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A variant of Hörmander's L^2 theorem for Dirac operator in Clifford analysis

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Abstract In this paper, we give the Hörmander's L^2 theorem for Dirac operator over open subset $\Omega \in \mathbb{R}^{n+1}$ with Clifford algebra. Some sufficient conditions on the existence of weak solutions for Dirac operators have been found in the sense of Clifford analysis. In particular, if Ω is bounded, then we prove that for any f in L^2 space with value in Clifford algebra, we can find a weak solution of Dirac operator such that

$$\overline{D}u = f$$

with the solution u in the L^2 space as well. The method is based on Hörmander's L^2 existence theorem in complex analysis and the L^2 weighted space is utilised.

Keywords Hörmander's L^2 theorem · Clifford analysis · weak solution · Dirac operator

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1 Introduction

The development of function theories over Clifford algebras has proved a useful setting for generalizing many aspects of one variable complex function theory to higher dimensions. The study of these function theories is referred to as Clifford analysis [Brackx et al(1982),Huang et al(2006),Gong et al(2009),Ryan(2000)]. This analysis is closely related to a number of studies made in mathematical physics, and many applications have been found in this area in recent years. In [Ryan(1995)], Ryan considered solutions of the polynomial Dirac operator, which affords an integral representation. Furthermore, the author gave a Pompeiu representation for C^1-functions in a Lipschitz bounded domain. In [Ryan(1990)], the author presented a classification of linear, conformally invariant, Clifford-algebra-valued differential operators over $\mathbb{C}^n$, which comprise the Dirac operator and its iterates. In [Qian and Ryan(1996)], Qian and Ryan used Vahlen matrices to study the conformal covariance of various types of Hardy spaces over hypersurfaces in $\mathbb{R}^n$. In [De Ridder et al(2012)], the discrete Fueter polynomials was introduced, which formed a basis of the space of discrete spherical monogenics. Moreover, the explicit construction for this discrete Fueter basis, in arbitrary dimension m and for arbitrary homogeneity degree k was presented as well.

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In [Hörmander(1965)], the famous Hörmander's L^2 existence and approximation theorems was given for the $\bar{\partial}$ operator in pseudo-convex domains in $\mathbb{C}^n$. When $n = 1$, the existence theorem of complex variable can be deduced. The aim of this paper is to establish a Hörmander's L^2 theorem in $\mathbb{R}^{n+1}$ with Clifford analysis, and provide sufficient conditions on the existence of the weak solutions for Dirac operator in the sense of Clifford algebra.

Let $\mathcal{A}$ be an real Clifford algebra over an (n+1)-dimensional real vector space $\mathbb{R}^{n+1}$ and the corresponding norm on $\mathcal{A}$ is given by $|\lambda|_0 = \sqrt{(\lambda, \lambda)_0}$ (see subsection 2.1). Let Ω be an open subset of $\mathbb{R}^{n+1}$, $L^2(\Omega, \mathcal{A}, \varphi)$ be a right Hilbert $\mathcal{A}$-module for a given function $\varphi \in C^2(\Omega, \mathbb{R})$ with the norm given by Definition 29. (see subsection 2.3). $\overline{D}$ denotes the Dirac differential operator and the operator $\overline{D}^*_\varphi$ is given by (9). Then we can present the main results of the paper as follows.

Theorem 11 *Given $f \in L^2(\Omega, \mathcal{A}, \varphi)$, there exists $u \in L^2(\Omega, \mathcal{A}, \varphi)$ such that*

$$\overline{D}u = f \tag{1}$$

with

$$\|u\|^2 = \int_\Omega |u|_0^2 e^{-\varphi} dx \le 2^n c \tag{2}$$

if

$$|(f, \alpha)_\varphi|_0^2 \le c\|\overline{D}^*_\varphi \alpha\|^2 = c \int_\Omega |\overline{D}^*_\varphi \alpha|_0^2 e^{-\varphi}, \ \forall \alpha \in C_0^\infty(\Omega, \mathcal{A}). \tag{3}$$

Conversely, if there exists $u \in L^2(\Omega, \mathcal{A}, \varphi)$ such that (1) is satisfied with

$$\|u\|^2 \le c$$

Then we can get (3).

The factor 2^n in (2) comes from the definition of the norm in Clifford analysis. If $n = 1$, then the factor would disappear which gives the necessary and sufficient condition in the theorem. From the above theorem, we give the following sufficient conditions on the existence of weak solutions for Dirac operator.

Theorem 12 *Given $\varphi \in C^2(\Omega, \mathbb{R})$ with Ω being an open subset of $\mathbb{R}^{n+1}$ and $n > 1$; $\Delta\varphi \ge 0$, and $\frac{\partial^2 \varphi}{\partial x_j \partial x_i} = 0, \ i \ne j, \ 1 \le i, j \le n$ and $\frac{\partial^2 \varphi}{\partial x_i^2} \le 0, \ 1 \le i \le n$. Then for all $f \in L^2(\Omega, \mathcal{A}, \varphi)$ with $\int_\Omega \frac{|f|_0^2}{\Delta\varphi} e^{-\varphi} dx = c < \infty$, there exists a $u \in L^2(\Omega, \mathcal{A}, \varphi)$ such that*

$$\overline{D}u = f$$

with

$$\|u\|^2 = \int_\Omega |u|_0^2 e^{-\varphi} dx \le 2^n \int_\Omega \frac{|f|_0^2}{\Delta\varphi} e^{-\varphi} dx.$$

Remark 13 *Assuming $x = (x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1}$, it is easy to see that $\varphi(x) = x_0^2$ satisfies the conditions in Theorem 12. Another simple example would be*

$$\varphi(x) = (n+1)x_0^2 - \sum_{i=1}^n x_i^2.$$

It is obvious that $\Delta\varphi(x) = 2$, $\frac{\partial^2 \varphi}{\partial x_i^2} = -2$, and $\frac{\partial^2 \varphi}{\partial x_j \partial x_i} = 0, \ i \ne j, \ 1 \le i, j \le n$.

Corollary 14 *Given $\varphi \in C^2(\Omega, \mathbb{R})$, and $\varphi(x) = \varphi(x_0)$ with $\varphi''(x_0) \ge 0$. Then for all $f \in L^2(\Omega, \mathcal{A}, \varphi)$ with $\int_\Omega \frac{|f|_0^2}{\varphi''} e^{-\varphi} dx = c < \infty$, there exists a $u \in L^2(\Omega, \mathcal{A}, \varphi)$ such that*

$$\overline{D}u = f$$

with

$$\|u\|^2 = \int_\Omega |u|_0^2 e^{-\varphi} dx \le 2^n \int_\Omega \frac{|f|_0^2}{\varphi''} e^{-\varphi} dx.$$

Furthermore, there is nothing to do with the boundary conditions of Ω in the above results. This phenomenon is totally different with the famous Hörmander's L^2 existence theorems of several complex variables in [Hörmander(1965)]. Then we can also have the following corollary on global solutions.

Theorem 15 *Given $\varphi \in C^2(\mathbb{R}^{n+1}, \mathbb{R})$ with all derivative conditions in Theorem 11 satisfied. Then for all $f \in L^2(\mathbb{R}^{n+1}, \mathcal{A}, \varphi)$ with $\int_{\mathbb{R}^{n+1}} \frac{|f|_0^2}{\Delta\varphi} e^{-\varphi} dx = c < \infty$, there exists a $u \in L^2(\mathbb{R}^{n+1}, \mathcal{A}, \varphi)$ satisfying*

$$\overline{D}u = f$$

with

$$\|u\|^2 = \int_{\mathbb{R}^{n+1}} |u|_0^2 e^{-\varphi} dx \leq 2^n \int_{\mathbb{R}^{n+1}} \frac{|f|_0^2}{\Delta\varphi} e^{-\varphi} dx.$$

On the other hand, if the boundary of Ω is concerned, we consider a special kind of $\Omega_0 = \{x \in \mathbb{R}^{n+1} : a \leq x_0 \leq b\}$ for any $a,\ b \in \mathbb{R}$ with $a < b$, then we can get the following theorem within L^2 space instead of L^2 weighted space.

Theorem 16 *Let $f \in L^2(\Omega_0, \mathcal{A})$. Then there exists a $u \in L^2(\Omega_0, \mathcal{A})$ such that*

$$\overline{D}u = f$$

with

$$\int_{\Omega_0} |u|_0^2 dx \leq 2^n c(a, b) \int_{\Omega_0} |f|_0^2 dx$$

and $c(a, b)$ is a factor depending on $a,\ b$.

Proof Let $\varphi(x) = x_0^2$. It can be obtained that $L^2(\Omega_0, \mathcal{A}) = L^2(\Omega_0, \mathcal{A}, \varphi)$ for the boundary of x_0. Then the theorem is proved with Theorem 12.

Remark 17 *In particular, any bounded domain Ω in $\mathbb{R}^{n+1}$ can be regarded as one type of Ω_0. Therefore, it comes from Theorem 16 that for any $f \in L^2(\Omega, \mathcal{A})$, we can find a weak solution of Dirac operator $\overline{D}u = f$ with $u \in L^2(\Omega, \mathcal{A})$.*

2 Preliminaries

To make the paper self-contained, some basic notations and results used in this paper are included.

