Using Calculus to Make Kids

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + ar^4 + \ldots = \frac{a}{1-r} \text{ if } |r| < 1$$

BIG DADDY

1. Consider $f(w) = \sum_{n=0}^{\infty} w^n = 1 + w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^8 + w^9 + w^{10} + w^{11} + \ldots$

a. Replace w with $-x^2$ in the above expression:

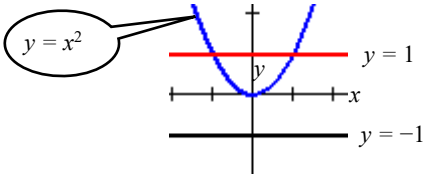
i. $f(-x^2) = \sum_{n=0}^{\infty} \left(\boxed{-x^2}\right)^n = 1 + \boxed{-x^2} + \left(\boxed{-x^2}\right)^2 + \left(\boxed{-x^2}\right)^3 + \left(\boxed{-x^2}\right)^4 + \ldots$

ii. Let's call the above child series $a(x)$. Simplify by removing all parentheses.

$a(x) = 1 + \boxed{-x^2} + \boxed{} + \boxed{} + \boxed{} + \ldots$

iii. Write the expression in sigma notation so it is simplified. $a(x) = \sum_{n=0}^{\infty} (-1)^n \boxed{\phantom{x^{2n}}}$

b. Report the radius of convergence of $a(x)$. **Tip:** Replace $r = w$ with $-x^2$ in the inequality $|r| < 1$. Since $y = |-x^2| = x^2$, you can solve this graphically finding when the parabola $y = x^2$ is below the line $y = 1$.



Complete: the radius of convergence of the series $a(x)$ is $R =$ $\boxed{}$.

Explore the **left** endpoint. Use your expanded form of the series in **a.ii.**

Complete: $a(\boxed{}) = 1 + \boxed{} + \boxed{} + \boxed{} + \boxed{} + \ldots = \boxed{}$

Therefore, the **left** endpoint of the interval of convergence, $x =$ _____ , is _____ in the interval.
{ included, excluded}

So on the graph of $a(x)$ at the left endpoint there is a _____
{ hole, point on the graph}

Explore the **right** endpoint. Use your expanded form of the series in **a.ii.**

Complete: $a(\boxed{}) = 1 + \boxed{} + \boxed{} + \boxed{} + \boxed{} + \ldots = \boxed{}$

Therefore, the **right** endpoint of the interval of convergence, $x =$ _____ , is _____ in the interval.
{ included, excluded}

So on the graph of $a(x)$ at the right endpoint there is a _____
{ hole, point on the graph}

We conclude the interval of convergence of the series $a(x)$ in part **1a** is the following.
Circle one and complete the blanks to the choice you selected.

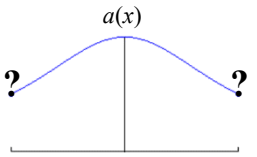
A. $\boxed{} \le x \le \boxed{}$ B. $\boxed{} < x \le \boxed{}$ C. $\boxed{} \le x < \boxed{}$ D. $\boxed{} < x < \boxed{}$

c. Within its interval of convergence, the series $a(x)$ in **1a** is equivalent to the closed form of the function below:

$$a(x) = \frac{1}{\boxed{}}$$

Tip: Replace $r = w$ with $-x^2$ in the closed form of yo' Big Daddy in the box at the top of the page.

d. Enter your closed form expression from part **1c** in a grapher using a viewing window where Xmin and Xmax would match your interval in part **1b**.
You should see the graph to the right. The graph has even symmetry.
Why would you expect this to be the case given what you wrote for part **1a.ii** and part **1c**?



Notice the *even* symmetry
(about the *y-axis*)

Fun Facts: The Genetics of Power Series!

We can create a new child series using

- **Algebraic Operations** (Substitution, Scaling, or Shifting):
  The child series can end up with a completely different **Center, Radius** and **Interval of Convergence** than its parent.
- **Calculus Operations** (Differentiation or Integration): The **Center** and **Radius of Convergence** are locked in and will match the parent series exactly.

