Practice Questions from HW 27-29 KEY

1. Answer the following for the power series $\sum c_n (x-a)^n$. Complete the blanks.

a. The power series $\sum c_n (x-a)^n$ is centered at the value $x = \underline{\ a\ }$.

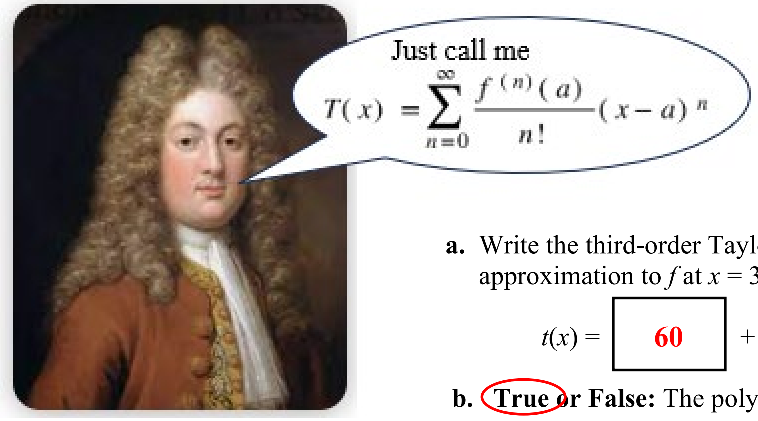
b. Suppose the interval of convergence is **all real numbers**. Then the radius of convergence is $R = \underline{\ \infty\ }$.

c. Suppose the interval of convergence is only **the value $x = a$**. Then the radius of convergence is $R = \underline{\ 0\ }$.

d. Suppose the interval of convergence is $| x - a | < b$, i.e. $a - b < x < a + b$.
Then the radius of convergence is $R = \underline{\ b\ }$.
The geometric representation of $| x - a | < b$ is the set of all x which are b units from a.

2. Suppose we have the following

Brook Taylor



$f(34) = 60$,

$f'(34) = 43$,

$f''(34) = 22$, and

$f'''(34) = 30$.

a. Write the third-order Taylor polynomial approximation to f at $x = 34$ Simplify please but do not multiply out.

$$t(x) = \boxed{60} + \boxed{43(x-34)} + \underset{\boxed{\frac{22(x-34)^2}{2!}}}{\boxed{11(x-34)^2}} + \underset{\boxed{\frac{30(x-34)^3}{3!}}}{\boxed{5(x-34)^3}}$$

b. [✓ **True**] **or False:** The polynomial $t(x)$ is the tangent cubic to f at the value $x = 34$.

3. A function $f(x)$ centered at 0 has the unique property that $f^{(k)}(0) = 12$ for $k = 0, 1, 2, 3, \ldots$

The Maclaurin Series for $f(x)$ is $\sum_{k=0}^{\infty} \boxed{\frac{12x^n}{n!}} = \boxed{12} + \boxed{12x} + \boxed{6x^2} + \boxed{2x^3} + \ldots$

$$\boxed{\frac{12x^2}{2!} = \frac{12x^2}{2}} \qquad \boxed{\frac{12x^3}{3!} = \frac{12x^3}{6}}$$

What function $f(x)$ has this property? $f(x) = \boxed{12e^x}$

For all x we have $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ $\quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

For $-1 < x < 1$ we have $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$

For $-1 < x \le 1$ we have $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $\quad$ For $-1 \le x \le 1$ we have $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

4. Write the first four nonzero terms of the series for $f(w) = e^{-w}$.

From the Fun Fact, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ so replace x with $-w$: $e^{-w} = 1 + (-w) + \frac{(-w)^2}{2!} + \frac{(-w)^3}{3!} + \ldots$

$$e^{-w} = 1 - w + \frac{w^2}{2!} - \frac{w^3}{3!} + \ldots$$

The interval of convergence for e^x and e^{-w} is all real numbers.

If $h(w) = \int f(w)\,dw$, assuming $h(0) = -1$, we have $h(w) = \int 1\,dw - \int w\,dw + \frac{1}{2!}\int w^2\,dw - \frac{1}{3!}\int w^3\,dw + \ldots$

$$h(w) = w - \frac{1}{2}w^2 + \frac{1}{3\cdot 2!}w^3 - \frac{1}{4\cdot 3!}w^4 + \ldots + C$$

$$h(w) = w - \frac{w^2}{2!} + \frac{w^3}{3!} - \frac{w^4}{4!} + \ldots + C$$

When we substitute 0 in for w, we have $-1 = 0 - 0 + 0 - 0 + \ldots + C$ so $C = -1$.

