z_1, r) \setminus P$ for a relatively closed polar set P, then a direct computation shows $c_\beta^2(z_0) = \pi^{-1} v(\mathbb{C}_\infty \setminus j_{z_0}(\Omega))$. The proof is complete.

□

The proof of Theorem 2.4 indicates the center of Ω may not necessarily be z_0, at which the equality (2.5) holds. Using the lemma below, we will show that if the stronger equality $c_B^2(z_0) = \pi v^{-1}(\Omega)$ holds for $v(\Omega) < \infty$, then in the conclusion of the preceding theorem the center of Ω must be z_0.

Lemma 2.5. *Let $\Omega \subset \mathbb{C}$ be a domain with $v(\Omega) < \infty$. Then for all $z \in \Omega$,*

$$\frac{\pi}{v(\Omega)} \le \frac{v(\mathbb{C}_\infty \setminus j_z(\Omega))}{\pi}, \tag{2.11}$$

and equality holds at some $z_0 \in \Omega$ if and only if Ω is a disk centered at z_0 less a relatively closed set of measure zero.

Proof. If $B \subset \mathbb{C}$ is a set with $v(B) = 0$, then $v(j_{z_0}(B)) = 0$. With this fact it is straightforward to verify that equality holds for a disk less a relatively closed set of measure 0. Since $v(\Omega) = v(\Omega - \{z_0\})$ and $j_{z_0}(\Omega) = j(\Omega - \{z_0\})$, without loss of generality we may suppose $z_0 = 0$. Notice that for any $r > 0$, $v(r\Omega) = r^2 v(\Omega)$ and

$$v(\mathbb{C}_\infty \setminus j(r\Omega)) = v(\mathbb{C}_\infty \setminus r^{-1} j(\Omega)) = v(r^{-1}(\mathbb{C}_\infty \setminus j(\Omega))) = r^{-2} v(\mathbb{C}_\infty \setminus j(\Omega)).$$

Here for any set $B \subset \mathbb{C}$, $rB := \{rz : z \in B\}$. We may further assume that $v(\Omega) = \pi$. So the inequality (2.11) is equivalent to

$$v(j(\mathbb{C}_\infty \setminus \Omega)) = v(\mathbb{C}_\infty \setminus j(\Omega)) \ge \pi.$$

Set

$$S_1 = \mathbb{D} \setminus \Omega, \quad S_2 = \Omega \setminus \mathbb{D}.$$

So $v(S_1)=v(S_2)$. Since $\mathbb{C}_\infty\setminus\Omega=S_1\sqcup((\mathbb{C}_\infty\setminus\mathbb{D})\setminus S_2)$ and $j(\mathbb{C}_\infty\setminus\mathbb{D})=\overline{\mathbb{D}}$,

$$j(\mathbb{C}_\infty\setminus\Omega)=j(S_1)\sqcup(\overline{\mathbb{D}}\setminus j(S_2)),$$

where $\sqcup$ denotes the disjoint union. Noticing $j(S_2)\subset\overline{\mathbb{D}}$, we further have

$$v(j(\mathbb{C}_\infty\setminus\Omega))-\pi=v(j(S_1))+v(\overline{\mathbb{D}})-v(j(S_2))-\pi=v(j(S_1))-v(j(S_2)).$$

Applying change of variables formula, one gets

$$v(j(\mathbb{C}_\infty\setminus\Omega))-\pi=\int_{S_1}\frac{1}{|z|^4}dv(z)-\int_{S_2}\frac{1}{|z|^4}dv(z)\geq\int_{S_1}1dv(z)-\int_{S_2}1dv(z)=0. \tag{2.12}$$

Here we have used the fact that $|z|<1$ on S_1 and $|z|\geq 1$ on S_2. This completes the proof of the inequality part.

If equality holds in (2.11), then the inequality in (2.12) becomes equality and

$$\int_{S_1}\frac{1}{|z|^4}dv(z)=\int_{S_1}1dv(z),\quad\int_{S_2}\frac{1}{|z|^4}dv(z)=\int_{S_2}1dv(z).$$

Since $|z|<1$ on S_1, the first equation implies $v(S_1)=0$. Thus, $v(S_2)=0$. By definitions of S_1 and S_2, we know that Ω is the unit disk centered at 0 less a relatively closed set of measure zero.

□

Proof of Theorem 1.2 *(Rigidity theorem of c_B and c_β).* If $v(\Omega)=\infty$, then the inequality is trivial. By [2, p. 107], $c_B(z_0)=0$ if and only if $c_B\equiv 0$, and thus if and only if $\Omega=\mathbb{C}_\infty\setminus P$ where $P\in\mathcal{N}_B$ by definition.

We now assume $v(\Omega)<\infty$. Equation (1.2) follows from (2.4) and (2.11). If $\Omega=\mathbb{D}(z_0,\sqrt{\pi^{-1}v(\Omega)})\setminus Q$, where

$$Q\cap\overline{\mathbb{D}(z_0,s)}\in\mathcal{N}_B,\quad\text{for all } s<\sqrt{\pi^{-1}v(\Omega)},$$

then by the classical Schwarz lemma, $c_B^2(z_0)=\pi v(\Omega)^{-1}$.

Conversely, if equality holds at $z_0\in\Omega$, then

$$c_B^2(z_0)=\frac{v(\mathbb{C}_\infty\setminus j_{z_0}(\Omega))}{\pi}=\frac{\pi}{v(\Omega)}.$$

By the equality part of Lemma 2.5, $\Omega=\mathbb{D}(z_0,\sqrt{\pi^{-1}v(\Omega)})\setminus P$, where P is a relatively closed set of measure 0. By the equality part of Theorem 2.4, we further conclude that $P\cap\overline{\mathbb{D}(z_0,s)}\in\mathcal{N}_B$ for all $s<\sqrt{\pi^{-1}v(\Omega)}$. The rest of the theorem follows from the second part of Theorem 2.4.

□

Remark 2.6. If $c_\beta^2(z_0)>c_B^2(z_0)=\pi v^{-1}(\Omega)$, then P in the preceding theorem may not be polar. Indeed, let Q be the compact four-corner Cantor set defined in [18]. As shown therein, $Q\in\mathcal{N}_B$, but is not polar. Let $\Omega=\mathbb{D}(z_0,r)\setminus Q$ where z_0 and r are chosen such that $z_0\notin Q\subset\mathbb{D}(z_0,r)$. Since $Q\in\mathcal{N}_B$, all bounded holomorphic functions on Ω extend across Q. Thus,

$$c_{B;\Omega}(z_0)=c_{B;\mathbb{D}(z_0,r)}(z_0)=\sqrt{\frac{\pi}{v(\mathbb{D}(z_0,r))}}=\sqrt{\frac{\pi}{v(\Omega)}},$$

where the last equality used the fact that sets of class $\mathcal{N}_B$ have two-dimensional Lebesgue measure 0.

3 Applications of the rigidity theorem of c_B and c_β

Let Ω be a domain in $\mathbb{C}^n, n \geq 1$. The Bergman space of a domain Ω is the Hilbert space

$$A^2(\Omega) = L^2(\Omega) \cap \mathcal{O}(\Omega)$$

with $L^2(\Omega)$-norm denoted by $\|\cdot\|_\Omega$. The Bergman kernel is defined by

$$K(z) = \sup\{|f(z)|^2 : f \in A^2(\Omega), \|f\|_\Omega \leq 1\}, \quad z \in \Omega.$$

By considering constant functions in the defining set of the kernel, we get

$$K(z) \geq \frac{1}{v(\Omega)}, \ z \in \Omega,$$

which is sharp when $\Omega = \mathbb{D}(z_0, r) \setminus P$, where P is a relatively closed polar set and $z = z_0$.

As an application of Theorem 1.2, we show that this sharp example is in fact the only possible domain where the equality can be achieved. Corollary 3.1 below, which is the main result of [14], was originally proved by the first and second named authors using the equality part of the Suita Conjecture. Here we provide a new proof based on our Theorem 1.2 and Suita's Theorem.