2.1 The Clifford algebra $\mathcal{A}$

Let $\mathcal{A}$ be an real Clifford algebra over an (n+1)-dimensional real vector space $\mathbb{R}^{n+1}$ with orthogonal basis $e := \{e_0, e_1, ..., e_n\}$, where $e_0 = 1$ is a unit element in $\mathbb{R}^{n+1}$. Furthermore,

$$\begin{cases} e_i e_j + e_j e_i = 0,\ i \neq j \\ e_i^2 = -1,\ i = 1, ..., n. \end{cases} \tag{4}$$

Then $\mathcal{A}$ has its basis

$$\{e_A = e_{h_1 \cdots h_r} = e_{h_1} \cdots e_{h_r} : 1 \leq h_1 < ... < h_r \leq n, 1 \leq r \leq n\}.$$

If $i \in \{h_1, ..., h_r\}$, we denote $i \in A$ and if $i \notin \{h_1, ..., h_r\}$, we denote $i \notin A$. $A - i$ means $\{h_1, ..., h_r\} \setminus \{i\}$ and $A + i$ means $\{h_1, ..., h_r\} \cup \{i\}$. So real Clifford algebra is composed of elements having the type $a = \sum\limits_A x_A e_A$, in which $x_A \in \mathbb{R}$ are real numbers. For $a \in \mathcal{A}$, we give the inversion in the Clifford algebra as follows: $a^* = \sum\limits_A x_A e_A^*$ where $e_A^* = (-1)^{|A|} e_A$ and $|A| = n(A)$ is the $r \in \mathbb{Z}^+$ as $e_A = e_{h_1 \cdots h_r}$. When $A = \emptyset$, $|A| = 0$. Next, we define the reversion in the Clifford algebra, which is given by $a^\dagger = \sum\limits_A x_A e_A^\dagger$

where $e_A^\dagger = (-1)^{(|A|-1)|A|/2}e_A$. Now we present the involution which is a combination of the inversion and the reversion introduced above.

$$\bar{a} = \sum_A x_A \bar{e}_A$$

where $\bar{e}_A = e_A^{*\dagger} = (-1)^{(|A|+1)|A|/2}e_A$. From the definition, one can easily deduce that $e_A\bar{e}_A = \bar{e}_A e_A = 1$. Furthermore, we have

$$\overline{\lambda\mu} = \bar{\mu}\bar{\lambda}, \quad \forall \lambda, \mu \in \mathcal{A}.$$

Let $a = \sum\limits_A x_A e_A$ be a Clifford number. The coefficient x_A of the e_A-component will also be denoted by $[a]_A$. In particular the coefficient x_0 of the e_0-component will be denoted by $[a]_0$, which is called the scalar part of the Clifford number a. An inner product on $\mathcal{A}$ is defined by putting for any $\lambda, \mu \in \mathcal{A}$, $(\lambda, \mu)_0 := 2^n[\lambda\bar{\mu}]_0 = 2^n \sum\limits_A \lambda_A \mu_A$. The corresponding norm on $\mathcal{A}$ reads $|\lambda|_0 = \sqrt{(\lambda, \lambda)_0}$.

We define a real functional on $\mathcal{A}$ that $\tau_{e_A} : \mathcal{A} \to \mathbb{R}$

$$\langle \tau_{e_A}, \mu \rangle = 2^n(-1)^{(|A|+1)|A|/2}\mu_A.$$

In the special case where $A = \emptyset$ we have

$$\langle \tau_{e_0}, \mu \rangle = 2^n \mu_0.$$

Let Ω be an open subset of $\mathbb{R}^{n+1}$. Then functions f defined in Ω and with values in $\mathcal{A}$ are considered. They are of the form

$$f(x) = \sum_A f_A(x) e_A$$

where $f_A(x)$ are functions with real value. Let $\overline{D}$ denotes the Dirac differential operator

$$\overline{D} = \sum_{i=0}^n e_i \partial_{x_i},$$

its action on functions from the left and from the right being governed by the rules

$$\overline{D}f = \sum_{i,A} e_i e_A \partial_{x_i} f_A \text{ and } f\overline{D} = \sum_{i,A} e_A e_i \partial_{x_i} f_A.$$

f is called left-monogenic if $\overline{D}f = 0$ and it is called right-monogenic if $f\overline{D} = 0$. The conjugate operator is given by

$$D = \sum_{i=0}^n \bar{e}_i \partial_{x_i}.$$

It can be found that

$$\overline{D}D = D\overline{D} = \Delta$$

where Δ denotes the classical Laplacian in $\mathbb{R}^{n+1}$. When $n = 1$, one can think of x_0 as the real part and of x_1 as the imaginary part of the variable and to identify e_1 with i. the operator $\overline{D}$ then take the form $\overline{D} = \partial_{x_0} + i\partial_{x_1}$, which is similar the operator $\bar{\partial}$ in complex analysis.

2.2 Modules over Clifford algebras

This subsection is to give some general information concerning a class of topological modules over Clifford algebras. In the sequel definitions and properties will be stated for left $\mathcal{A}$-module and their duals, the passage to the case of right $\mathcal{A}$-module being straight-forward.

Definition 21 (unitary left $\mathcal{A}$-module) *Let X be a unitary left $\mathcal{A}$-module, i.e. X is abelian group and a law $(\lambda, f) \to \lambda f : \mathcal{A} \times X \to X$ is defined such that $\forall \lambda, \mu \in \mathcal{A}$, and $f,\ g \in X$*

(i) $(\lambda + \mu)f = \lambda f + \mu f$,
(ii) $\lambda\mu f = \lambda(\mu f)$,
(iii) $\lambda(f + g) = \lambda f + \lambda g$,
(iv) $e_0 f = f$.

Moreover, when speaking of a submodule E of the unitary left $\mathcal{A}$-module X, we mean that E is a non empty subset of X which becomes a unitary left $\mathcal{A}$-module too when restricting the module operations of X to E.

Definition 22 (left $\mathcal{A}$-linear operator) *If X, Y are unitary left $\mathcal{A}$-modules, then $T : X \to Y$ is said to be a left $\mathcal{A}$-linear operator, if $\forall\ f,\ g \in X$ and $\lambda \in \mathcal{A}$*

$$T(\lambda f + g) = \lambda T(f) + T(g).$$

The set of all "T" is denoted by $L(X, Y)$. If $Y = \mathcal{A}$, $L(X, \mathcal{A})$ is called the algebraic dual of X and denoted by X^{*alg}. Its elements are called left $\mathcal{A}$-linear functionals on X and for any $T \in X^{*alg}$ and $f \in X$, we denote by $\langle T, f\rangle$ the value of T at f.

Definition 23 (bounded functional) *An element $T \in X^{*alg}$ is called bounded, if there exist a semi-norm p on X and $c > 0$ such that for all $f \in X$*

$$|\langle T, f\rangle|_0 \leq c \cdot p(f).$$

Theorem 24 *[Brackx et al(1982)]***(Hahn-Banach type theorem)** *Let X be a unitary left $\mathcal{A}$-module with semi-norm p, Y be a submodule of X, and T be a left $\mathcal{A}$-linear functional on Y such that for some $c > 0$,*

$$|\langle T, g\rangle|_0 \leq c \cdot p(g), \quad \forall g \in Y$$

Then there exists a left $\mathcal{A}$-linear functional $\widetilde{T}$ on X such that

(i) $\widetilde{T}\ |_Y = T$,
(ii) *for some* $c^* > 0$, $|\langle \widetilde{T}, f\rangle|_0 \leq c^* \cdot p(f)$, $\forall f \in X$.

Definition 25 (inner product on a unitary right $\mathcal{A}$-module) *Let H be a unitary right $\mathcal{A}$-module, then a function $(\ ,\ ) :\ H \times H \to \mathcal{A}$ is said to be a inner product on H if for all $f, g, h \in H$ and $\lambda \in \mathcal{A}$,*

(i) $(f, g + h) = (f, g) + (f, h)$,
(ii) $(f, g\lambda) = (f, g)\lambda$,
(iii) $(f, g) = \overline{(g, f)}$,
(iv) $\langle \tau_{e_0}, (f, f)\rangle \geq 0$ *and* $\langle \tau_{e_0}, (f, f)\rangle = 0$ *if and only if* $f = 0$,
(v) $\langle \tau_{e_0}, (f\lambda, f\lambda)\rangle \leq |\lambda|_0^2 \langle \tau_{e_0}, (f, f)\rangle$.

From the definition on inner product, putting for each $f \in H$

$$\|f\|^2 = \langle \tau_{e_0}, (f, f)\rangle,$$

then it can be obtained that for any $f, g \in H$,

$$|\langle \tau_{e_0},\ (f, g)\rangle| \leq \|f\|\|g\|, \|f + g\| \leq \|f\| + \|g\|. \tag{5}$$

Hence, $\|\cdot\|$ is a proper norm on H turning it into a normed right A-module. Moreover, we have the following Cauchy-Schwarz inequality.

Proposition 26 *[Brackx et al(1982)] For all* $f, g \in H$, $|(f, g)|_0 \le \|f\|\|g\|$.