Thus $h(w) = -1 + w - \frac{w^2}{2!} + \frac{w^3}{3!} - \frac{w^4}{4!} + \ldots$

The interval of convergence of $h(w)$ is all real numbers and the series converges to $\int e^{-w}dw = -e^{-w}$.

We could have also obtained this series by multiplying each term of the series for $f(w) = e^{-w} = 1 - w + \frac{w^2}{2!} - \frac{w^3}{3!} + \ldots$ by −1 to get $-e^{-w} = -1 + w - \frac{w^2}{2!} + \frac{w^3}{3!} + \ldots$

5. Write the first four nonzero terms of the series for $f(w) = \sin(w^2)$

From the Fun Fact, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ so replace x with w^2: $\sin w^2 = w^2 - \frac{w^6}{3!} + \frac{w^{10}}{5!} - \frac{w^{14}}{7!} + \ldots$

$g(w) = f'(w) = 2w - \frac{6w^5}{3!} + \frac{10w^9}{5!} - \frac{14w^{13}}{7!} + \ldots = \frac{2w^1}{0!} - \frac{2w^5}{2!} + \frac{2w^9}{4!} - \frac{2w^{13}}{6!} + \ldots$

The interval of convergence of $f(w)$ and $g(w)$ is all real numbers.
The series $g(w) = f'(w)$ converges to the derivative of $f(w) = \sin(w^2)$ which is **$2w\cos w^2$.** (Use the chain rule.)
We can check by creating the series for **$\cos w^2$** and multiplying each term by $2w$.

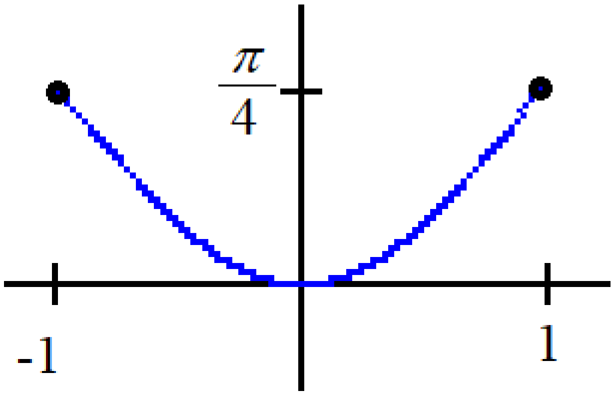
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ so $\cos w^2 = 1 - \frac{w^4}{2!} + \frac{w^8}{4!} - \frac{w^{12}}{6!} + \ldots$ and $2w\cos w^2 = 2w - \frac{2w^5}{2!} + \frac{2w^9}{4!} - \frac{2w^{13}}{6!} + \ldots$

6. Write the first four nonzero terms of the series for $f(w) = \tan^{-1}(w^2)$

From the Fun Fact, $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ so replace x with w^2: $\tan^{-1} w^2 = w^2 - \frac{w^6}{3} + \frac{w^{10}}{5} - \frac{w^{14}}{7} + \ldots$

Compare this with the previous question.
The radius of convergence is $R = 1$.
To find the interval of convergence we test each endpoint. The powers are even so we can test both endpoints at once.
At $w = \pm 1$, the series for $f(w) = \tan^{-1}(w^2)$ is $(\pm 1)^2 - \frac{(\pm 1)^6}{3} + \frac{(\pm 1)^{10}}{5} - \frac{(\pm 1)^{14}}{7} + \ldots = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

This is an alternating series $\sum_{k=0}^{\infty} (-1)^{k+1} \cdot \frac{1}{2k+1}$ that converges conditionally by the Alternating Series Test, since $\frac{1}{2k+1}$ decreases and approaches 0. Each endpoint is a defined point.
The interval of convergence is $[-1, 1]$ or $-1 \le w \le 1$



7. Write the first four nonzero terms of the series for $f(w) = 10\tan^{-1}(\frac{w}{2})$

From the Fun Fact, $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ so replace x with $\frac{w}{2}$:

