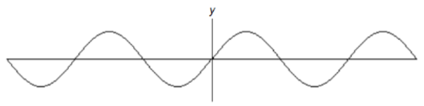
Thanks, Slick Lenny, for the Result $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$

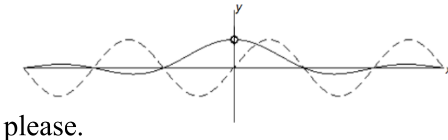
Name

(+3 Rhino Participation Bonus)

1. The functions $y = \sin x$ and $y = \frac{\sin x}{x}$ share infinitely many zeros. List those they share shown in the graph.

$x = \pm$ ☐, $\pm$ ☐, $\pm$ ☐, …





2. Write the first six terms of the Taylor polynomial for sine. Use factorial notation please.

$\sin x = \square \cdot x^{\square} - \square \cdot x^{\square} + \square \cdot x^{\square} - \square \cdot x^{\square} + \square \cdot x^{\square} - \square \cdot x^{\square} + \ldots$

3. Something cool happens next.

Divide each term by x. Assume x ≠ 0 and everything will be chill.

$\frac{\sin x}{x} = \square \cdot x^{\square} - \square \cdot x^{\square} + \square \cdot x^{\square} - \square \cdot x^{\square} + \square \cdot x^{\square} - \square \cdot x^{\square} + \ldots$

4. Lenny used what you wrote in **#1** to write the infinite degree polynomial in **#3** in *factored* form as a product. Complete the boxes with ***positive*** real numbers. Combine each of the pairs of factors of the same color. Use $(A-B)(A+B) = A^2 - B^2$.

$\frac{\sin x}{x} = \left(1 - \frac{x}{\square}\right)\left(1 + \frac{x}{\square}\right)\left(1 - \frac{x}{\square}\right)\left(1 + \frac{x}{\square}\right)\left(1 - \frac{x}{\square}\right)\left(1 + \frac{x}{\square}\right)\cdots$

Slick trick!

$= \left(1 - \frac{x^2}{\square}\right)\left(1 - \frac{x^2}{\square}\right)\left(1 - \frac{x^2}{\square}\right)\cdots$

5. Lenny expanded the infinite product you reported in **#4** and collected like terms to compare it with **#3**.
TIP: Notice the pattern of the coefficients of the x^2 terms for several partial products of this form.

$(1-A^2x^2)(1-B^2x^2) = 1 - (A^2+B^2)x^2 + (A^2B^2)x^4.$

$(1-A^2x^2)(1-B^2x^2)(1-C^2x^2) = 1 - (A^2+B^2+C^2)x^2 + (A^2B^2 + A^2C^2 + B^2C^2)x^4 - (A^2B^2C^2)x^6.$

$(1-A^2x^2)(1-B^2x^2)(1-C^2x^2)(1-D^2x^2) = 1 - (A^2+B^2+C^2+D^2)x^2$
$+ (A^2B^2 + A^2C^2 + A^2D^2 + B^2C^2 + B^2D^2 + C^2D^2)x^4$
$- (A^2B^2C^2 + A^2B^2D^2 + A^2C^2D^2 + B^2C^2D^2)x^6$
$+ (A^2B^2C^2D^2)x^8.$

For the infinite product in **#4**, $\frac{\sin x}{x} = (1-A^2x^2)(1-B^2x^2)(1-C^2x^2)(1-D^2x^2)\cdots$, we would have

$A =$ ☐, $B =$ ☐, $C =$ ☐, $D =$ ☐, etc.

6. Lenny noticed some golden treasure with the ***coefficient of the x^2 term*** of the *expanded* form of $\frac{\sin x}{x}$.
Report it in the box below. (It involves π and k.)

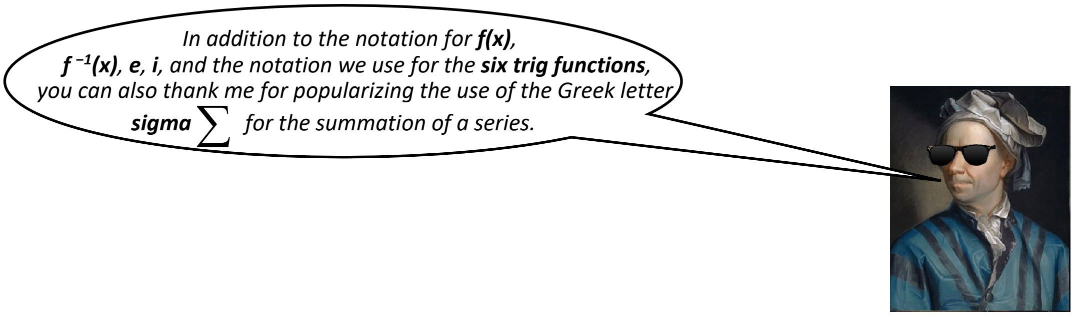
Set what you reported in Question 6 to what is in the dashed box in Question 3.

$\frac{\sin x}{x} = 1 - \sum_{k=\square}^{\square} \square \, x^2 + \ldots = 1 - \square \, x^2 + \ldots$

7. Follow Lenny's tip to show the result below. Equate the coefficients of the x^2 terms and solve.

(OVER PLEASE)

Pi Is Good.
More Pi Is Better.



8. It has been convenient to use the capital Greek letter Sigma (equivalent to our letter S) for a sum. Write the expanded form of $\frac{\sin x}{x}$ using Sigma notation.

$$\frac{\sin x}{x} = \frac{x^0}{1!} - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \ldots = \sum_{k=\square}^{\infty} \square$$

9. It is also convenient use the capital Greek letter Pi (equivalent to our letter P) for a product. Write the factored form of $\frac{\sin x}{x}$ using Pi notation.

$$\frac{\sin x}{x} = \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{16\pi^2}\right)\cdots = \prod_{k=\square}^{\square} \square$$

10. Recall how, in **#5**, we saw the expanded form of these partial products.

$$(1-A^2x^2)(1-B^2x^2) = 1 - (A^2+B^2)x^2 + (A^2B^2)x^4.$$

$$(1-A^2x^2)(1-B^2x^2)(1-C^2x^2) = 1 - (A^2+B^2+C^2)x^2 + (A^2B^2+A^2C^2+B^2C^2)x^4 - (A^2B^2C^2)x^6.$$

$$\begin{aligned}(1-A^2x^2)(1-B^2x^2)(1-C^2x^2)(1-D^2x^2) = 1 &- (A^2+B^2+C^2+D^2)x^2 \\ &+ (A^2B^2+A^2C^2+A^2D^2+B^2C^2+B^2D^2+C^2D^2)x^4. \\ &- (A^2B^2C^2+A^2B^2D^2+A^2C^2D^2+B^2C^2D^2)x^6 + (A^2B^2C^2D^2)x^8.\end{aligned}$$

$$(1-A^2x^2)(1-B^2x^2)(1-C^2x^2)(1-D^2x^2)(1-E^2x^2) = 1 - (A^2+B^2+C^2+D^2+E^2)x^2 + \ldots + (A^2B^2C^2D^2E^2)x^{10}.$$

In **#5** we found the values of A, B, C, and D for which $\frac{\sin x}{x} = (1-A^2x^2)(1-B^2x^2)(1-C^2x^2)(1-D^2x^2)\cdots$

Complete the boxes for the nth partial product.

$$\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{16\pi^2}\right)\cdots\left(1-\frac{x^2}{n^2\pi^2}\right) = 1 - \sum_{k=\square}^{\square} \square + \ldots + \prod_{k=\square}^{\square} \square$$