Definition 27 (right Hilbert $\mathcal{A}$-module) *Let H be a unitary right $\mathcal{A}$-module provided with an inner product $(\ ,\ )$. Then is it called a right Hilbert $\mathcal{A}$-module if it is complete for the norm topology derived from the inner product.*

Theorem 28 *[Brackx et al(1982)]***(Riesz representation theorem)** *Let H be a right Hilbert $\mathcal{A}$-modules and $T \in H^{*alg}$. Then T is bounded if and only if there exists a (unique) element $g \in H$ such that for all $f \in H$,*

$$T(f) := \langle T, f\rangle = (g, f).$$

2.3 Hilbert space of square integrable functions

Now we extend the standard Hilbert space of square integrable functions to Clifford algebra. First, we denote $L^1(\Omega, \mu)$ and $L^2(\Omega, \mu)$ be the sets of all integrable or square integrable functions defined on the domain $\Omega \in \mathbb{R}^{n+1}$ with respect to the measure μ. Then $L^1(\Omega, \mathcal{A}, \mu)$ and $L^2(\Omega, \mathcal{A}, \mu)$ are defined as the sets of functions $f : \Omega \to \mathcal{A}$ which are integrable or square integrable with respect to μ, i.e., if $f = \sum_A f_A e_A$, then for each A, $f_A \in L^1(\Omega, \mu)$, respectively $f_A^2 \in L^1(\Omega, \mu)$. Then **one may easily check that $L^1(\Omega, \mathcal{A}, \mu)$ and $L^2(\Omega, \mathcal{A}, \mu)$ are unitary bi-$\mathcal{A}$-module, i.e., unitary left-$\mathcal{A}$-module and unitary right-$\mathcal{A}$-module**. Furthermore, for any $f, g \in L^2(\Omega, \mathcal{A}, \mu)$, $\bar{f} \in L^2(\Omega, \mathcal{A}, \mu)$ while $\bar{f}g \in L^1(\Omega, \mathcal{A}, \mu)$, where $\bar{f}(x) = \overline{f(x)}$ and $(\bar{f}g)(x) = \bar{f}(x)g(x),\ x \in \Omega$. Consider as a right $\mathcal{A}$-module, define for $f, g \in L^2(\Omega, \mathcal{A}, \mu)$ that

$$(f, g) = \int_\Omega \bar{f}(x)g(x)d\mu.$$

Furthermore for any real linear functional T on $\mathcal{A}$

$$\langle T, (f, g)\rangle = \langle T, \int_\Omega \bar{f}(x)g(x)d\mu\rangle = \int_\Omega \langle T, \bar{f}(x)g(x)\rangle d\mu.$$

Consequently, taking $T = \tau_{e_0}$ we find that

$$\begin{aligned}\langle \tau_{e_0}, (f, f)\rangle &= \langle \tau_{e_0}, \int_\Omega \bar{f}(x)f(x)d\mu\rangle = \int_\Omega \langle \tau_{e_0}, \bar{f}(x)f(x)\rangle d\mu \\ &= \int_\Omega |f(x)|_0^2 d\mu.\end{aligned} \tag{6}$$

Hence, for all $f \in L^2(\Omega, \mathcal{A}, \mu)$, $\langle \tau_{e_0}, (f, f)\rangle \ge 0$ and $\langle \tau_{e_0}, (f, f)\rangle = 0$ if and only if $f = 0$ a.e. in Ω. Then it is easy to see that under the inner product defined, all conditions for $L^2(\Omega, \mathcal{A}, \mu)$ to be a unitary right inner product $\mathcal{A}$-module are satisfied. Since $L^2(\Omega, \mathcal{A}, \mu) = \prod_A L^2(\Omega, \mu)$, we have that $L^2(\Omega, \mathcal{A}, \mu)$ is complete; in other words $L^2(\Omega, \mathcal{A}, \mu)$ is a right Hilbert $\mathcal{A}$-module, with the norm

$$\|f\|^2 = \langle \tau_{e_0}, (f, f)\rangle = \int_\Omega |f(x)|_0^2 d\mu$$

for $f \in L^2(\Omega, \mathcal{A}, \mu)$.

Definition 29 (weighted L^2 space) *Similar with $L^2(\Omega, \mathcal{A}, \mu)$, we can define the weighted $L^2(H, \mathcal{A}, \varphi)$ for a given function $\varphi \in C^2(\Omega, \mathbb{R})$. First, let*

$$L^2(\Omega, \varphi) = \big\{f | f : \Omega \to \mathbb{R},\ \int_\Omega |f(x)|^2 e^{-\varphi}\ dx < +\infty\big\}.$$

Then we denote

$$L^2(H, \mathcal{A}, \varphi) = \{f | f : \Omega \to \mathcal{A},\ f = \sum_A f_A e_A,\ f_A \in L^2(\Omega, \varphi)\}.$$

Moreover, for all $f, g \in L^2(H, \mathcal{A}, \varphi)$, we define

$$(f, g)_\varphi = \int_\Omega \bar{f}(x) g(x) e^{-\varphi} dx.$$

Then it is also easy to see $L^2(\Omega, \mathcal{A}, \varphi)$ is a right Hilbert $\mathcal{A}$-module, with the norm

$$\|f\|^2 = \langle \tau_{e_0}, (f, f)_\varphi \rangle = \int_\Omega |f(x)|_0^2 e^{-\varphi} dx \tag{7}$$

for $f \in L^2(\Omega, \mathcal{A}, \varphi)$.

2.4 Cauchy's integral formula

Let M be an (n+1)-dimensional differentiable and oriented manifold contained in some open subset Σ of $\mathbb{R}^{n+1}$. By means of the n-forms

$$d\hat{x}_i = dx_0 \wedge \cdots \wedge dx_{i-1} \wedge dx_{x_{i+1}} \wedge \cdots \wedge dx_n,\ i = 0, 1, ..., n,$$

an $\mathcal{A}$-valued n-form is introduced by putting

$$d\sigma = \sum_{i=0}^{n} (-1)^i e_i d\hat{x}_i,$$

similarly, denote

$$d\bar{\sigma} = \sum_{i=0}^{n} (-1)^i \bar{e}_i d\hat{x}_i.$$

Furthermore the volume-element

$$dx = dx_0 \wedge \cdots \wedge dx_n$$

is used.

Proposition 210 *[Brackx et al(1982)]***(Stokes-Green Theorem)** *If $f, g \in C^1(\Sigma, \mathcal{A})$ then for any (n+1)-chain Ω on $M \subset \Sigma$,*

$$\int_{\partial\Omega} f d\sigma g = \int_\Omega (f\overline{D}) g dx + \int_\Omega f(\overline{D} g) dx,$$

$$\int_{\partial\Omega} f d\bar{\sigma} g = \int_\Omega (fD) g dx + \int_\Omega f(Dg) dx.$$

Remark 211 *Denote $C_0^\infty(\Omega, \mathbb{R})$ as the set of all smooth real-valued functions with compact support in Ω and $C_0^\infty(\Omega, \mathcal{A}) := \{f | f : \Omega \to \mathcal{A},\ f = \sum_A f_A e_A,\ f_A \in C_0^\infty(\Omega, \mathbb{R})\}$. If f or $g \in C_0^\infty(\Omega, \mathcal{A})$, then we have from the Stokes-Green theorem that*

$$\int_\Omega (f\overline{D}) g dx = -\int_\Omega f(\overline{D} g) dx,$$

$$\int_\Omega (fD) g dx = -\int_\Omega f(Dg) dx.$$

Lemma 212 *If $u(x) \in C^1(\Omega, \mathcal{A})$, then $\overline{\overline{D}u} = \bar{u}D$.*

Proof Let $u(x) = \sum_A e_A u_A$. Then

$$\overline{\overline{D}u} = \sum_{i,A} \overline{e_i e_A} \partial_{x_i} u_A = \sum_{i,A} \bar{e}_A \bar{e}_i \partial_{x_i} u_A = \bar{u} D. \tag{8}$$

Lemma 213 *[Huang et al(2006)] If $u(x) = \sum_A e_A u_A$, $v(x) = \sum_{i=0}^n e_i v_i$, then*

$$\overline{D}(uv) = (\overline{D}u)v + u(\overline{D}v) + \sum_{j=1}^{n} (e_j u - u e_j) \partial_{x_j} v.$$

2.5 Weak solutions

Definition 214 **($\overline{D}$ solution in weak sense)** *If $f \in L^1_{loc}(\Omega, \mathcal{A})$, $u : \Omega \to \mathcal{A}$ is a weak solution of*

$$\overline{D}u = f \ (or\ Du = f)$$

if for any $\alpha \in C_0^\infty(\Omega, \mathcal{A})$,

$$\int_\Omega \alpha f dx = -\int_\Omega (\alpha\overline{D})u dx \ (or \ \int_\Omega \alpha f dx = -\int_\Omega (\alpha D)u dx).$$

It should be noticed that if u is a weak solution of $\overline{D}u = 0$, in addition, if u is smooth in Ω, then it is left-monogenic. Now it is natural to give the definition of Δ solution in weak sense.

Definition 215 **(Δ solution in weak sense)** *If $f \in L^1_{loc}(\Omega, \mathcal{A})$, $u : \Omega \to \mathcal{A}$ is a weak solution of*

$$\Delta u = f$$

if for any $\alpha \in C_0^\infty(\Omega, \mathcal{A})$,

$$\int_\Omega \alpha f dx = \int_\Omega (\Delta\alpha)u dx.$$

Theorem 216 *If $f \in L^1_{loc}(\Omega, \mathcal{A})$, and $\overline{D}f = 0$ in weak sense, then f is left-monogenic at any point of Ω.*

Proof : Since $\overline{D}f = 0$ in weak sense, then $\Delta f = 0$ in weak sense. By Weyl's lemma, f is smooth in Ω and has $\Delta f = 0$ in classical sense, then of course f is left-monogenic at any point of Ω.

Remark 217 *This is useful to deal with uniqueness of weak solutions. for example, if $u,\ v \in L^1_{loc}(\Omega, \mathcal{A})$ are two weak solutions of $\overline{D}u = f$, then $u = v + w$ with any w left-monogenic.*

Remark 218 *An important example of a left monogenic function is the generalized Cauchy kernel*

$$G(x) = \frac{1}{\omega_{n+1}} \frac{\overline{x}}{|x|^{n+1}},$$

where ω_{n+1} denotes the surface area of the unit ball in $\mathbb{R}^{n+1}$. This function obviously belongs to $L^1_{loc}(\Omega, \mathcal{A})$ and is a fundamental solution of the Dirac operator in the classical sense at any point of $\mathbb{R}^{n+1}$ except 0. However, it is not a weak solution of the Dirac operator. In fact, if it satisfies $\overline{D}f = 0$ in weak sense, then from Theorem 216, it must be left-monogenic in the any point of Ω which could include 0. Therefore, we get a contradiction.