Corollary 3.1 (Originally proved in [14])**.** *Let $\Omega \subset \mathbb{C}$ be a domain. Then there exists a $z_0 \in \Omega$ such that*

$$K(z_0) = \frac{1}{v(\Omega)}, \tag{3.1}$$

if and only if either of the following holds true:

1. *$v(\Omega) < \infty$ and $\Omega = \mathbb{D}(z_0, r) \setminus P$, where P is a relatively closed polar set and with $r^2 = \pi^{-1} v(\Omega)$.*
2. *$v(\Omega) = \infty$ and $\Omega = \mathbb{C} \setminus P$, where P is a possibly empty, closed polar set.*

Proof. First assume $v(\Omega) < \infty$. By Suita's Theorem (Theorem 1.1) and Theorem 1.2,

$$\pi K(z_0) \geq c_B^2(z_0) \geq \frac{\pi}{v(\Omega)} = \pi K(z_0).$$

By the equality part of Theorem 1.2,

$$\Omega = \mathbb{D}(z_0, r) \setminus P, \quad P \cap \mathbb{D}(z_0, s) \in \mathcal{N}_B, \quad s < r = \sqrt{\pi^{-1} v(\Omega)}.$$

The equality part of Suita's Theorem (Theorem 1.1) implies additionally that P is polar. The case when $v(\Omega) = \infty$ is already known (see [7, Theorem 4]).

□

Remark 3.2. The case $v(\Omega) < \infty$ in the preceding proof used Suita's Theorem and not the Suita Conjecture. The proof of Suita's Theorem, which was proved using Riemann surface theory, is much simpler than the proof of the Suita Conjecture. Thus, the proof given here is simpler than the original proof in [14].

Remark 3.3. After the initial version of the paper, Boas [9] kindly pointed out to us that Corollary 3.1 fails for domains in $\mathbb{C}^n, n \geq 2$, as indicated by the following examples. Let Ω be a domain in $\mathbb{C}^n, n \geq 2$ satisfying (3.1) (for instance, a ball centered at z_0), and consider images $F(\Omega)$ under maps of the form $F : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$F(z) = (z', z_n + f_0(z')), \quad z' = (z_1, \dots, z_{n-1}), \tag{3.2}$$

where $f_0 : \mathbb{C}^{n-1} \to \mathbb{C}$ is any holomorphic function. Such maps are called shears and were considered extensively by Rosay and Rudin [28]. F is biholomorphic, and the determinant $\det J_{\mathbb{C}} F$ of its complex Jacobian is constantly 1. Hence F is volume preserving with $v(F(\Omega)) = v(\Omega)$. On the other hand, by the transformation rule of the Bergman kernel, $K_{F(\Omega)}(F(z_0)) = K_\Omega(z_0)$, so (3.1) holds for $F(\Omega)$ at $F(z_0)$. Due to the arbitrariness of f_0, a large degree of freedom is afforded to the geometries of such $F(\Omega)$, in stark contrast to the situation in $\mathbb{C}$. Moreover, the domains Ω that satisfy (3.1) include the bounded complete Reinhardt domains, bounded complete circular domains, bounded quasi-circular domains containing the origin, and bounded quasi-Reinhardt domains containing the origin [23], which form a strictly increasing sequence under the set containment relation in $\mathbb{C}^n, n \geq 2$. By selecting Ω from one of these classes, we can produce additional domains $F(\Omega)$ satisfying (3.1) that are not biholomorphically equivalent to the ball. Lastly, for a bounded quasi-Reinhardt Ω containing the origin, by choosing a non-polynomial mapping F in (3.2) such that $F(\Omega)$ is bounded and F(0) = 0, we may get a minimal domain centered at 0 that is not quasi-Reinhardt (see [12, 23]).

For domains in $\mathbb{C}$, we also have the following more precise estimate on the on-diagonal Bergman kernel. Recall that given $z \in \Omega$, $j_z = \frac{1}{\cdot - z}$.

Corollary 3.4. *Let $\Omega \subset \mathbb{C}$ be a domain with $v(\Omega) < \infty$. Then for all $z \in \Omega$,*

$$K(z) \geq \frac{v(\mathbb{C}_\infty \setminus j_z(\Omega))}{\pi^2},$$

and equality holds at some $z_0 \in \Omega$ if and only if Ω is a disk less a relatively closed polar set.

Proof. By Suita's Theorem (Theorem 1.1), the Ahlfors-Beurling Inequality (2.4), and Theorem 2.4,

$$\pi K(z) \geq c_B^2(z) \geq \frac{v(\mathbb{C}_\infty \setminus j_z(\Omega))}{\pi}$$

for $z \in \Omega$. The equality part is a consequence of Theorems 1.1 and 2.4.

□

For $j = 1, 2, \dots$, let

$$K^{(j)}(z) = \sup\{|f^{(j)}(z)|^2 : f \in A^2(\Omega), \|f\|_\Omega \leq 1, f^{(k)}(z) = 0, k = 0, \dots, j-1\}$$

denote the Bergman kernels for higher order derivatives, and set $K^{(0)} = K$. Błocki and Zwonek [7] established that for $z \in \Omega \subset \mathbb{C}$,

$$K^{(j)}(z) \geq \frac{j!(j+1)!}{\pi}(c_\beta(z))^{2j+2}, \tag{3.3}$$

which is sharp for a disk less a relatively closed polar set $\mathbb{D}(z_0, r) \setminus P$ with $z = z_0$.

Proof of Corollary 1.3 *(Rigidity theorem of higher-order Bergman kernels).* The case $v(\Omega) = \infty$ is already known, cf. [7, Theorem 4]. For $v(\Omega) < \infty$, this follows from (3.3) and Theorem 1.2. □

One important property of the analytic capacity c_B is that it is distance decreasing with respect to any given holomorphic map $f : X \to Y$, where X, Y are hyperbolic Riemann surfaces, cf. [21, Chapter 2]:

$$f^*\left(c_{B;Y}(z)|dz|\right) \le c_{B;X}(z)|dz|, \tag{3.4}$$

where $f^*\left(c_{B;Y}(z)|dz|\right)$ denotes the pull-back to X via f of the analytic capacity on Y.

Remark 3.5. For a hyperbolic Riemann surface X with a local coordinate z, if there exists $z_0 \in X$ and a non-constant holomorphic function $f : X \to \mathbb{D}$ such that

$$K(z_0)|dz|^2 = \frac{|df(z_0)|^2}{\pi(1-|f(z_0)|^2)^2}, \tag{3.5}$$

then by (1.1) and (3.4), we have

$$\sqrt{\pi K(z_0)}|dz| \ge c_{B;X}(z_0)|dz| \ge f^*\left(c_{B;\mathbb{D}}(z_0)|dz|\right) = \frac{|df(z_0)|}{1-|f(z_0)|^2}.$$

Without loss of generality, assume that $df(z_0) > 0$. Therefore, (3.5) forces the equality condition in Theorem 1.1 to hold true, and there exists a biholomorphism $h : X \to \mathbb{D} \setminus P$ such that $h(z_0) = 0$ and $\frac{\partial h}{\partial z}(z_0) > 0$, where P is a relatively closed polar subset. By the transformation rule of the Bergman kernel, $\sqrt{\pi K(z_0)} = \frac{\partial h}{\partial z}(z_0)$. Take $\varphi(z) := \frac{z-f(z_0)}{1-\overline{f(z_0)}z} \in \mathrm{Aut}(\mathbb{D})$. Then $\varphi \circ f \circ h^{-1}$ is bounded, so it extends to a holomorphic function $F : \mathbb{D} \to \mathbb{D}$ such that $F(0) = 0$. Moreover, $F'(0) = \sqrt{\pi K(z_0)}^{-1} \cdot \sqrt{\pi K(z_0)}(1-|f(z_0)|^2) \cdot (1-|f(z_0)|^2)^{-1} = 1$. The Schwarz lemma implies that $F(z) \equiv z$ on $\mathbb{D}$, so f is in fact biholomorphic. Consequently, the identity (3.5) extends to all of X, and f is a holomorphic isometry with respect to the Bergman kernel.

