Title A dualization of Hoffman's circulation theorem.

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Abstract This is joint work with Oliver Pretzel of Imperial College, London. Consider a directed graph G and a ring R with $\mathbb{Z} \leq R \leq \mathbb{R}$. Given an edge e in G, let $I(e) = [a(e), b(e)]$ be a closed R-interval. In linear-programming terminology, $I(e)$ is called a *capacity interval*. We denote the opposite direction along e by $-e$ and we say that $I(-e) = [-b(e), -a(e)]$. Setting $(-1)e = -e$ we can now extend I to R-linear combinations of edges by

$$I\left(\sum_i \lambda_i e_i\right) = \left[\sum_i \lambda_i a(e_i)\,, \sum_i \lambda_i b(e_i)\right].$$

Let $C(G, R)$ denote the R-module of all R-linear combinations of edges of G, let $Z(G, R) \leq C(G, R)$ be the submodule generated by cycles, and $K(G, R) \leq C(G, R)$ be the submodule generated by edge cuts. By linearity, a homomorphism $f\colon C(G, R) \to R$ satisfies $f(c) \in I(c)$ for all $c \in C(G, R)$ iff $f(e) \in I(e)$ for all edges e of G. Such a homomorphism is said to be *capacity respecting*. Of course, a capacity-respecting homomorphism on $C(G, R)$ can be restricted to any submodule of $C(G, R)$ and it will still be capacity respecting.

Hoffman's circulation theorem (1960) states that any capacity-respecting homomorphism on the submodule $K(G, R)$ extends to a capacity-respecting homomorphism on all of $C(G, R)$. Our main result is that this extendability property also holds holds for $Z(G, R)$ when $\mathbb{Z} \leq R \leq \mathbb{Q}$. Since $Z(G, R)^\perp = K(G, R)$ this can be thought of as a dualization of Hoffman's theorem. The proof of our theorem is more complicated than the proof of Hoffman's. This is due to the fact that $K(G, R)$ has a canonical choice of basis while $Z(G, R)$ does not. We will discuss this and other related results.