For $f \in L^2(\Omega, \mathcal{A}, \varphi)$, $u : \Omega \to \mathcal{A}$. If $\overline{D}u = f$, based on the Stokes-Green theorem, we can define the dual operator $\overline{D}^*_\varphi$ of $\overline{D}$ under the inner product of $L^2(\Omega, \mathcal{A}, \varphi)$. For any $\alpha \in C_0^\infty(\Omega, \mathcal{A})$,

$$\begin{aligned} (\alpha, f)_\varphi &= \int_\Omega \bar{\alpha} f e^{-\varphi} dx = \int_\Omega \bar{\alpha} e^{-\varphi} f dx \\ &= \int_\Omega (\bar{\alpha} e^{-\varphi})(\overline{D}u) dx \\ &= -\int_\Omega \big((\bar{\alpha} e^{-\varphi})\overline{D}\big) u dx \\ &= -\int_\Omega \big((\bar{\alpha} e^{-\varphi})\overline{D}\big) e^\varphi u e^{-\varphi} dx \\ &= \int_\Omega \overline{-e^\varphi D(\alpha e^{-\varphi})} u e^{-\varphi} dx \\ &= (-e^{-\varphi} D(\alpha e^{-\varphi}), u)_\varphi \triangleq (\overline{D}^*_\varphi \alpha, u)_\varphi, \end{aligned} \tag{9}$$

where $\overline{D}^*_\varphi\alpha = -e^\varphi D(\alpha e^{-\varphi}) = \alpha(D\varphi) - D\alpha$, i.e.

$$(\alpha, \overline{D}u)_\varphi = (\overline{D}^*_\varphi\alpha, u)_\varphi.$$

In the same way, we also have

$$(\overline{D}u, \alpha)_\varphi = (u, \overline{D}^*_\varphi\alpha)_\varphi.$$

3 The proof of Theorem 11

Now we are in the position of proving Theorem 11.

Proof $(Sufficiency)$ From the definition of dual operator and Cauchy-Schwarz inequality in Proposition 26, we have

$$\begin{aligned}|(f,\alpha)_\varphi|_0^2 =&|(\overline{D}u,\alpha)_\varphi|_0^2 = |(u,\overline{D}^*_\varphi\alpha)_\varphi|_0^2\\ &\leq \|u\|^2\cdot\|\overline{D}^*_\varphi\alpha\|^2\\ &\leq c\cdot\|\overline{D}^*_\varphi\alpha\|^2.\end{aligned}$$

$(necessity)$ We aim to prove the necessity with Riesz representation theorem. First, we denote the submodule

$$E = \{\overline{D}^*_\varphi\alpha,\ \alpha\in C_0^\infty(\Omega,\mathcal{A}),\ \varphi\in C^2(\Omega,\mathbb{R})\}\subset L^2(\Omega,\mathcal{A},\varphi).$$

Then we define a linear functional L_f on E, i.e., $L_f\in E^{*alg}$ for a fixed $f\in L^2(\Omega,\mathcal{A},\varphi)$ as follows,

$$\langle L_f, \overline{D}^*_\varphi\alpha\rangle = (f,\alpha)_\varphi = \int_\Omega \bar{f}\cdot\alpha\cdot e^{-\varphi}dx\in\mathcal{A}.$$

From (3), we have

$$|\langle L_f,\overline{D}^*_\varphi\alpha\rangle|_0 = |(f,\alpha)_\varphi|_0 \leq \sqrt{c}\cdot\|\overline{D}^*_\varphi\alpha\|,$$

which meas that L_f is a bounded functional from Definition 23. By the Hahn-Banach type theorem in Theorem 24, L_f can be extended to a linear functional $\widetilde{L}_f$ on $L^2(\Omega,\mathcal{A},\varphi)$, and with

$$|\langle\widetilde{L}_f, g\rangle|_0 \leq \sqrt{c^*}\|g\|,\ \forall g\in L^2(\Omega,\mathcal{A},\varphi), \tag{10}$$

where $\sqrt{c^*} = \sqrt{c}\cdot|e_0|_0$, since $|e_A|_0 = 2^{n/2}$, then $c^* = 2^n c$ from [Brackx et al(1982)]. Now we are in the position to use the Riesz representation theorem for the operator $\widetilde{L}_f$. From Theorem 28, there exists a $u\in L^2(\Omega,\mathcal{A},\varphi)$ such that

$$\langle\widetilde{L}_f, g\rangle = (u,g)_\varphi,\ \forall g\in L^2(\Omega,\mathcal{A},\varphi). \tag{11}$$

For $\forall\alpha\in C_0^\infty(\Omega,\mathcal{A})$, let $g = \overline{D}^*_\varphi\alpha$. Then

$$(f,\alpha)_\varphi =\langle L_f^*, \overline{D}^*_\varphi\alpha\rangle = (u,\overline{D}^*_\varphi\alpha)_\varphi = (\overline{D}u,\alpha)_\varphi,$$

which deduces that

$$\int_\Omega \bar{f}\alpha e^{-\varphi}dx = \int_\Omega \overline{(\overline{D}u)}\alpha e^{-\varphi}dx.$$

Conjugating both sides of above equation leads to

$$\int_\Omega \bar{\alpha}f\cdot e^{-\varphi}dx = \int_\Omega \bar{\alpha}(\overline{D})ue^{-\varphi}dx.$$

Let $\alpha = \bar{\alpha}e^\varphi$, then it can be obtained that

$$\int_\Omega \alpha f dx = \int_\Omega \alpha(\overline{D}u)dx,\ \forall\alpha\in C_0^\infty(\Omega,\mathcal{A}).$$

Hence,

$$\overline{D}u = f$$

is proved from the definition of weak solutions.

Next, we give the bound for the norm of u. Let $g = u = \sum_A e_A u_A \in L^2(\Omega, \mathcal{A}, \varphi)$, from (10) and (11), we get that

$$|(u,u)_\varphi|_0 \le \sqrt{c^*}\|u\|. \tag{12}$$

On the other hand,

$$\begin{aligned}
|(u,u)_\varphi|_0^2 =& \Big|\int_\Omega \bar{u}ue^{-\varphi}dx\Big|_0^2 \\
&= 2^n \cdot \Big[\int_\Omega \bar{u}ue^{-\varphi}dx \cdot \overline{\int_\Omega \bar{u}ue^{-\varphi}dx}\Big]_0 \\
&= 2^n \Big[\int_\Omega (\sum_A u_A^2 + \sum_{A\neq B} \bar{e}_A e_B u_A u_B)e^{-\varphi}dx \cdot \overline{\int_\Omega (\sum_A u_A^2 + \sum_{A\neq B} \bar{e}_A e_B u_A u_B)e^{-\varphi}dx}\Big]_0 \\
&= 2^n \Big[(\int_\Omega \sum_A u_A^2 e^{-\varphi}dx)^2 + (\int_\Omega \sum_{A\neq B} u_A u_B e^{-\varphi}dx)^2\Big],
\end{aligned} \tag{13}$$

and

$$\|u\|^2 = \int_\Omega |u|_0^2 e^{-\varphi}dx = 2^n \int_\Omega [\bar{u}u]_0 e^{-\varphi}dx = 2^n \int_\Omega \sum_A u_A^2 \cdot e^{-\varphi}dx \tag{14}$$

So we have $\|u\|^4 = 2^{2n} \cdot (\int_\Omega \sum\limits_A u_A^2 \cdot e^{-\varphi}dx)^2$. Hence,

$$|(u,u)_\varphi|_0^2 = 2^n[(\int_\Omega \sum_A u_A^2 \cdot e^{-\varphi}dx)^2 + (\int_\Omega \sum_{A\neq B} u_A u_B e^{-\varphi}dx)^2] \ge 2^{-n}\|u\|^4.$$

Combining with (12), it is obtained that

$$\|u\|^2 \le 2^{n/2}|(u,u)_\varphi|_0 \le 2^{n/2}\sqrt{c^*}\|u\|,$$

and

$$\|u\|^2 \le 2^n c.$$

The proof is completed.

4 The proof of Theorem 12

It should be noticed that inequality (3) in Theorem 11 is related with $\alpha \in C_0^\infty(\Omega, \mathcal{A})$. In the following, we will give another sufficient condition that has nothing to do with the space $C_0^\infty(\Omega, \mathcal{A})$. First, we need to compute the norm of $\|\overline{D}_\varphi^* \alpha\|$ for any $\alpha \in C_0^\infty(\Omega, \mathcal{A})$.