In the remaining part of the section, we focus on a bounded domain Ω in $\mathbb{C}$ with Lipschitz boundary. The Hardy space $H^2(\partial\Omega)$ is defined to be the set of holomorphic functions on Ω which admit a non-tangential boundary limit function and a maximal function on the boundary which belong to $L^2(\partial\Omega)$. See Appendix B or [22] for more details. Let $\|\cdot\|_{\partial\Omega}$ denote the $L^2(\partial\Omega)$ norm and $S(\cdot,\cdot)$ denote the Szegö kernel. When restricted to the diagonal, it satisfies

$$S(z,z) := S(z) = \sup\{|f(z)|^2 : f \in H^2(\partial\Omega), \|f\|_{\partial\Omega} \le 1\}.$$

By considering constant functions in the defining set of the kernel, for any $z \in \Omega$, we get

$$S(z) \ge \frac{1}{\sigma(\partial\Omega)},$$

where $\sigma(\partial\Omega)$ denotes the arclength of $\partial\Omega$. The lower bound is sharp for $\Omega = \mathbb{D}(z_0, r)$ and $z = z_0$. As applications of the Rigidity theorem of c_B and c_β, Theorem 1.2, we will show that these are the only possible domains where the equality can be achieved.

Let $\{\Omega_j\}_{j=1}^{\infty}$ be a family of exhausting subdomains of Ω, and $S_j(\cdot,\cdot)$ be the corresponding Szegö kernels. Boas showed in [8, Theorem 2.1] that if in addition Ω has C^∞-smooth boundary and is exhausted by sublevel sets Ω_j of its defining function, then for $a, z \in \Omega_j$,

$$\lim_{j\to\infty} S_j(z,a) = S(z,a).$$

For a bounded domain $\Omega \subset \mathbb{C}$ with Lipschitz boundary, there are subdomains $\{\Omega_j\}_{j=1}^{\infty}$ with C^∞ boundary that approximate Ω well uniformly and non-tangentially in the sense of Nečas (see [22, p. 539] or Appendix B for more details). In Proposition B.3, we extend Boas' stability result of the Szegö kernel to Lipschitz domains in $\mathbb{C}$ with respect to the Nečas approximation.

It is known for a finitely connected domain with C^∞ boundary that the analytic capacity and (on-diagonal) Szegö kernel satisfy the relation $c_B(z) = 2\pi S(z)$. For $\{\Omega_j\}_{j=1}^{\infty}$ given above, as shown by Ahlfors and Beurling [2, Theorem 1], the analytic capacities $c_{j;B}$ and c_B of these domains, respectively, similarly satisfy

$$\lim_{j\to\infty} c_{j;B}(z) = c_B(z), \quad z \in \Omega.$$

Consequently, we can conclude

Proposition 3.6. *If $\Omega \subset \mathbb{C}$ is bounded with Lipschitz boundary, then*

$$c_B(z) = 2\pi S(z), \quad z \in \Omega. \tag{3.6}$$

The Rigidity theorem of c_B and c_β, Theorem 1.2, together with Proposition 3.6, enables us to prove the rigidity phenomenon of the Szegö kernel.

Proof of Theorem 1.4 *(Rigidity theorem of the Szegö kernel).* The equality conditions in Suita's Theorem (Theorem 1.1) and in $c_\beta \geq c_B$ (Theorem A.1) imply that the equality condition in Case 1 holds if and only if Ω is biholomorphic to a disk less a relatively closed polar set. Since Ω is Lipschitz, it is a regular domain for the Dirichlet problem. Thus, the polar set is empty, cf. [27].

For Case 2, it suffices to prove the 'only if' direction. By the Rigidity theorem of c_B and c_β (Theorem 1.2), $2\pi S(z_0) = \sqrt{\pi v^{-1}(\Omega)}$ if and only if

$$\Omega = \mathbb{D}(z_0, r) \setminus Q, \quad Q \cap \overline{\mathbb{D}(z_0, s)} \in \mathcal{N}_B, \quad \text{for all } s < r = \sqrt{\pi^{-1} v(\Omega)}.$$

Since Ω has Lipschitz boundary, its boundary has no singleton connected components. Thus $Q = \emptyset$. The equality case $2\pi S(z_0) = 2\pi\sigma^{-1}(\partial\Omega)$ follows now as well.

By (3.6), $2\pi S(z_0) = \delta^{-1}(z_0)$ is equivalent to $c_B(z_0) = \delta^{-1}(z_0)$. Let f_0 be an extremal function for $c_B(z_0)$. By the Schwarz lemma, $f_0|_{\mathbb{D}(z_0,\delta(z_0))} : \mathbb{D}(z_0, \delta(z_0)) \to \mathbb{D}$ is extremal for $\mathbb{D}(z_0, \delta(z_0))$, and thus equals $e^{i\theta}\delta^{-1}(z_0)(z - z_0)$ for some $\theta \in [0, 2\pi)$. Since $|f_0| < 1$, $\Omega = \mathbb{D}(z_0, \delta(z_0))$.

□

Theorem 2.4 combined with Proposition 3.6 also gives another rigidity result concerning the Szegö kernel below.

Corollary 3.7. *Suppose $\Omega \subset \mathbb{C}$ is a bounded domain with Lipschitz boundary. Then for $z \in \Omega$,*

$$S^2(z) \geq \frac{v(\mathbb{C}_\infty \setminus j_z(\Omega))}{4\pi^3}.$$

Moreover, equality holds at some $z_0 \in \Omega$ if and only if Ω is a disk.

Remark 3.8. In comparison with (1.4) and Theorem 1.4, concerning the inequality between the on-diagonal Bergman and Szegö kernels and the equality conditions, we would like to point out that when the domain is bounded and simply-connected with C^∞-boundary,

$$K(z,a) = 4\pi S^2(z,a), \quad z,a \in \Omega,$$

cf. [3, Theorem 25.1].

Remark 3.9. Let $f : \Omega_1 \to \Omega_2$ be a holomorphic map, where $\Omega_1, \Omega_2 \subset \mathbb{C}$ are bounded domains with Lipschitz boundaries. Then, by Proposition 3.6 and (3.4), the Szegö kernel is decreasing with respect to f, namely

$$|f'(z)|S_{\Omega_2}(f(z)) \le S_{\Omega_1}(z),$$

and equality holds if f is a biholomorphism.

4 Rigidity of sublevel sets of Green's function

For a fixed $z_0 \in \Omega$, let $G_{z_0}(\cdot) = G(\cdot, z_0)$ and

$$\Omega_t = \{z \in \Omega : G_{z_0}(z) < t\}, \quad t \in (-\infty, 0]$$

denote the sublevel sets of the Green's function. The following monotonic property was proved by Błocki and Zwonek.

Theorem 4.1. [6] *Let Ω be a bounded domain in $\mathbb{C}$. Then for any $z_0 \in \Omega$,*

$$f(t) := \frac{\pi e^{2t}}{v(\Omega_t)}$$

is non-increasing in $t \in (-\infty, 0)$.

Remark 4.2. Note that

$$\lim_{t\to 0^-} \frac{e^{2t}}{v(\Omega_t)} = \frac{1}{v(\Omega)}.$$

As $t \to -\infty$, Ω_t is approximable by the set $\{\log(c_\beta(z_0)|z-z_0|) < t\}$ (see [4,6]). This implies

$$\lim_{t\to-\infty} \frac{e^{2t}}{v(\Omega_t)} = \lim_{t\to-\infty} \frac{e^{2t}}{v(\{|z-z_0| < e^t c_\beta^{-1}(z_0)\})} = \frac{c_\beta^2(z_0)}{\pi}.$$

This gives the inequalities (1.5) between the logarithmic capacity and the sublevel sets of the Green's function.