$10\tan^{-1}\left(\frac{w}{2}\right) = 10 \cdot \left(\frac{w}{2} - \frac{1}{3} \cdot \frac{w^3}{2^3} + \frac{1}{5} \cdot \frac{w^5}{2^5} - \frac{1}{7} \cdot \frac{w^7}{2^7} + \ldots\right) = 10\tan^{-1}\left(\frac{w}{2}\right) = 5w - \frac{5w^3}{3 \cdot 2^2} + \frac{5w^5}{5 \cdot 2^4} - \frac{5w^7}{7 \cdot 2^6} + \ldots$

The radius of convergence of tan$^{-1}(x)$ is 1, and, ignoring endpoints, the interval of convergence is $-1 < x < 1$.
Ignoring endpoints, the interval of convergence tan$^{-1}(w/2)$ is $-1 < \frac{w}{2} < 1$. Multiply all parts by 2, and we have $-2 < w < 2$.

Now test endpoints.

At $w = -2$, we have $5(-2) - \frac{5(-2)^3}{3\cdot 2^2} + \frac{5(-2)^5}{5\cdot 2^4} - \frac{5(-2)^7}{7\cdot 2^6} + \ldots = 5(-2) + \frac{5\cdot 2}{3} - \frac{5\cdot 2}{5} + \frac{5\cdot 2}{7} + \ldots$

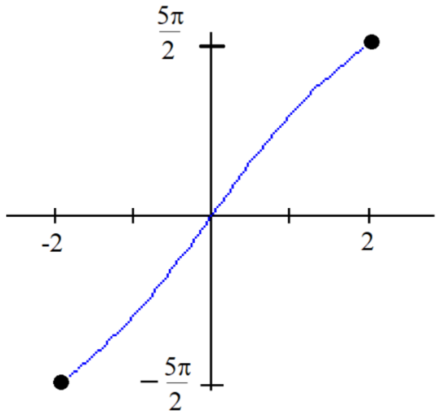
This is the alternating series $-\frac{5\cdot 2}{1} + \frac{5\cdot 2}{3} - \frac{5\cdot 2}{5} + \frac{5\cdot 2}{7} + \ldots$ which converges conditionally by AST.

$\frac{10}{2n+1}$ decreases and approaches 0.

At $w = 2$, we have $5(2) - \frac{5(2)^3}{3\cdot 2^2} + \frac{5(2)^5}{5\cdot 2^4} - \frac{5(2)^7}{7\cdot 2^6} + \ldots$

This is an alternating series $-\frac{5\cdot 2}{1} + \frac{5\cdot 2}{3} - \frac{5\cdot 2}{5} + \frac{5\cdot 2}{7} + \ldots$ which also converges conditionally by AST.

Each endpoint is a defined point. Evaluate $f(w) = 10\tan^{-1}(\frac{w}{2})$ at -2 and 2 to find what the series converges to at the endpoints.
Recall $\tan^{-1} 1 = \frac{\pi}{4}$ so $10\tan^{-1} 1 = 10\cdot\frac{\pi}{4} = \frac{5\pi}{2}$
The interval of convergence is $[-2, 2]$ or $-2 \le w \le 2$.



8. Write the first four nonzero terms of the series for $f(x) = e^{-(x-70)^2}$.

From the Fun Fact, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ so replace x with $-(x-70)^2$.
Even powers will have positive coefficients. Odd powers will have negative coefficients. The terms are:

$$1 - \frac{(x-70)^2}{1!} + \frac{(x-70)^4}{2!} - \frac{(x-70)^6}{3!} + \ldots$$

In sigma notation, $f(x) = \sum_{k=0}^{\infty} \frac{(x-70)^{2k}}{k!}\cdot(-1)^k$. It converges for all x because it inherited the same radius of convergence as $f(x)$, which is $R = \infty$. Although not necessary, we could also use the ratio test to show that series converges for all x.

$$\left|\frac{a_{k+1}}{1}\cdot\frac{1}{a_k}\right| = \frac{(x-70)^{2(k+1)}}{(k+1)!}\cdot\frac{k!}{(x-70)^{2k}} = \frac{\cancel{(x-70)^{2k}}(x-70)^2}{(k+1)\cancel{k!}}\cdot\frac{\cancel{k!}}{\cancel{(x-70)^{2k}}} = \frac{1}{k+1}\cdot(x-70)^2$$

As $k \to \infty$, this expression approaches 0 which is less than 1, regardless of the value of x. $\lim_{k\to\infty}\frac{1}{k+1}\cdot(x-70)^2 = 0$.