$$
\begin{aligned}
\|\overline{D}_{\varphi}^{*}\alpha\|^2 =& \int_{\Omega} |\overline{D}_{\varphi}^{*}\alpha|_0^2 e^{-\varphi} dx \\
=& \int_{\Omega} \langle \tau_{e_0}, \overline{\overline{D}_{\varphi}^{*}\alpha} \cdot \overline{D}_{\varphi}^{*}\alpha \rangle e^{-\varphi} dx \\
=& \langle \tau_{e_0}, \int_{\Omega} \overline{\overline{D}_{\varphi}^{*}\alpha} \cdot \overline{D}_{\varphi}^{*}\alpha e^{-\varphi} dx \rangle \\
=& \langle \tau_{e_0}, (\overline{D}_{\varphi}^{*}\alpha, \overline{D}_{\varphi}^{*}\alpha)_{\varphi} \rangle \\
=& \langle \tau_{e_0}, (\alpha, \overline{D}\overline{D}_{\varphi}^{*}\alpha)_{\varphi} \rangle \\
=& \langle \tau_{e_0}, (\alpha, \overline{D}(\alpha(D\varphi) - D\alpha))_{\varphi} \rangle \\
=& \langle \tau_{e_0}, (\alpha, \overline{D}\alpha(D\varphi) + \alpha\Delta\varphi - \Delta\alpha + \sum_{j=1}^{n}(e_j\alpha - \alpha e_j)\frac{\partial}{\partial x_j}(D\varphi))_{\varphi} \rangle \\
=& \langle \tau_{e_0}, (\alpha, \overline{D}_{\varphi}^{*}(\overline{D}\alpha) + \alpha\Delta\varphi + \sum_{j=1}^{n}(e_j\alpha - \alpha e_j)\frac{\partial}{\partial x_j}(D\varphi))_{\varphi} \rangle \\
=& \langle \tau_{e_0}, (\alpha, \overline{D}_{\varphi}^{*}(\overline{D}\alpha))_{\varphi} + (\alpha, \alpha\Delta\varphi)_{\varphi} + (\alpha, \sum_{j=1}^{n}(e_j\alpha - \alpha e_j)\frac{\partial}{\partial x_j}(D\varphi))_{\varphi} \rangle \\
=& \langle \tau_{e_0}, (\alpha, \overline{D}_{\varphi}^{*}(\overline{D}\alpha))_{\varphi} \rangle + \langle \tau_{e_0}, (\alpha, \alpha\Delta\varphi)_{\varphi} \rangle + \langle \tau_{e_0}, (\alpha, \sum_{j=1}^{n}(e_j\alpha - \alpha e_j)\frac{\partial}{\partial x_j}(D\varphi))_{\varphi} \rangle \\
=& I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 =& \langle \tau_{e_0}, (\alpha, \overline{D}_{\varphi}^{*}(\overline{D}\alpha))_{\varphi} \rangle = \langle \tau_{e_0}, (\overline{D}\alpha, \overline{D}\alpha)_{\varphi} \rangle = \|\overline{D}\alpha\|^2, \\
I_2 =& \langle \tau_{e_0}, (\alpha, \alpha\Delta\varphi)_{\varphi} \rangle = \int_{\Omega} |\alpha|_0^2 \Delta\varphi e^{-\varphi} dx,
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 =& \langle \tau_{e_0}, (\alpha, \sum_{j=1}^{n}(e_j\alpha - \alpha e_j)\frac{\partial}{\partial x_j}(D\varphi))_{\varphi} \rangle \\
=& \langle \tau_{e_0}, (\alpha, \sum_{j=1}^{n}(e_j\alpha - \alpha e_j)\frac{\partial}{\partial x_j}(\sum_{i=0}^{n}\bar{e}_i\frac{\partial\varphi}{\partial x_i}))_{\varphi} \rangle \\
=& \langle \tau_{e_0}, (\alpha, \sum_{j=1}^{n}\sum_{i=0}^{n}(e_j\alpha\bar{e}_i - \alpha e_j\bar{e}_i)\frac{\partial^2\varphi}{\partial x_j \partial x_i})_{\varphi} \rangle \qquad (15) \\
=& \langle \tau_{e_0}, \int_{\Omega} \bar{\alpha} \sum_{j=1}^{n}\sum_{i=0}^{n}(e_j\alpha\bar{e}_i - \alpha e_j\bar{e}_i)\frac{\partial^2\varphi}{\partial x_j \partial x_i} e^{-\varphi} dx \rangle \\
=& \int_{\Omega} \langle \tau_{e_0}, \bar{\alpha} \sum_{j=1}^{n}\sum_{i=0}^{n}(e_j\alpha\bar{e}_i - \alpha e_j\bar{e}_i)\frac{\partial^2\varphi}{\partial x_j \partial x_i} \rangle e^{-\varphi} dx.
\end{aligned}
$$

It should be noticed that if $n = 1$, i.e., the space $\mathbb{R}^2$ is considered, then $I_3 = 0$.

Since for $1 \leq i, j \leq n$ and $i \neq j$, $e_j\bar{e}_i = -e_je_i = e_ie_j = -e_i\bar{e}_j$. For simplicity, let

$$
\begin{aligned}
I_4 =&\langle \tau_{e_0}, \bar{\alpha}\sum_{j=1}^{n}\sum_{i=0}^{n}(e_j\alpha\bar{e}_i - \alpha e_j\bar{e}_i)\frac{\partial^2\varphi}{\partial x_j\partial x_i}\rangle\\
=&\langle \tau_{e_0}, \sum_{j=1}^{n}\sum_{i=1}^{n}(\bar{\alpha}e_j\alpha\bar{e}_i - \bar{\alpha}\alpha e_j\bar{e}_i)\frac{\partial^2\varphi}{\partial x_j\partial x_i}\rangle + \langle \tau_{e_0}, \sum_{j=1}^{n}(\bar{\alpha}e_j\alpha\bar{e}_0 - \bar{\alpha}\alpha e_j\bar{e}_0)\frac{\partial^2\varphi}{\partial x_j\partial x_0}\rangle\\
=&\langle \tau_{e_0}, \sum_{i=1}^{n}(\bar{\alpha}e_i\alpha\bar{e}_i - \bar{\alpha}\alpha e_i\bar{e}_i)\frac{\partial^2\varphi}{\partial x_i^2}\rangle + \langle \tau_{e_0}, \sum_{j\neq i}^{n}(\bar{\alpha}e_j\alpha\bar{e}_i)\frac{\partial^2\varphi}{\partial x_j\partial x_i}\rangle\\
&+ \langle \tau_{e_0}, \sum_{j=1}^{n}(\bar{\alpha}e_j\alpha\bar{e}_0 - \bar{\alpha}\alpha e_j\bar{e}_0)\frac{\partial^2\varphi}{\partial x_j\partial x_0}\rangle\\
=&\langle \tau_{e_0}, \sum_{i=1}^{n}(\bar{\alpha}e_i\alpha\bar{e}_i - \bar{\alpha}\alpha)\frac{\partial^2\varphi}{\partial x_i^2}\rangle + \langle \tau_{e_0}, \sum_{j\neq i}^{n}(\bar{\alpha}e_j\alpha\bar{e}_i)\frac{\partial^2\varphi}{\partial x_j\partial x_i}\rangle\\
&+ \langle \tau_{e_0}, \sum_{j=1}^{n}(\bar{\alpha}e_j\alpha\bar{e}_0 - \bar{\alpha}\alpha e_j\bar{e}_0)\frac{\partial^2\varphi}{\partial x_j\partial x_0}\rangle\\
=&I_5 + I_6 + I_7.
\end{aligned}
\tag{16}
$$

Assume $\alpha = \sum\limits_A \alpha_A e_A \in \mathcal{A}$, $\bar{\alpha} = \sum\limits_A \alpha_A \bar{e}_A$, then for any $1 \leq i \leq n$,

$$
\begin{aligned}
\bar{\alpha}e_i\alpha\bar{e}_i =& \sum_A \alpha_A\bar{e}_Ae_i \cdot \sum_A \alpha_A e_A\bar{e}_i\\
=& \sum_A (-1)^{\frac{|A|(|A|+1)}{2}}\alpha_A e_A e_i \cdot \sum_A (-1)\alpha_A e_A e_i
\end{aligned}
\tag{17}
$$

Then

$$
\begin{aligned}
I_5 =&\langle \tau_{e_0}, \sum_{i=1}^{n} (\bar{\alpha} e_i \alpha \bar{e}_i - \bar{\alpha}\alpha) \frac{\partial^2 \varphi}{\partial x_i^2} \rangle \\
=&\langle \tau_{e_0}, \sum_{i=1}^{n} (\bar{\alpha} e_i \alpha \bar{e}_i) \frac{\partial^2 \varphi}{\partial x_i^2} \rangle - \langle \tau_{e_0}, \sum_{i=1}^{n} (\bar{\alpha}\alpha) \frac{\partial^2 \varphi}{\partial x_i^2} \rangle \\
=&\langle \tau_{e_0}, \sum_{i=1}^{n} (\sum_{A} (-1)^{\frac{|A|(|A|+1)}{2}} \alpha_A e_A e_i \cdot \sum_{A} (-1) \alpha_A e_A e_i) \frac{\partial^2 \varphi}{\partial x_i^2} \rangle - \langle \tau_{e_0}, \sum_{i=1}^{n} (\bar{\alpha}\alpha) \frac{\partial^2 \varphi}{\partial x_i^2} \rangle \\
=&2^n \sum_{i=1}^{n} (\sum_{A} (-1)^{\frac{|A|(|A|+1)}{2}+1} \alpha_A^2 e_A e_i e_A e_i) \frac{\partial^2 \varphi}{\partial x_i^2} - \sum_{i=1}^{n} |\alpha|_0^2 \frac{\partial^2 \varphi}{\partial x_i^2} \\
=&2^n \sum_{i=1}^{n} (\sum_{i \notin A} (-1)^{\frac{|A|(|A|+1)}{2}+1} \alpha_A^2 \cdot \overline{e_A e_i} \cdot e_A e_i \cdot (-1)^{\frac{(|A|+1)(|A|+2)}{2}} \\
&+ \sum_{i \in A} (-1)^{\frac{|A|(|A|+1)}{2}+1} \cdot \alpha_A^2 \cdot \overline{e_{A-i}} \cdot e_{A-i} \cdot (-1)^{\frac{(|A|-1)(|A|)}{2}}) \frac{\partial^2 \varphi}{\partial x_i^2} - \sum_{i=1}^{n} |\alpha|_0^2 \frac{\partial^2 \varphi}{\partial x_i^2} \\
=&2^n \sum_{i=1}^{n} (\sum_{i \notin A} (-1)^{\frac{|A|(|A|+1)}{2}+1+\frac{(|A|+1)(|A|+2)}{2}} \cdot \alpha_A^2 \\
&+ \sum_{i \in A} (-1)^{\frac{|A|(|A|+1)}{2}+1+\frac{(|A|-1)(|A|)}{2}} \cdot \alpha_A^2) \frac{\partial^2 \varphi}{\partial x_i^2} - \sum_{i=1}^{n} |\alpha|_0^2 \frac{\partial^2 \varphi}{\partial x_i^2} \\
=&2^n \sum_{i=1}^{n} (\sum_{i \notin A} (-1)^{|A|^2} \cdot \alpha_A^2 + \sum_{i \in A} (-1)^{|A|^2+1} \cdot \alpha_A^2) \frac{\partial^2 \varphi}{\partial x_i^2} - \sum_{i=1}^{n} |\alpha|_0^2 \frac{\partial^2 \varphi}{\partial x_i^2} \\
=&2^n \sum_{i=1}^{n} (\sum_{i \notin A, |A|^2 \text{ is odd}} (-2) \alpha_A^2 + \sum_{i \in A, |A|^2 \text{ is even}} (-2) \alpha_A^2) \frac{\partial^2 \varphi}{\partial x_i^2} \\
=&-2^{n+1} \sum_{i=1}^{n} (\sum_{i \notin A, |A|^2 \text{ is odd}} \alpha_A^2 + \sum_{i \in A, |A|^2 \text{ is even}} \alpha_A^2) \frac{\partial^2 \varphi}{\partial x_i^2}.
\end{aligned}
\tag{18}
$$