The next theorem discusses the second inequality in (1.5), which states that either f in Theorem 4.1 is strictly decreasing, or the domain has to be rigid.

Theorem 4.3. *Let Ω be a bounded domain in $\mathbb{C}$. If there exist $z_0 \in \Omega$ and $t_1 < t_2$ in $(-\infty, 0)$ such that*

$$\frac{\pi e^{2t_1}}{v(\Omega_{t_1})} = \frac{\pi e^{2t_2}}{v(\Omega_{t_2})},$$

then $G_{z_0}(z) = \log|z-z_0|/R$ for some constant $R > 0$. Consequently, Ω is a disk centered at z_0 with radius R possibly less a relatively closed polar subset.

Proof. By scaling and translating if necessary, we may assume that $z_0 = 0$ and the diameter of Ω is 2. By the inequality part of Theorem 4.1, $f(t_1) = f(t_2)$ implies that there exists a constant C such that

$$v(\Omega_t) = Ce^{2t},$$

for $t \in (t_1, t_2)$. Thus

$$\frac{d}{dt}v(\Omega_t) = 2v(\Omega_t).$$

Following [6,7], we see from the Cauchy-Schwarz inequality that for almost every $t \in (-\infty, 0)$,

$$\sigma^2(\partial\Omega_t) \leq \int_{\partial\Omega_t} \frac{1}{|\nabla G_0|} d\sigma \int_{\partial\Omega_t} |\nabla G_0| d\sigma. \tag{4.1}$$

Note that due to the harmonicity of G_0 away from 0,

$$\int_{\partial\Omega_t} |\nabla G_0| d\sigma = \int_{\partial\Omega_t} \frac{\partial G_0}{\partial \nu} d\sigma = 2\pi. \tag{4.2}$$

On the other hand, by the co-area formula $v(\Omega_t) = \int_{-\infty}^{t} \int_{\partial\Omega_s} \frac{d\sigma}{|\nabla G_0|} ds$, which further leads to

$$\int_{\partial\Omega_t} \frac{1}{|\nabla G_0|} d\sigma = \frac{d}{dt} v(\Omega_t). \tag{4.3}$$

for almost every $t \in (-\infty, 0)$. Pick up a point $t_0 \in (t_1, t_2)$ such that (4.1-4.3) hold. Then,

$$\sigma^2(\partial\Omega_{t_0}) \leq 4\pi v(\Omega_{t_0}).$$

According to the classical isoperimetric inequality (see [11] and the references therein), Ω_{t_0} is equivalent (two sets E_1 and E_2 are equivalent if and only if $v(E_1 \cup E_2 \setminus E_1 \cap E_2) = 0$) to a disk centered at a point $a \in \Omega$ with radius re^{t_0} for some $r > 0$, and in particular, the involved Cauchy-Schwarz inequality (4.1) attains the equality. This means $\frac{1}{|\nabla G_0|}$ is necessarily a constant multiple of $|\nabla G_0|$, or equivalently, $|\nabla G_0|$ is a constant on $\partial\Omega_{t_0}$. Combining this with (4.2), one obtains $\frac{\partial}{\partial\nu} G_0 = |\nabla G_0| = r^{-1}e^{-t_0}$ on $\partial\Omega_{t_0}$.

Now G_0 is harmonic on $\Omega \setminus \Omega_{t_0}$ and satisfies the following Cauchy data

$$G_0 = t_0, \quad \frac{\partial G_0}{\partial \nu} = \frac{1}{re^{t_0}}$$

on some smooth piece in $\partial\Omega_{t_0}$ contained in $\partial\mathbb{D}(a, re^{t_0})$. As a consequence of the unique continuation property of harmonic functions for local Cauchy data (see [32]), we get $G_0(z) = \log \frac{|z-a|}{r}$ on $\Omega \setminus \Omega_{t_0}$, and further on Ω by the uniqueness. Since G_0 has a pole 0 and the diameter of Ω is 2, we see that $a = 0$ and $r = 1$, with $G_0(z) = \log |z|$ on Ω. Then we complete the proof using Lemma A.2.

□

In particular, the corollary below follows from Theorem 4.3 directly. Here we provide an alternative proof adopting the property of the Bergman kernel, instead of using the unique continuation property of harmonic functions as in Theorem 4.3.

Corollary 4.4. *Let Ω be a bounded domain in $\mathbb{C}$. Then, there exist $z_0 \in \Omega$ and $t_0 \in (-\infty, 0]$ such that*

$$c_\beta^2(z_0) = \frac{\pi e^{2t_0}}{v(\Omega_{t_0})},$$

if and only if Ω is a disk centered at z_0 possibly less a relatively closed polar subset. In particular,

$$c_\beta^2(z_0) = \frac{\pi}{v(\Omega)}$$

for some $z_0 \in \Omega$ if and only if Ω is a disk centered at z_0 possibly less a relatively closed polar subset.

Proof. It suffices to prove the necessity. By scaling and translating, we may assume $z_0 = 0$ and $c_\beta(0) = 1$. By the isoperimetric inequality as in the first part of the proof of Theorem 4.3, Ω_t is equivalent to a disk of radius e^t for almost every $t \in (-\infty, t_0)$. Let $t^\sharp < t_0$ be a negatively large constant such that when $t < t^\sharp$, $\partial\Omega_t$ is smooth. Hence Ω_t is precisely a disk of radius e^t for each such arbitrarily fixed t. Denote by a_t the center of Ω_t.

We claim that $a_t = 0$. To see this, let K_t and c_t stand for the corresponding Bergman kernel and logarithmic capacity of Ω_t, respectively. Then for $z \in \Omega_t$,

$$K_t(z) = \frac{e^{2t}}{\pi(e^{2t} - |z - a_t|^2)^2}.$$

Since the Green's function on Ω_t with a pole 0 is $G_0(z) - t$, by definition,

$$c_t(0) = \exp \lim_{z \to 0} \{(G_0(z) - t) - \log |z|\} = e^{-t} c(0) = e^{-t}.$$

Making use of the fact that $\pi K_t = c_t^2$ on the disk Ω_t, we further have at $z = 0$ that

$$\frac{e^{2t}}{(e^{2t} - |a_t|^2)^2} = \pi K_t(0) = c_t^2(0) = e^{-2t}.$$

It immediately tells us that $a_t = 0$, so

$$\Omega_t = \mathbb{D}(0, e^t), \quad \text{for} \quad t < t^\sharp. \tag{4.4}$$

Lastly let $\rho(z) := G_0(z) - \log |z|$ on Ω. Then $\Omega_t = \{z \in \mathbb{C} : |z| < \frac{e^t}{e^{\rho(z)}}\}$, $t < 0$. Comparing this with (4.4) for $t < t^\sharp$, we have $\rho|_{\partial\Omega_t} = 0$. Since ρ is harmonic on Ω, $\rho \equiv 0$ on Ω_t. By the uniqueness again, $\rho \equiv 0$ on Ω, where $G_0(z) = \log |z|$. The proof is complete in view of Lemma A.2. □

Proof of Theorem 1.5: The 'only if' direction is straightforward. For the 'if' direction, Case 2 follows from Theorem 4.3. Case 1 follows from Corollary 4.4, or alternatively, from Theorem 4.3. Indeed, if Case 1 holds, then by (1.5) we have for all $t < t_1$

$$\frac{\pi e^{2t}}{v(\Omega_t)} = \frac{\pi e^{2t_1}}{v(\Omega_{t_1})}.$$

Case 1 is thus reduced to Theorem 4.3 and so Ω is a disk centered at z_0 possibly less a relatively closed polar subset. Case 3 can be proved similarly from Theorem 4.3. □