The Endpoint Catch: While the radius survives calculus operations, the **Interval boundaries** might not.
Taking a *derivative* can cause an *included* endpoint to become *excluded*. Taking an *integral* may introduce an alternator, which can flip an *excluded* endpoint to *included* thanks to the Alternating Series Test. Always test your endpoints!

2. We have $a(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^{10} - \ldots = \frac{1}{1+x^2}$ on $-1 < x < 1$.

a. Then $b(x) = \int a(x)dx = \square - \square + \square - \square + \square - \ldots = \int \frac{1}{1+x^2}dx$

b. Integrate the closed form of the representation of the function $\int \frac{1}{1+x^2}dx =$ _____ $+ C$.

c. If $x = 0$ then, from **2b**, we have $b(0) =$ ___ $+ C$. Hint. See the graph below. Thus, equating **2a** and **2b**, we have $C =$ ___.

d. Write the expression for the series $b(x)$ in sigma notation so it is simplified. ⟶ $b(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \square$

TIP: You can see the pattern in part **2a** or integrate the expression (without the alternator) for $a(x) = \sum (-1)^n \cdot x^{2n}$

e. Report the radius of convergence of $b(x)$. $R =$ _____ (No Ratio Test needed. See the Fun Facts on the previous page.)

Explore the **left** endpoint:

i. Complete: $b(\boxed{-1}) = -1 + \square + \square + \square + \square + \ldots = \sum_{n=0}^{\infty} (-1)^n \cdot \square$ will _____ .
{ converge, diverge}

Your claim is true because of the _____ .
{ Give the name of your convergence test}

Therefore, the **left** endpoint of the interval of convergence, $x = -1$, is _____ in the interval.
{ included, excluded}
So on the graph of $b(x)$ at the left endpoint there is a _____ .
{ hole, point on the graph}

ii. Explore the **right** endpoint:

Complete: $b(\boxed{1}) = 1 + \square + \square + \square + \square + \ldots = \sum_{n=0}^{\infty} (-1)^{n+1} \cdot \square$ will _____ .
{ converge, diverge}

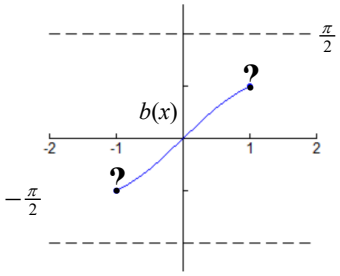
Your claim is true because of the _____ .
{ Give the name of your convergence test}

Therefore, the **right** endpoint of the interval of convergence, $x = 1$, is _____ in the interval.
{ included, excluded}
So on the graph of $b(x)$ at the right endpoint there is a _____ .
{ hole, point on the graph}

We conclude the series $b(x)$ in part will converge to the function to its closed form representation on the interval below. Circle one.

A. $\boxed{-1} \le x \le \boxed{1}$ B. $\boxed{-1} < x \le \boxed{1}$ C. $\boxed{-1} \le x < \boxed{1}$ D. $\boxed{-1} < x < \boxed{1}$

f. Enter your closed form expression in a grapher using a viewing window where Xmin = −1 and Xmax = 1.
You should see the graph below.
The graph has odd symmetry. Why would you expect this to be the case given what you wrote for part **2b**?



Rhino Bonus Opportunity (+1 point)**: A Gift from Gottfried (and Big Daddy)**

g. Recall in part **2e** you substituted $x = 1$ in your series expression in part **2b** to see it converges.

Substitute $x = 1$ in the closed form representation of $b(x)$ and find its exact value. Write it in the box below.

This remarkable result was published in the 17th century by Gottfried Wilhelm Leibniz.

i. Enter exact numerical values in each box.

$$1 + \square + \square + \square + \square + \square + \ldots = \frac{\square}{\square}$$

ii. Enter exact values in each box on the graph below.