To consider I_7, we first study $\bar{\alpha} e_j \alpha$ for any $1 \le j \le n$. Without loss of generality, let $e_j = e_1,\ \bar{\alpha} = \sum\limits_{A} \alpha_A \bar{e}_A,\ \alpha = \sum\limits_{A} \alpha_A e_A$. Then $\bar{\alpha} e_1 \alpha = (\sum\limits_{A} \alpha_A \bar{e}_A) e_1 (\sum\limits_{A} \alpha_A e_A)$.

When $e_A = e_1 e_{h_2} e_{h_3} \cdots e_{h_r}$, where $1 < h_2 < h_3 < \cdots < h_r$ and $1 < r \le n$.

$$
\begin{aligned}
\alpha_A \bar{e}_A e_1 =&\alpha_{1h_2 \cdots h_r} (-1)^{\frac{r(r+1)}{2}} \cdot e_1 e_{h_2} e_{h_3} \cdots e_{h_r} \cdot e_1 \\
=&\alpha_{1h_2 \cdots h_r} (-1)^{\frac{r(r+1)}{2}+r} e_{h_2} e_{h_3} \cdots e_{h_r} \\
\alpha_A e_A e_1 =&\alpha_{1h_2 \cdots h_r} e_1 e_{h_2} \cdots e_{h_r} \cdot e_1 = \alpha_{1h_2 \cdots h_r} (-1)^r e_{h_2} \cdots e_{h_r}.
\end{aligned}
\tag{19}
$$

When $e_A = e_1$,

$$
\begin{aligned}
\alpha_A \bar{e}_A e_1 =&\alpha_1 \\
\alpha_A e_A e_1 =&-\alpha_1.
\end{aligned}
\tag{20}
$$

When $e_A = e_{h_2} e_{h_3} \cdots e_{h_r}$, where $1 < h_2 < h_3 < \cdots < h_r$ and $1 < r \le n$.

$$
\begin{aligned}
\alpha_A \bar{e}_A e_1 =&\alpha_{h_2 \cdots h_r} (-1)^{\frac{(r-1)(r)}{2}} \cdot e_{h_2} e_{h_3} \cdots e_{h_r} \cdot e_1 \\
=&\alpha_{h_2 \cdots h_r} (-1)^{\frac{(r-1)(r)}{2}+r-1} e_1 e_{h_2} \cdots e_{h_r} \\
\alpha_A e_A e_1 =&\alpha_{h_2 \cdots h_r} e_{h_2} \cdots e_{h_r} \cdot e_1 = \alpha_{h_2 \cdots h_r} (-1)^{r-1} e_1 e_{h_2} \cdots e_{h_r}.
\end{aligned}
\tag{21}
$$

When $e_A = e_0$,

$$
\begin{aligned}
\alpha_A \bar{e}_A e_1 =&\alpha_0 e_1 \\
\alpha_A e_A e_1 =&\alpha_0 e_1.
\end{aligned}
\tag{22}
$$

To compute I_7, one needs to know the coefficient for e_0 of $\bar{\alpha}e_1\alpha - \bar{\alpha}\alpha e_1$. It means that we should find out the corresponding terms of $e_1e_{h_2}e_{h_3}\cdots e_{h_r}$ and $e_{h_2}\cdots e_{h_r}$ in $\bar{\alpha}e_1$ and α, in $\bar{\alpha}$ and αe_1.

Case a1. For $\bar{\alpha}e_1\alpha$, from (21), the corresponding terms of $e_1e_{h_2}e_{h_3}\cdots e_{h_r}$ with $1 < h_2 < h_3 < \cdots < h_r$ and $1 < r \leq n$ in $\bar{\alpha}e_1 = (\sum\limits_A \alpha_A\bar{e}_A)e_1$ and $\alpha = \sum\limits_A \alpha_A e_A$ are $\alpha_{h_2\cdots h_r}(-1)^{\frac{(r-1)(r)}{2}+r-1}e_1e_{h_2}\cdots e_{h_r}$ and $\alpha_{1h_2\cdots h_r}e_1e_{h_2}\cdots e_{h_r}$, respectively. Multiplying these terms leads to

$$\begin{aligned}
&(-1)^{\frac{(r-1)(r)}{2}+r-1}e_1e_{h_2}\cdots e_{h_r}\cdot e_1e_{h_2}\cdots e_{h_r}\cdot\alpha_{1h_2\cdots h_r}\cdot\alpha_{h_2\cdots h_r}\\
&\quad=(-1)^{\frac{(r-1)(r)}{2}+r-1}(-1)^{\frac{(r)(r+1)}{2}}\cdot\overline{e_1\cdots e_{h_r}}\cdot e_1e_{h_2}\cdots e_{h_r}\cdot\alpha_{1h_2\cdots h_r}\alpha_{h_2\cdots h_r}\\
&\quad=(-1)^{\frac{(r)(r+1)}{2}+r-1+\frac{(r-1)(r)}{2}}\cdot\alpha_{1h_2\cdots h_r}\alpha_{h_2\cdots h_r}.
\end{aligned}\tag{23}$$

On the other hand, for $\bar{\alpha}e_1\alpha$, from (19), the corresponding terms of $e_{h_2}e_{h_3}\cdots e_{h_r}$ with $1 < h_2 < h_3 < \cdots < h_r$ and $1 < r \leq n$ in $\bar{\alpha}e_1$ and α are $\alpha_{1h_2\cdots h_r}(-1)^{\frac{r(r+1)}{2}+r}e_{h_2}e_{h_3}\cdots e_{h_r}$ and $\alpha_{h_2\cdots h_r}e_{h_2}\cdots e_{h_r}$, respectively. Multiplying these terms leads to

$$\begin{aligned}
&(-1)^{\frac{(r)(r+1)}{2}+r}e_{h_2\cdots h_r}\cdot e_{h_2\cdots h_r}\cdot\alpha_{1h_2\cdots h_r}\cdot\alpha_{h_2\cdots h_r}\\
&\quad=(-1)^{\frac{(r)(r+1)}{2}+r}(-1)^{\frac{(r-1)(r)}{2}}\cdot\overline{e_{h_2\cdots h_r}}\cdot e_{h_2\cdots h_r}\cdot\alpha_{1h_2\cdots h_r}\alpha_{h_2\cdots h_r}\\
&\quad=(-1)^{\frac{(r)(r+1)}{2}+r+\frac{(r-1)(r)}{2}}\cdot\alpha_{1h_2\cdots h_r}\alpha_{h_2\cdots h_r}.
\end{aligned}\tag{24}$$

From (23) and (24), these two terms cancel.

Case a2. For $\bar{\alpha}e_1\alpha$, from (22), the corresponding terms of e_1 in $\bar{\alpha}e_1$ and α are $\alpha_0 e_1$ and $\alpha_1 e_1$, respectively. Multiplying these terms leads to

$$\alpha_0e_1\alpha_1e_1 = -\alpha_0\alpha_1.\tag{25}$$

On the other hand, for $\bar{\alpha}e_1\alpha$, from (20), the corresponding terms of e_0 in $\bar{\alpha}e_1$ and α are α_1 and α_0, respectively. Multiplying these terms leads to $\alpha_0\alpha_1$. Combining with (25), these two terms also cancel.