5 Rigidity properties of the distance function in $\mathbb{C}^n$

Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2-boundary. It is known that there exists a constant C depending only on n such that

$$K(z) \le C\delta^{-n-1}(z)$$

for all $z \in \Omega$; if in addition Ω is pseudoconvex, by [16, 26] there exists a constant C_Ω depending only on Ω such that

$$K(z) \ge \frac{C_\Omega}{\delta^2(z)}.$$

In particular, the Bergman kernel of $\mathbb{B}^n(z_0, r)$, the ball in $\mathbb{C}^n$ centered at z_0 with radius r, is given by

$$K_{\mathbb{B}^n(z_0,r)}(z) = \frac{n!r^2}{\pi^n(r^2 - |z - z_0|^2)^{n+1}}. \tag{5.1}$$

Denote by $PSH^-(\Omega)$ the space of negative plurisubharmonic functions on Ω. Let $G_z(\cdot)$ be the pluricomplex Green's function of Ω with pole $z \in \Omega$ given by

$$G_z(w) = \sup\{u(w) : u \in PSH^-(\Omega), \limsup_{\zeta\to z} u(\zeta) - \log|\zeta - z| < \infty\}.$$

The Azukawa indicatrix for $z \in \Omega \subset \mathbb{C}^n$ is defined as

$$I^A_\Omega(z) := \{X \in \mathbb{C}^n : \limsup_{\zeta\to 0} (G_z(z + \zeta X) - \log|\zeta|) < 0\}. \tag{5.2}$$

Straightforward calculations show $I^A_{\mathbb{B}^n(z,r)}(z) = \mathbb{B}^n(z, r)$, and when $n = 1$, $I^A_\Omega(z) = \mathbb{D}(0, c^{-1}_{\beta;\Omega}(z))$.

Proof of Theorem 1.6. Firstly, we deal with the relation between the Bergman kernel and the distance function. Let $z \in \Omega$ be fixed and consider $\mathbb{B}^n(z, \delta(z))$. Then $\mathbb{B}^n(z, \delta(z)) \subset \Omega$. By (5.1) and the monotonic decreasing property of the Bergman kernels with respect to domains,

$$K(z) \le K_{\mathbb{B}^n(z,\delta(z))}(z) \le \frac{n!}{\pi^n\delta^{2n}(z)},$$

which yields (1.7).

Then we prove the rigidity part in (1.7) and without loss of generality assume equality is attained at $z_0 = 0$, namely $K(0) = n!\pi^{-n}\delta^{-2n}(0)$. Let f be an extremal holomorphic function on Ω such that $\|f\|_{L^2(\Omega)} = 1$ and $K(0) = |f(0)|^2$. Then the restriction of f to $\Omega_1 := \mathbb{B}^n(0, \delta(0))$ is also holomorphic with $\|f\|_{L^2(\Omega_1)} \le \|f\|_{L^2(\Omega)} = 1$. Therefore, the Bergman kernel on the ball Ω_1 satisfies

$$\frac{n!}{\pi^n\delta^{2n}(0)} = K_{\Omega_1}(0) \ge \frac{|f(0)|^2}{\|f\|^2_{L^2(\Omega_1)}} \ge |f(0)|^2 = K(0) = \frac{n!}{\pi^n\delta^{2n}(0)},$$

which forces both inequalities above to become equalities. In particular,

$$\|f\|_{L^2(\Omega)} = 1 = \|f\|_{L^2(\Omega_1)}. \tag{5.3}$$

This implies $\Omega = \Omega_1$, i.e., Ω is a ball. In fact, if $\Omega \ne \Omega_1$, then $f = 0$ almost everywhere on the non-empty open set $\Omega \setminus \overline{\Omega_1}$ (of positive Lebesgue measure) by (5.3). By the holomorphicity we know that $f \equiv 0$ almost everywhere on Ω, which contradicts the fact that $\|f\|_{L^2(\Omega)} = 1$.

Secondly, we deal with the relation between the Azukawa indicatrix and the distance function. Once again, we may suppose $z = 0$. Since $\mathbb{B}^n(0, \delta(0)) \subset \Omega$,

$$\mathbb{B}^n(0, \delta(0)) = I^A_{\mathbb{B}^n(0,\delta(0))}(0) \subset I^A_\Omega(0), \tag{5.4}$$

which gives the inequality (1.6).

Assume equality in (1.6) holds. Then

$$v(I^A_\Omega(0) \setminus \mathbb{B}^n(0, \delta(0))) = 0. \tag{5.5}$$

We first claim that for all $X \in \partial\mathbb{B}^n(0, \delta(0))$,

$$\limsup_{\lambda\to 0} G_0(\lambda X) - \log|\lambda| \le 0. \tag{5.6}$$

Indeed, if $\limsup_{\lambda\to 0} G_0(\lambda X) - \log|\lambda| = \epsilon_0 > 0$, then for $t > 0$ sufficiently close to 1^- (say $t > e^{-\epsilon_0}$),

$$\begin{aligned}
\limsup_{\lambda\to 0} G_0(\lambda(tX)) - \log|\lambda| &= \limsup_{\lambda\to 0} G_0(\lambda tX) - \log|\lambda t| + \log t \\
&= \left(\limsup_{\lambda t\to 0} G_0((\lambda t) X) - \log|\lambda t|\right) + \log t \\
&= \epsilon_0 + \log t > 0.
\end{aligned}$$

This would mean $tX \notin I^A_\Omega(0)$, which contradicts (5.4). The claim is proved. The same argument also shows that if $X \in I^A_\Omega(0)$, then so is tX for all $0 \le t \le 1$.

By (5.6), there is a function $f : \partial\mathbb{B}^n(0, \delta(0)) \to [1, \infty)$ such that

$$\left(I^A_\Omega(0) \cup \partial\mathbb{B}^n(0, \delta(0))\right) \setminus \mathbb{B}^n(0, \delta(0)) = \{r\omega : \omega \in \partial\mathbb{B}^n(0, \delta(0)), 1 \le r \le f(\omega)\}.$$

Then by (5.5),

$$0 = \int_{\partial\mathbb{B}^n(0,\delta(0))} \int_1^{f(\omega)} r^{2n-1} dr d\sigma(\omega).$$

Thus $f(\omega) = 1$ almost everywhere with respect to $\sigma(\partial\mathbb{B}^n(0, \delta(0)))$. Thus, for almost every $X \in \partial\mathbb{B}^n(0, \delta(0))$,

$$\limsup_{\lambda\to 0} G_0(\lambda X) - \log|\lambda| = 0. \tag{5.7}$$

For fixed $X \in \partial\mathbb{B}^n(0, \delta(0))$ satisfying (5.7), consider $u : \mathbb{D}(0, 1) \setminus \{0\} \to \mathbb{R}$ by

$$u(\lambda) = G_0(\lambda X) - \log|\lambda|.$$

By (5.7), u extends subharmonically to equal 0 at 0. On the other hand, by the monotonicity of the pluri-complex Green's functions,

$$u(\lambda) \le \log\left|\frac{\lambda X}{\delta(0)}\right| - \log|\lambda| = 0.$$

Thus, $u \in SH^-(\mathbb{D}(0, 1) \setminus \{0\})$. By the maximum principle $u \equiv 0$, or equivalently,

$$G_0(\lambda X) \equiv \log\left|\frac{\lambda X}{\delta(0)}\right|$$

for almost every $X \in \partial\mathbb{B}^n(0,\delta(0))$. Since $G_0 < 0$ on Ω, there is a (possibly empty) $\sigma(\partial\mathbb{B}^n(0,\delta(0)))$-null set $\mathcal{E}$ such that

$$\partial\mathbb{B}^n(0,\delta(0)) \setminus \mathcal{E} \subset \partial\Omega.$$

Since $\partial\Omega$ is closed and $\mathbb{B}^n(0,\delta(0)) \subset \Omega$,

$$\partial\Omega = \partial\mathbb{B}^n(0,\delta(0)), \quad \Omega = \mathbb{B}^n(0,\delta(0)).$$

□

In the case of pseudoconvex domains, the rigidity result concerning the Azukawa indicatrix and the distance function follows immediately from the multi-dimensional Suita conjecture and the first part of our proof of Theorem 1.6. Recently, the first author and Wong [15] used curvature properties of the Bergman metric to characterize pseudoconvex domains that are biholomorphic to a ball $\mathbb{B}^n$ possibly less a relatively closed pluripolar set.

