The Taylor Polynomial for the Square Root Function Centered at 1

1. Find the first three derivatives of $f(x) = f(x) = \sqrt{x}$

$f(x) = \sqrt{x} = x^{1/2}$

$f'(x) =$

$f''(x) =$

$f'''(x) =$

2. Evaluate the function and its first three derivatives at $x = 1$ to find the Taylor coefficients c_0, c_1, c_2, and c_3. Simplify.

$f(1) = \sqrt{1} = 1$ $\qquad c_0 = \frac{f(1)}{0!} =$

$f'(1) =$ $\qquad c_1 = \frac{f'(1)}{1!} =$

$f''(1) =$ $\qquad c_2 = \frac{f''(1)}{2!} =$

$f'''(1) =$ $\qquad c_3 = \frac{f'''(1)}{3!} =$

3. Write the third-degree Taylor polynomial $T_3(x)$, i.e., the "tangent cubic" to the graph at $x = 1$.
The values in the boxes should be whole numbers or fractions in lowest terms.

$T_3(x) = c_0(x-1)^0 + c_1(x-1)^1 + c_2(x-1)^2 + c_3(x-1)^3.$

$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!} \cdot (x-1)^k = \sum_{n=0}^{3} c_n \cdot (x-1)^n = \square + \square \cdot (x-1) + \square \cdot (x-1)^2 + \square \cdot (x-1)^3$$

4. Fun Fact: It can be shown that, for $n \geq 2$, the value of the nth derivative at $x = 1$ follows that pattern

$$f^{(n)}(1) = \frac{(-1)^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n}.$$

a. Use this formula to find the value of $f^{(4)}(1)$ for $n = 4$.

$f^{(4)}(1) =$

b. Using your third derivative in Question 1, differentiate one more time to find $f^{(4)}(x)$.

$f^{(4)}(x) =$

c. Show that if $x = 1$, then $f^{(iv)}(1)$ is your result in part **4a**.

d. Use the value of $f^{(4)}(1)$ to find $c_4 = \frac{f^{(4)}(1)}{4!}$ and simplify. Insert an integer in the box. $c_4 = -\frac{5}{\square}$

Rhino Bonus (+2)

5. Using the Fun Fact in Question **4**, the general Taylor series for $f(x) = \sqrt{x}$ centered at 1 can be written as:

$$\sqrt{x} = T_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!}(x-1)^n = 1 + \tfrac{1}{2}(x-1)^2 - \tfrac{1}{8}(x-1)^3 + \tfrac{1}{16}(x-1)^3 - \ldots$$

Follow the steps to use the Ratio Test find the radius of convergence. Most terms will "one out."

a. Find $\left|\frac{a_{n+1}}{a_n}\right|$ and simplify. Enter an expression involving x in the box. Some of the work has been done for you below.

$$a_n = \frac{(-1)^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!}(x-1)^n$$

$$a_{n+1} = \frac{(-1)^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdots (2(n+1)-3)}{2^{n+1}(n+1)!}(x-1)^{n+1}$$

$$= \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n+1)}{2^n 2(n+1)n!}(x-1)^n(x-1)$$

$$= \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)(2n+1)}{2^n 2(n+1)n!}(x-1)^n(x-1)$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)(2n+1)(x-1)^n(x-1)}{2^n 2(n+1)n!}}{\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)(x-1)^n}{2^n n!}}\right| = \left|\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)(2n+1)(x-1)^n(x-1)2^n n!}{1 \cdot 3 \cdot 5 \cdots (2n-3)(x-1)^n \;\; 2^n 2(n+1)n!}\right|$$

$$= \left|\boxed{}\right|$$

b. Find the limit. Your answer involves x and contains absolute value signs.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \boxed{}$$

c. Solve the inequality $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ for x.

d. The *radius* of convergence is R = ____.

e. The *center* of the interval of convergence is ____.

6. It can be shown that, after testing each endpoint, they are both *included*.

Therefore, the interval of convergence is $\boxed{} \leq x \leq \boxed{}$ (Insert numbers in each box.)

Sketch the interval of convergence on a number line. 