From Cases a1 and a2, one can obtain that the coefficient for e_0 of $\bar{\alpha}e_1\alpha$ equals zero, i.e.,

$$\langle\tau_{e_0},\sum_{j=1}^{n}(\bar{\alpha}e_j\alpha\bar{e}_0)\frac{\partial^2\varphi}{\partial x_j\partial x_0}\rangle = 0.\tag{26}$$

Case b1. For $\bar{\alpha}\alpha e_1$, from (21), the corresponding terms of $e_1e_{h_2}e_{h_3}\cdots e_{h_r}$ with $1 < h_2 < h_3 < \cdots < h_r$ and $1 < r \leq n$ in $\alpha e_1 = (\sum\limits_A \alpha_A e_A)e_1$ and $\bar{\alpha} = \sum\limits_A \alpha_A\bar{e}_A$ are $\alpha_{h_2\cdots h_r}(-1)^{r-1}e_1e_{h_2}\cdots e_{h_r}$ and $\alpha_{1h_2\cdots h_r}\overline{e_1e_{h_2}\cdots e_{h_r}}$, respectively. Multiplying these terms leads to

$$\begin{aligned}
&(\alpha_{1h_2\cdots h_r}\overline{e_1e_{h_2}\cdots e_{h_r}})\cdot(\alpha_{h_2\cdots h_r}e_{h_2}\cdots e_{h_r}\cdot e_1)\\
&\quad=(\alpha_{1h_2\cdots h_r}\overline{e_1e_{h_2}\cdots e_{h_r}})\cdot((-1)^{r-1}e_1e_{h_2}\cdots e_{h_r}\cdot\alpha_{h_2\cdots h_r})\\
&\quad=(-1)^{r-1}\alpha_{1h_2\cdots h_r}\cdot\alpha_{h_2\cdots h_r}.
\end{aligned}\tag{27}$$

On the other hand, for $\bar{\alpha}\alpha e_1$, from (19), the corresponding terms of $e_{h_2}e_{h_3}\cdots e_{h_r}$ with $1 < h_2 < h_3 < \cdots < h_r$ and $1 < r \leq n$ in αe_1 and $\bar{\alpha}$ are $\alpha_{1h_2\cdots h_r}(-1)^r e_{h_2}\cdots e_{h_r}$ and $\alpha_{h_2\cdots h_r}\overline{e_{h_2}\cdots e_{h_r}}$, respectively. Multiplying these terms leads to

$$\begin{aligned}
&(\alpha_{h_2\cdots h_r}\overline{e_{h_2}\cdots e_{h_r}})\cdot(\alpha_{1h_2\cdots h_r}e_1\cdots e_{h_r}\cdot e_1)\\
&\quad=(\alpha_{h_2\cdots h_r}\overline{e_{h_2}\cdots e_{h_r}})\cdot((-1)^r e_{h_2}\cdots e_{h_r}\cdot\alpha_{1h_2\cdots h_r})\\
&\quad=(-1)^r\alpha_{h_2\cdots h_r}\cdot\alpha_{1h_2\cdots h_r}.
\end{aligned}\tag{28}$$

From (27) and (28), these two terms cancel.

Case b2. For $\bar{\alpha}\alpha e_1$, from (22), the corresponding terms of e_1 in αe_1 and $\bar{\alpha}$ are $\alpha_0 e_1$ and $\alpha_1 \bar{e}_1$, respectively. Multiplying these terms leads to

$$\alpha_0 e_1 \alpha_1 \bar{e}_1 = \alpha_0 \alpha_1. \tag{29}$$

On the other hand, for $\bar{\alpha}\alpha e_1$, from (20), the corresponding terms of e_0 in αe_1 and $\bar{\alpha}$ are $-\alpha_1$ and α_0, respectively. Multiplying these terms leads to $-\alpha_0\alpha_1$. Combining with (29), these two terms also cancel.

From Cases b1 and b2, one can obtain that the coefficient for e_0 of $\bar{\alpha} e_1 \alpha$ equals zero, i.e.,

$$\langle \tau_{e_0}, \sum_{j=1}^{n} (\bar{\alpha}\alpha e_j \bar{e}_0) \frac{\partial^2 \varphi}{\partial x_j \partial x_0} \rangle = 0. \tag{30}$$

Thus, $I_7 = 0$ from (26) and (30).

To compute I_6, i.e., to get $[\bar{\alpha} e_i \alpha \bar{e}_j]_0$ for $i \neq j$, similar with the analysis of I_7, we should divide the vectors in $\bar{\alpha} e_i$ and $\alpha \bar{e}_j$ into four cases.

Case c1. $i \in A,\ j \notin A$ for e_A in $\bar{\alpha}$ and $i \notin B,\ j \in B$ for e_B in α with $A - i = B - j$.

For this case, firstly, we assume $e_A = e_{h_1 \cdots h_{p(i)} \cdots h_r}$ and $h_{p(i)} = i$, $e_B = e_{h_1 \cdots h_{p(j)} \cdots h_r}$ and $h_{p(j)} = j$. We have

$$\begin{aligned}
\alpha_A \bar{e}_A e_i =& \alpha_A (-1)^{\frac{r(r+1)}{2}} \cdot e_{h_1} \cdots e_i \cdots e_{h_r} \cdot e_i \\
=& \alpha_A (-1)^{\frac{r(r+1)}{2} + r - p(i)} e_{h_1} \cdots e_i^2 \cdots e_{h_r}, \\
=& \alpha_A (-1)^{\frac{r(r+1)}{2} + r - p(i) + 1} e_{A-i}, \\
\alpha_B e_B \bar{e}_j =& \alpha_B e_{h_1} \cdots e_j \cdots e_{h_r} \cdot \bar{e}_j \\
=& \alpha_B (-1)^{r - p(j)} e_{h_1} \cdots e_j \bar{e}_j \cdots e_{h_r}, \\
=& \alpha_B (-1)^{r - p(j)} e_{B-j}.
\end{aligned} \tag{31}$$

Then

$$\begin{aligned}
\alpha_A \bar{e}_A e_i \alpha_B e_B \bar{e}_j =& \alpha_A (-1)^{\frac{r(r+1)}{2} + r - p(i) + 1} e_{A-i} \alpha_B (-1)^{r - p(j)} e_{B-j} \\
=& \alpha_A \alpha_B (-1)^{\frac{r(r+1)}{2} + r - p(i) + 1 + r - p(j) + \frac{r(r-1)}{2}} \overline{e_{A-i}} e_{B-j} \\
=& \alpha_A \alpha_B (-1)^{r^2 + 1 - p(i) - p(j)}.
\end{aligned} \tag{32}$$

Case c2. $i \notin A,\ j \in A$ for e_A in $\bar{\alpha}$ and $i \in B,\ j \notin B$ for e_B in α with $A + i = B + j$.

We assume $e_A = e_{h_1 \cdots h_{p(j)} \cdots h_r}$ and $h_{p(j)} = j$, $e_B = e_{h_1 \cdots h_{p(i)} \cdots h_r}$ and $h_{p(i)} = i$. We have

$$\begin{aligned}
\alpha_A \bar{e}_A e_i =& \alpha_A (-1)^{\frac{r(r+1)}{2}} \cdot e_{h_1} \cdots e_j \cdots e_{h_r} \cdot e_i, \\
\alpha_B e_B \bar{e}_j =& \alpha_B e_{h_1} \cdots e_i \cdots e_{h_r} \cdot \bar{e}_j \\
=& -\alpha_B e_{h_1} \cdots e_i \cdots e_{h_r} \cdot e_j \\
=& \alpha_B e_{h_1} \cdots e_j \cdots e_{h_r} \cdot e_i.
\end{aligned} \tag{33}$$

Then

$$\begin{aligned}
\alpha_A \bar{e}_A e_i \alpha_B e_B \bar{e}_j =& \alpha_A (-1)^{\frac{r(r+1)}{2}} \cdot e_{h_1} \cdots e_j \cdots e_{h_r} \cdot e_i \alpha_B e_{h_1} \cdots e_j \cdots e_{h_r} \cdot e_i \\
=& \alpha_A \alpha_B (-1)^{\frac{r(r+1)}{2} + \frac{(r+1)(r+2)}{2}} \overline{e_{h_1} \cdots e_j \cdots e_{h_r} \cdot e_i} e_{h_1} \cdots e_j \cdots e_{h_r} \cdot e_i \\
=& \alpha_A \alpha_B (-1)^{r^2 + 1}.
\end{aligned} \tag{34}$$

Case c3. $i \in A,\ j \in A$ for e_A in $\bar{\alpha}$ and $i \notin B,\ j \notin B$ for e_B in α with $A - i = B + j$.

For this case, we assume $e_A = e_{h_1 \cdots h_{p(i)} \cdots h_{p(j)} \cdots h_{r+2}}$ with $h_{p(i)} = i,\ h_{p(j)} = j$. Without loss of generality, we assume $i < j$. Furthermore, let $e_B = e_{h_1 \cdots h_r}$. We have

$$\begin{aligned}
\alpha_A \bar{e}_A e_i =& \alpha_A (-1)^{\frac{(r+2)(r+3)}{2}} \cdot e_{h_1} \cdots e_i \cdots e_j \cdots e_{h_{r+2}} \cdot e_i \\
=& \alpha_A (-1)^{\frac{(r+2)(r+3)}{2} + r + 2 - h(i)} \cdot e_{h_1} \cdots e_j \cdots e_{h_{r+2}} \cdot e_i^2 \\
=& \alpha_A (-1)^{\frac{(r+2)(r+3)}{2} + r + 1 - h(i)} \cdot e_{h_1} \cdots e_j \cdots e_{h_{r+2}} \\
=& \alpha_A (-1)^{\frac{(r+2)(r+3)}{2} + r + 1 - h(i) + r + 2 - h(j)} \cdot e_{h_1} \cdots e_{h_{r+2}} \cdot e_j, \\
\alpha_B e_B \bar{e}_j =& \alpha_B e_{h_1} \cdots e_{h_r} \cdot \bar{e}_j \\
=& -\alpha_B e_{h_1} \cdots e_{h_r} \cdot e_j.
\end{aligned} \tag{35}$$

Then

$$\begin{aligned}
\alpha_A \bar{e}_A e_i \alpha_B e_B \bar{e}_j =& \alpha_A (-1)^{\frac{(r+2)(r+3)}{2} + r + 1 - h(i) + r + 2 - h(j)} \cdot e_{h_1} \cdots e_{h_{r+2}} \cdot e_j (-1) \alpha_B e_{h_1} \cdots e_{h_r} \cdot e_j \\
=& \alpha_A \alpha_B (-1)^{\frac{(r+2)(r+3)}{2} - h(i) - h(j)} \cdot e_{h_1} \cdots e_{h_{r+2}} \cdot e_j e_{h_1} \cdots e_{h_r} \cdot e_j \\
=& \alpha_A \alpha_B (-1)^{\frac{(r+2)(r+3)}{2} - h(i) - h(j) + \frac{(r+1)(r+2)}{2}} \cdot \overline{e_{h_1} \cdots e_{h_{r+2}} \cdot e_j} e_{h_1} \cdots e_{h_r} \cdot e_j \\
=& \alpha_A \alpha_B (-1)^{r^2 - h(j) - h(i)}.
\end{aligned} \tag{36}$$

Case c4. $i \notin A,\ j \notin A$ for e_A in $\bar{\alpha}$ and $i \in B,\ j \in B$ for e_B in α with $A + i = B - j$.