When $n = 1$, since $I^A_\Omega(z) = \mathbb{D}(0, c^{-1}_{\beta;\Omega}(z))$, Theorem 1.6 implies the corollary below. Here we provide an alternative proof using more elementary methods.

Corollary 5.1. *Let Ω be a bounded domain in $\mathbb{C}$. Then for any $z \in \Omega$,*

$$\delta^{-1}(z) \geq c_\beta(z). \tag{5.8}$$

Moreover, equality in (5.8) holds at some $z_0 \in \Omega$ if and only if Ω is a disk centered at z_0.

Proof. For any $z \in \Omega$, write the Green's function as $G_z(w) = \log|z-w| + \rho(w)$. Then ρ is harmonic in Ω and $\rho(z) = \log c_\beta(z)$. For any positive number $R < \delta(z)$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} G_z(z+Re^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} \log R + \rho(z+Re^{it})dt.$$

By the mean value theorem for harmonic functions, $\frac{1}{2\pi}\int_0^{2\pi} \rho(z+Re^{it})dt = \rho(z) = \log c_\beta(z)$. Thus

$$\frac{1}{2\pi}\int_0^{2\pi} G_z(z+Re^{it})dt = \log(Rc_\beta(z)). \tag{5.9}$$

Recall that for $w \in \bar{\Omega}\setminus\{z\}$, $G_z(w) \leq 0$. This implies from (5.9) that

$$c_\beta(z) \leq R^{-1}.$$

Letting $R \to \delta(z)^-$, we obtain (5.8).

Assume $c_\beta(z_0) = \delta^{-1}(z_0)$ at some point $z_0 \in \Omega$. Then by (5.9) and the nonpositivity of G_{z_0}, $G_{z_0}(w) \to 0^-$ for all $w \in \partial\mathbb{D}(z_0,R)$ as $R \to \delta(z_0)^-$. The continuity and nonpositivity of G_{z_0} in Ω further concludes that this can only happen when $\partial\Omega$ coincides with $\partial\mathbb{D}(z_0,\delta(z_0))$.

□

Remark 5.2. When Ω is a bounded domain in $\mathbb{C}$, Theorem 1.6 reduces to

$$\frac{1}{\delta^2(z)} \geq \pi K(z)$$

for all $z \in \Omega$ and equality holds at some $z_0 \in \Omega$ if and only if Ω is a disk centered at z_0. Combining this with the previous inequality chains considered in the paper, we know that there exists some $z_0 \in \Omega$ (and some $t_0 \le 0$) such that

$$\frac{1}{\delta^2(z_0)} = c_B^2(z_0) \quad \text{or} \quad \frac{\pi e^{2t_0}}{v(\{G(\cdot, z_0) < t_0\})} \quad \text{or} \quad \frac{v(\mathbb{C}_\infty \setminus j_{z_0}(\Omega))}{\pi}$$

if and only if Ω is a disk centered at z_0.

Appendices

A Rigidity phenomenon between c_B and c_β

The aim of this appendix is to prove the following rigidity theorem characterizing the equality conditions for $c_B \le c_\beta$.

Theorem A.1. *For an open hyperbolic Riemann surface X, $c_\beta(z_0) = c_B(z_0)$ for some $z_0 \in X$ if and only if X is biholomorphic to the unit disk $\mathbb{D}$ possibly less a relatively closed polar subset P; in this case, the extremal functions of the analytic capacity $c_B(z_0)$ equal $\varphi \circ f$, where $f : X \to \mathbb{D} \setminus P$ is a biholomorphism and $\varphi \in Aut(\mathbb{D})$ such that $\varphi \circ f(z_0) = 0$.*

Given two Riemann surfaces X and Y with X admitting a Green's function, Minda in [24, Theorem 3] proved that if $f : X \to Y$ is holomorphic, then

$$f^*(c_{\beta;Y}(w)|dw|) \le c_{\beta;X}(\zeta)|d\zeta|.$$

Moreover, if equality holds at a single point, then f is biholomorphic onto its image and $Y \setminus f(X)$ is a closed polar set. Although Minda's approach does not mention the analytic capacity, it can be used to deduce Theorem A.1 as follows. For any $f \in Hol(X, \mathbb{D})$,

$$|f'(z)||dz| \le \frac{|f'(z)|}{1 - |f(z)|^2}|dz| = f^*(c_{\beta;\mathbb{D}}(z))|dz| \le c_\beta(z)|dz|.$$

If we suppose $c_\beta(z_0) = c_B(z_0)$ and set $f = f_0$ to be an extremal function for $c_B(z_0)$, then all inequalities above become equalities at z_0. By Minda's result, f_0 is a biholomorphism onto $\mathbb{D} \setminus P$ where P is a relatively closed polar set. Thus, Theorem A.1 is proved. The key ingredient of Minda's proof is the "sharp form of the Lindelöf principle" for the Green's function. In the following, we shall reprove Theorem A.1 without resorting to this principle.

Let f be a holomorphic function on an open Riemann surface X such that $f(X) \subset \mathbb{C}$ admits a Green's function. Then the Green's function satisfies a **subordination property**:

$$G_X(z, z') \ge G_{f(X)}(f(z), f(z')) \tag{A.1}$$

for all $(z, z') \in X \times X$. Moreover, if there exists some $(z_0, z_0') \in X \times X$ with $z_0 \ne z_0'$ such that equality in (A.1) holds, then

$$G_X(z, z') \equiv G_{f(X)}(f(z), f(z')) \tag{A.2}$$

for all $(z, z') \in X \times X$ and f is injective. See for instance [27, Theorem 4.4.4] for the proof for planar domains. The cases for Riemann surface can be proved similarly. A consequence of the property is the following rigidity property Lemma A.2 of the Green's function. We note that strengthened forms of the subordination property and Lemma A.2 were proved in [24, Theorem 1] by Minda using the Lindelöf principle for the Green's function.

Lemma A.2 (Rigidity lemma of the Green's function)**.** *On an open Riemann surface X, the Green's function with a pole $z_0 \in X$ is*

$$G(z, z_0) = \log |f(z)|$$

for some holomorphic function f on X if and only if f is a biholomorphism from X to the unit disk possibly less a relatively closed polar subset such that $f(z_0) = 0$.

Proof. Since $G_X(z, z_0) < 0$ for $z \in X$, $f(X) \subset \mathbb{D}$. The image $f(X)$ admits a Green's function because $f(X)$ is bounded. Also, observe that $f(z_0) = 0$. Applying the subordination property (A.1) to f and the identity map, we get

$$\log |f(z)| = G_X(z, z_0) \geq G_{f(X)}(f(z), 0) \geq G_{\mathbb{D}}(f(z), 0) = \log |f(z)|.$$

By the subordination property (A.2), f is injective and $G_{f(X)}(\zeta, 0) = \log |\zeta|$ for $\zeta \in f(X)$. Let $\eta \in \partial f(X) \cap \mathbb{D}$ and $\zeta_n (\in f(X)) \to \eta$. Since

$$\lim_{n\to\infty} G_{f(X)}(\zeta_n, 0) = \lim_{n\to\infty} \log |\zeta_n| = \log |\eta| < 0,$$

η is an irregular boundary point. By Kellogg's Theorem, cf. [27, Theorem 4.2.5], $P = \partial f(X) \cap \mathbb{D}$ is a relatively closed polar set in $\mathbb{D}$. Suppose $z_0 \in \mathbb{D} \backslash \overline{f(X)}$. Then for some $\epsilon > 0$, $f(X) \subset \mathbb{D} \backslash \overline{\mathbb{D}(z_0, \epsilon)}$. Let k be the harmonic function defined on the latter set with Dirichlet boundary data

$$k(z) = \begin{cases} 0, & z \in \partial \mathbb{D} \\ -\log(|z_0| + \epsilon), & z \in \partial \mathbb{D}(z_0, \epsilon). \end{cases}$$

Since $G_{f(X)}(z, 0) \leq G_{f(X)}(z, 0) + k(z)$ and $G_{f(X)}(z, 0) + k(z)$ is in the defining set of (2.1) for the domain $f(X)$, we have arrived at a contradiction unless $\mathbb{D} \setminus \overline{f(X)}$ is empty. Thus, $\mathbb{D} = f(X) \sqcup P$, where $\sqcup$ denotes the disjoint union. The proof of the theorem is complete. □

In [19, Theorem 3.1, p. 1196] of the equality part of the Suita conjecture, Guan and Zhou showed that if $\pi K(z_0) = c_\beta^2(z_0)$, then by their optimal L^2 extension theorem, $\exp G(z, z_0) = |g(z)|$, for some holomorphic function g on Ω. By proving further $c_\beta^2(z_0) = c_B^2(z_0)$ if $\exp G(z, z_0) = |g(z)|$, they were able to apply Suita's Theorem (Theorem 1.1) to yield that X is biholomorphic to a disk less a relatively closed polar set.

In fact, Lemma A.2 says that $|g(z)| = \exp G(z, z_0)$ if and only if the surface is biholomorphic to a disk less a relatively closed polar set. Consequently, one does not need to involve the analytic capacity or Suita's Theorem to prove the equality part of the Suita conjecture as in [19]. See also [13] for related work.

Next, by relying more explicitly on the analytic capacity, we shall give a proof of Theorem A.1 by making use of Lemma A.2 and following an idea of Guan and Zhou in [19, Lemma 4.25].

Proof of Theorem A.1. If X is biholomorphic to a disk less a relatively closed polar set, then $c_\beta \equiv c_B$, since the polar part is negligible.

Conversely, write $G_{z_0}(z)$ for $G(z, z_0)$, and let $u(z) := \log(|f_0(z)|)$ for $z \in X$, where f_0 is an extremal function of the analytic capacity at z_0. Since $|f_0| < 1$ on X, $u \in SH^-(X)$. By definition of the Green's function, we further see that $u - G_{z_0} \in SH^-(X)$. We will show that $u - G_{z_0}|_{z=z_0} = 0$. In the local coordinate $w(z)$ of X near z_0, $\lim_{z\to z_0} \log|f_0(z)| - \log|w(z)| = \log|\frac{df_0}{dw}(z_0)| = \log c_B(z_0)$ by definition of f_0. Thus for $z \in X$ near z_0,

$$u(z) - G_{z_0}(z) = (\log(|f_0(z)|) - \log|w(z)|) - (G_{z_0}(z) - \log|w(z)|) \to \log c_B(z_0) - \log c_\beta(z_0) = 0.$$

As a consequence of the maximum principle of subharmonic functions, we have

$$G_{z_0}(z) = \log(|f_0(z)|).$$

By Lemma A.2, f_0 is a biholomorphism from X to the unit disk less a relatively closed polar set, with $f_0(z_0) = 0$.

To complete the second part of the theorem, let $f : X \to \mathbb{D} \setminus P$ be any biholomorphism, and let $\varphi(z) := \frac{z - f(z_0)}{1 - \overline{f(z_0)} z} \in \mathrm{Aut}(\mathbb{D})$. Then, $F := \varphi \circ f : X \to \mathbb{D} \setminus \varphi(P)$ is also a biholomorphism such that $F(z_0) = 0$, and $G_{z_0} = \log|F|$ by Lemma A.2. We will show that F is an extremal function for $c_B(z_0)$. For any $h \in Hol(X, \mathbb{D})$ with $h(z_0) = 0$ and $h'(z_0) \neq 0$, we have that $\log|h| \in SH^-(X)$. By the definition of the Green's function, $\log|h| \leq G_{z_0} = \log|F|$ and so $|h'(z_0)| \leq |F'(z_0)|$. Therefore,

$$c_B(z_0) = |F'(z_0)|,$$

which means F is extremal.

□

It is known (see [29, VII.5H]) that if X is a regular region of connectivity greater than or equal to 2, then

$$c_\beta(z) > c_B(z), \quad \text{for all } z \in X. \tag{A.3}$$

Theorem A.1 says that (A.3) in fact holds for any hyperbolic Riemann surface X which is not biholomorphic to a disk possibly less a relatively closed polar subset.

B Stability of the Szegö kernel on Lipschitz domains

The Szegö projection and its kernel on planar domains with Lipschitz boundaries were studied extensively by Lanzani [22]. One of the key steps in extending results from smoothly bounded domains to those with Lipschitz boundaries is by approximating the bounded Lipschitz domain Ω by smoothly bounded subdomains in the sense of Nečas (see [22, Theorem 2.3]).

Theorem B.1. [22] *Let $\Omega \subset \mathbb{C}$ be a bounded domain with Lipschitz boundary. Then there exists a sequence of subdomains $\{\Omega_j\}_{j=1}^\infty$ with C^∞ smooth boundaries such that*

1. *There is a sequence of Lipschitz homeomorphisms $\Lambda_j : \partial\Omega \to \partial\Omega_j$ such that $\Lambda_j(P) \in \Gamma(P)$ (see Definition B.2) and $\lim_{j\to\infty} \Lambda_j(P) = P$.*

2. *There exists functions* $\omega_j : \partial\Omega \to (0,\infty)$ *that are uniformly bounded away from 0 and* ∞*, converge to 1 almost everywhere and*

$$\int_{\partial\Omega} h(\Lambda_j(w))\omega_j(w)d\sigma(w) = \int_{\partial\Omega_j} h(\zeta)d\sigma_j(\zeta)$$

for any $h \in L^1(\partial\Omega_j)$.

3. *The unit tangent vector* T_j *for* $\partial\Omega_j$ *and the unit tangent vector* T *for* $\partial\Omega$ *satisfy that* $T_j(\Lambda_j(\cdot)) \to T(\cdot)$ *almost everywhere on* $\partial\Omega$.

If Ω has C^∞ boundary with a defining function ρ and Szegö kernel S, define the sublevel sets of the defining function for small $\epsilon > 0$

$$\Omega_\epsilon = \{z \in \Omega : \rho(z) < -\epsilon\},$$

and consider the Szegö kernels $S_\epsilon(\cdot,\cdot)$ of these respective domains. In the proof of Theorem 2.1 of [8], Boas showed that for $a, z \in \Omega_\epsilon$,

$$\lim_{\epsilon\to 0^+} S_\epsilon(z,a) = S(z,a).$$

The purpose of this section is to extend this stability result of the Szegö kernel to Lipschitz domains in $\mathbb{C}$ with respect to the Nečas approximation. We begin by giving notations along the lines of [22].

Definition B.2. 1. *For* $\lambda > 0$, $P \in \partial\Omega$*, the non-tangential approach region to* P *is*

$$\Gamma(P) = \{\zeta : |\zeta - P| \le (1+\lambda)dist(\zeta, \partial\Omega)\}$$

2. *For a function* f *defined on a Lipschitz domain, the non-tangential limit* f^+ *and non-tangential maximal function* f^**, if they exist, are defined respectively as*

$$f^*(P) = \sup_{w\in\Gamma(P)} |f(w)|, \quad f^+(P) = \lim_{\substack{w\to P \\ w\in\Gamma(P)}} f(w), \quad P \in \partial\Omega.$$

3. *The Hardy space of a bounded domain* Ω *with Lipschitz boundary is*

$$H^2(\partial\Omega) = \{f^+ : f \in \mathcal{O}(\Omega), f^* \in L^2(\partial\Omega)\}.$$

A useful characterization of the Szegö kernel for our purposes is

$$S(z,a) = \frac{f(z)}{\|f^+\|^2_{\partial\Omega}}, \tag{B.1}$$

where f is the unique function with minimal $L^2(\partial\Omega)$-norm among all functions in $H^2(\partial\Omega)$ with $f(a) = 1$. Using Theorem B.1 we can adapt the proof of Boas [8, Theorem 2.1].