For this case, we assume $e_A = e_{h_1 \cdots h_r}$, $e_B = e_{h_1 \cdots h_{p(i)} \cdots h_{p(j)} \cdots h_{r+2}}$ with $h_{p(i)} = i,\ h_{p(j)} = j$ and $i < j$. We have

$$\begin{aligned}
\alpha_A \bar{e}_A e_i =& \alpha_A (-1)^{\frac{r(r+1)}{2}} \cdot e_{h_1} \cdots e_{h_r} \cdot e_i, \\
\alpha_B e_B \bar{e}_j =& \alpha_B e_{h_1} \cdots e_i \cdots e_j \cdots e_{h_{r+2}} \cdot \bar{e}_j \\
=& \alpha_B (-1)^{r+2-h(j)} \cdot e_{h_1} \cdots e_i \cdots e_{h_{r+2}} \cdot e_j \bar{e}_j \\
=& \alpha_B (-1)^{r+2-h(j)+r+2-h(i)-1} \cdot e_{h_1} \cdots e_{h_{r+2}} \cdot e_i \\
=& \alpha_B (-1)^{1-h(j)-h(i)} \cdot e_{h_1} \cdots e_{h_{r+2}} \cdot e_i
\end{aligned} \tag{37}$$

Then

$$\begin{aligned}
\alpha_A \bar{e}_A e_i \alpha_B e_B \bar{e}_j =& \alpha_A (-1)^{\frac{r(r+1)}{2}} \cdot e_{h_1} \cdots e_{h_r} \cdot e_i \alpha_B (-1)^{1-h(j)-h(i)} \cdot e_{h_1} \cdots e_{h_{r+2}} \cdot e_i \\
=& \alpha_A \alpha_B (-1)^{\frac{r(r+1)}{2} + 1 - h(j) - h(i)} \cdot e_{h_1} \cdots e_{h_r} \cdot e_i \cdot e_{h_1} \cdots e_{h_{r+2}} \cdot e_i \\
=& \alpha_A \alpha_B (-1)^{\frac{r(r+1)}{2} + 1 - h(j) - h(i) + \frac{(r+1)(r+2)}{2}} \cdot \overline{e_{h_1} \cdots e_{h_r} \cdot e_i} \cdot e_{h_1} \cdots e_{h_{r+2}} \cdot e_i \\
=& \alpha_A \alpha_B (-1)^{r^2 - h(j) - h(i)}.
\end{aligned} \tag{38}$$

Combining cases c1-c4, we have

$$
\begin{aligned}
I_6 =&\langle \tau_{e_0}, \sum_{j\neq i}^{n} (\bar{\alpha} e_j \alpha \bar{e}_i) \frac{\partial^2 \varphi}{\partial x_j \partial x_i} \rangle \\
=&\langle \tau_{e_0}, \sum_{j\neq i}^{n} \big((\sum_A \bar{e_A} \alpha_A) e_j (\sum_B e_B \alpha_B) \bar{e}_i \big) \frac{\partial^2 \varphi}{\partial x_j \partial x_i} \rangle \\
=&\langle \tau_{e_0}, \sum_{j\neq i}^{n} \big((\sum_A \bar{e_A} \alpha_A) e_i (\sum_B e_B \alpha_B) \bar{e}_j \big) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \rangle \\
=&\sum_{j\neq i}^{n} \langle \tau_{e_0}, (\sum_A \bar{e_A} \alpha_A) e_i (\sum_B e_B \alpha_B) \bar{e}_j \rangle \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \\
=&\sum_{j\neq i}^{n} \langle \tau_{e_0}, (\sum_A \bar{e_A} \alpha_A) e_i (\sum_B e_B \alpha_B) \bar{e}_j \rangle \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \\
=&2^n \sum_{j\neq i}^{n} \Big(\sum_{i\in A,\ j\notin A; A-i=B-j} \alpha_A \alpha_B (-1)^{r^2+1-p(i)-p(j)} \\
&+ \sum_{i\notin A,\ j\in A; A+i=B+j} \alpha_A \alpha_B (-1)^{r^2+1} \\
&+ \sum_{i\in A,\ j\in A; A-i=B+j} \alpha_A \alpha_B (-1)^{r^2-h(j)-h(i)} \\
&+ \sum_{i\notin A,\ j\notin A; A+i=B-j} \alpha_A \alpha_B (-1)^{r^2-h(j)-h(i)} \Big) \frac{\partial^2 \varphi}{\partial x_i \partial x_j}.
\end{aligned} \tag{39}
$$

In all,

$$
\begin{aligned}
I_3 =& \int_\Omega I_4 e^{-\varphi} dx \\
=& \int_\Omega (I_5 + I_6 + I_7) e^{-\varphi} dx \\
=& -2^{n+1} \int_\Omega \sum_{i=1}^{n} (\sum_{i\notin A, |A|^2 \text{ is odd}} \alpha_A^2 + \sum_{i\in A, |A|^2 \text{ is even}} \alpha_A^2) \frac{\partial^2 \varphi}{\partial x_i^2} e^{-\varphi} dx \\
&+ 2^n \int_\Omega \sum_{j\neq i}^{n} \Big(\sum_{i\in A,\ j\notin A; A-i=B-j} \alpha_A \alpha_B (-1)^{r^2+1-p(i)-p(j)} \\
&+ \sum_{i\notin A,\ j\in A; A+i=B+j} \alpha_A \alpha_B (-1)^{r^2+1} \\
&+ \sum_{i\in A,\ j\in A; A-i=B+j} \alpha_A \alpha_B (-1)^{r^2-h(j)-h(i)} \\
&+ \sum_{i\notin A,\ j\notin A; A+i=B-j} \alpha_A \alpha_B (-1)^{r^2-h(j)-h(i)} \Big) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} e^{-\varphi} dx.
\end{aligned} \tag{40}
$$

Then we have

$$
\|\overline{D}_\varphi^* \alpha\|^2 = \|\overline{D}\alpha\|^2 + \int_\Omega |\alpha|_0^2 \Delta\varphi e^{-\varphi} dx + I_3. \tag{41}
$$

It is obtained that if $\frac{\partial^2 \varphi}{\partial x_j \partial x_i} = 0,\ i \neq j,\ 1 \leq i, j \leq n$ and $\frac{\partial^2 \varphi}{\partial x_i^2} \leq 0,\ 1 \leq i \leq n$. Then we have $I_3 \geq 0$, and

$$
\|\overline{D}_\varphi^* \alpha\|^2 \geq \int_\Omega |\alpha|_0^2 \Delta\varphi e^{-\varphi} dx,
$$

With the above analysis, we can prove Theorem 12 easily.

Proof It is sufficient to prove the theorem if condition (3) in Theorem 11 is presented. By Cauchy-Schwarz inequality in Proposition 26, we have for any $\alpha \in C_0^\infty(\Omega, \mathcal{A})$ that

$$\begin{aligned}|(f,\alpha)_\varphi|_0^2 =& |\int_\Omega \bar{f}\cdot \alpha e^{-\varphi} dx|_0^2 \\ =& |\int_\Omega \bar{f}\cdot \frac{1}{\sqrt{\Delta\varphi}} \cdot \alpha \cdot \sqrt{\Delta\varphi} \cdot e^{-\varphi} dx|_0^2 \\ \leq& \|\bar{f}\frac{1}{\sqrt{\Delta\varphi}}\|^2 \cdot \|\alpha\cdot\sqrt{\Delta\varphi}\|^2 \\ =& \int_\Omega |\frac{\bar{f}}{\sqrt{\Delta\varphi}}|_0^2 e^{-\varphi} dx \cdot \int_\Omega |\alpha\cdot\sqrt{\Delta\varphi}|_0^2 e^{-\varphi} dx \\ \leq& c\|\overline{D}_\varphi^*\alpha\|^2.\end{aligned}$$

The proof is completed with Theorem 11.

It should be noticed that when $n = 1$, $I_3 = 0$. Then it comes from equation (41) that the Hörmander's L^2 theorem in $\mathbb{R}^2$ which equals the classical Hörmander's L^2 theorem in $\mathbb{C}$ could be described as follows.

Corollary 41 *Given $\varphi \in C^2(\Omega, \mathbb{R})$ with Ω being an open subset of $\mathbb{R}^2$; $\Delta\varphi \geq 0$. Then for all $f \in L^2(\Omega, \mathcal{A}, \varphi)$ with $\int_\Omega \frac{|f|_0^2}{\Delta\varphi} e^{-\varphi} dx = c < \infty$, there exists a $u \in L^2(\Omega, \mathcal{A}, \varphi)$ such that*

$$\overline{D}u = f$$

with

$$\|u\|^2 \leq \int_\Omega \frac{|f|_0^2}{\Delta\varphi} e^{-\varphi} dx.$$

5 Conclusion

In this paper, based on the Hörmander's L^2 theorem in complex analysis, the Hörmander's L^2 theorem for Dirac operator in $\mathbb{R}^{n+1}$ with $n > 1$ has been obtained by Clifford algebra. When $n = 1$, the result is equivalent to the classical Hörmander's L^2 theorem in complex variable from the proof of the main theorem. As a result, for any f in L^2 space over a bounded domain with value in Clifford algebra, we can find a weak solution of Dirac operator with the solution in the L^2 space as well. The potential applications of the result will be studied in our future work.

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