Proposition B.3. *Let* Ω *be a bounded domain with Lipschitz boundary and* $S(\cdot,\cdot)$ *be its Szegö kernel. Let* $\{\Omega_j\}_{j=1}^\infty$ *be a sequence of subdomains as in Theorem B.1. Denote by* $\{S_j(\cdot,\cdot)\}$ *the corresponding Szegö kernels for these domains. Then for each* $a, z \in \Omega$,

$$\lim_{j\to\infty} S_j(z,a) \to S(z,a).$$

Proof. Let $\{f_j\}$ and f be the extremal functions for the domains $\{\Omega_j\}_{j=1}^{\infty}$ and Ω as in (B.1). Since $|f\circ\Lambda_j|^2\omega_j \le C|f^*|^2 \in L^1(\partial\Omega)$,

$$\|f_j^+\|^2_{\partial\Omega_j} \le \|f\|^2_{\partial\Omega_j} = \|(f\circ\Lambda_j)\omega_j^{\frac{1}{2}}\|^2_{\partial\Omega} \le M < \infty \tag{B.2}$$

for some constant M, and by the dominated convergence theorem

$$\lim_{j\to\infty} \|(f\circ\Lambda_j)\omega_j^{\frac{1}{2}}\|^2_{\partial\Omega} = \|f^+\|^2_{\partial\Omega}. \tag{B.3}$$

By (B.2), on each Ω_k, $\{f_j\}_{j\ge k}$ is a normal family. After passing to a subsequence, there exists an $F\in\mathcal{O}(\Omega)$ such that $f_j\to F$ uniformly on compact subsets. Necessarily this implies $F(a)=1$.

By (B.2), after passing to a subsequence $(f_j^+\circ\Lambda_j)\omega_j^{1/2}$ converges weakly to f_∞ in $L^2(\partial\Omega)$. Let $\mathcal{C}$ denote the Cauchy transform. We claim that

$$F(z) = \mathcal{C}(f_\infty)(z)\left(:= \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f_\infty(w)}{w-z}dw \in H^2(\partial\Omega)\right).$$

In fact for any fixed $z\in\Omega_j$,

$$\begin{aligned}
|f_j(z)-\mathcal{C}(f_\infty)(z)| &= \left|\int_{\partial\Omega}\frac{f_j^+(\Lambda_j(w))T_j(\Lambda_j(w))\omega_j(w)}{\Lambda_j(w)-z}d\sigma - \int_{\partial\Omega}\frac{f_\infty(w)T(w)}{w-z}d\sigma\right| \\
&\le \int_{\partial\Omega}|f_j^+(\Lambda_j(w))\omega_j^{\frac{1}{2}}(w)|\left|\frac{T_j(\Lambda_j(w))\omega_j^{\frac{1}{2}}(w)}{\Lambda_j(w)-z} - \frac{T(w)}{w-z}\right|d\sigma \\
&\quad + \left|\int_{\partial\Omega}\left(f_j^+(\Lambda_j(w))\omega_j^{\frac{1}{2}}(w) - f_\infty(w)\right)\frac{T(w)}{w-z}d\sigma\right| := A+B.
\end{aligned}$$

Since $\frac{T(w)}{w-z}\in L^\infty(\partial\Omega)$ and $(f_j^+\circ\Lambda_j)\omega_j^{\frac{1}{2}}$ converges weakly to f_∞ in $L^2(\partial\Omega)$, we have $B\to 0$. For A, notice that by Theorem B.1 and the dominated convergence theorem, $\frac{T_j(\Lambda_j(w))\omega_j^{\frac{1}{2}}(w)}{\Lambda_j(w)-z}$ converges to $\frac{T(w)}{w-z}$ in $L^2(\partial\Omega)$ norm. Making use of Hölder inequality and the uniform boundedness of $\|(f_j^+\circ\Lambda_j)\omega_j^{1/2}\|_{\partial\Omega}$, we get

$$A \le \|(f_j^+\circ\Lambda_j)\omega_j^{\frac{1}{2}}\|_{\partial\Omega}\left\|\frac{T_j(\Lambda_j(w))\omega_j^{\frac{1}{2}}(w)}{\Lambda_j(w)-z} - \frac{T(w)}{w-z}\right\|_{\partial\Omega} \to 0.$$

Hence $f_j\to\mathcal{C}f_\infty$ pointwisely, and so $F=\mathcal{C}(f_\infty)$.

Using the fact that $\{\overline{TG} : G\in H^2(\partial\Omega)\} = H^2(\partial\Omega)^\perp$, we will show that $f_\infty\in H^2(\partial\Omega)$, cf. [22, Theorem 5.1], [3, Theorem 4.3]. Since $\langle f_j^+, \overline{T_jG}\rangle_{\partial\Omega_j}=0$,

$$\begin{aligned}
&|\langle f_\infty, \overline{TG}\rangle| \\
&= |\langle f_\infty - (f_j^+\circ\Lambda_j)\omega_j^{\frac{1}{2}}, \overline{TG}\rangle_{\partial\Omega} \\
&\qquad + \int_{\partial\Omega} f_j^+(\Lambda_j)(w))\omega_j^{\frac{1}{2}}(w)T(w)G(w) - (f_j^+(\Lambda_j)(w))T_j(\Lambda_j(w))G(\Lambda_j(w))\omega_j(w)d\sigma(w)| \\
&\le |\langle f_\infty - (f_j^+\circ\Lambda_j)\omega_j^{\frac{1}{2}}, \overline{TG}\rangle_{\partial\Omega}| + M^{\frac{1}{2}}\Big(\int_{\partial\Omega}|T(w)G(w) - T_j(\Lambda_j(w))G(\Lambda_j(w))\omega_j^{\frac{1}{2}}(w)|^2 d\sigma(w)\Big)^{\frac{1}{2}}.
\end{aligned}$$

The first term above approaches 0 as j approaches ∞ by weak-convergence. The integrand in the second term is dominated by $C|G^*|^2$ for some constant C. By the dominated convergence theorem, the integral approaches 0 as well. Thus, $f_\infty \in H^2(\partial\Omega)$. Since the Cauchy transform is a projection onto $H^2(\partial\Omega)$, $f_\infty = F^+$ on $\partial\Omega$. In particular $(f_j^+ \circ \Lambda_j){\omega_j}^{\frac{1}{2}}$ converges weakly to F^+ in $L^2(\partial\Omega)$ as well. Hence

$$\begin{aligned}\|F^+\|^2_{\partial\Omega} &\leq \liminf_{j\to\infty} \|(f_j \circ \Lambda_j)\omega_j^{\frac{1}{2}}\|^2_{\partial\Omega} = \liminf_{j\to\infty} \|f_j^+\|^2_{\partial\Omega_j} \\ &\leq \limsup_{j\to\infty} \|f_j^+\|^2_{\partial\Omega_j} \leq \limsup_{j\to\infty} \|f\|^2_{\partial\Omega_j} = \|f^+\|^2_{\partial\Omega}. \end{aligned} \tag{B.4}$$

Here the first inequality uses the weakly sequentially lower semicontinuity for the $L^2(\partial\Omega)$ norm, the third inequality applies (B.2) and the last equality uses (B.3). Since f is unique, $F = f$ on Ω, and all inequalities in (B.4) become equalities. In particular, f_j converges pointwisely to f on Ω and $\|f_j^+\|_{\partial\Omega_j} \to \|f^+\|_{\partial\Omega}$ by (B.4). By (B.1), $S_j(z,a) \to S(z,a)$.

□

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