9. Consider the Taylor series $-49x^2+\frac{49x^4}{2}-\frac{49x^8}{3}+\frac{49x^8}{4}-\frac{49x^{10}}{5}+\ldots$

a. Use the set of Fun Facts to write the function that represents this series on its interval of convergence.

$$\ln(1+w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \frac{w^5}{5} - \ldots \text{ first replace } w \text{ with } x^2$$

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} - \ldots \text{ and then multiply by } -49$$

$$-49x^2 + \frac{49x^4}{2} - \frac{49x^8}{3} + \frac{49x^8}{4} - \frac{49x^{10}}{5} + \ldots = -49\ln(1+x^2)$$

b. Write the series using summation notation. $\sum_{k=1}^{\infty}\frac{-49x^{2k}}{k}\cdot(-1)^{k+1}$ or $\sum_{k=1}^{\infty}\frac{49x^{2k}}{k}\cdot(-1)^{k}$

c. Report the radius R of convergence. $R = 1$

d. Test each endpoint for convergence.
Each term will have $x = \pm 1$ squared, and, since $(\pm 1)^2 = 1$,

the series $-49x^2+\frac{49x^4}{2}-\frac{49x^8}{3}+\frac{49x^8}{4}-\frac{49x^{10}}{5}+\ldots = -49+\frac{49}{2}-\frac{49}{3}+\frac{49}{4}-\frac{49}{5}+\ldots$

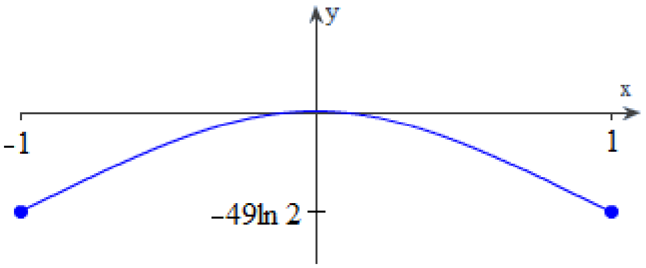
$$= -49\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots\right)$$

$= -49\ln 2$ for both endpoints.

The alternating harmonic series converges conditionally because of the AST (since $\frac{1}{n}$ decreases and approaches 0).

e. Report the interval of convergence. $[-1, 1]$ or $-1 \le x \le 1$.

f. Sketch a graph on the interval of convergence.



10. Consider the Taylor series $-5x+\frac{(5x)^2}{2}-\frac{(5x)^3}{3}+\frac{(5x)^4}{4}-\frac{(5x)^5}{5}+\ldots$

a. Use the set of Fun Facts to write the function that represents this series on its interval of convergence.

$$\ln(1+w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \frac{w^5}{5} - \ldots \text{ replace } w \text{ with } 5x$$

$$\ln(1+5x) = 5x - \frac{(5x)^2}{2} + \frac{(5x)^3}{3} - \frac{(5x)^4}{4} + \frac{(5x)^5}{5} + \ldots \text{ and then multiply by } -1$$

$$-5x + \frac{(5x)^2}{2} - \frac{(5x)^3}{3} + \frac{(5x)^4}{4} - \frac{(5x)^5}{5} + \ldots = -\ln(1+5x)$$

b. Write the series using summation notation. $\sum_{k=1}^{\infty} \frac{-(5x)^{k}}{k} \cdot (-1)^{k+1}$ or $\sum_{k=1}^{\infty} \frac{(5x)^{k}}{k} \cdot (-1)^{k}$

c. The radius of convergence of the parent function, ln(1 + *w*) = $w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \frac{w^5}{5} - \dots$ is $R = 1$.
The radius of convergence of ln(1 + *w*) is 1, and, ignoring endpoints, the interval of convergence is $-1 < w < 1$.
Similar to Problem 7, ignoring endpoints, when we replace *w* with 5*x*, the interval of convergence is $-1 < 5x < 1$.

Divide all parts by 5, and we have $-\frac{1}{5} < x < \frac{1}{5}$ so the radius of convergence of the child series $-\ln(1 + 5x)$ is $R = \boxed{\frac{1}{5}.}$

Note that multiplying terms of the series by a positive or negative constant does not change the radius of convergence.

d. Test each endpoint for convergence.

i. When $x = -R$, i.e., $x = -\frac{1}{5}$, the series $-5x + \frac{(5x)^2}{2} - \frac{(5x)^3}{3} + \frac{(5x)^4}{4} - \frac{(5x)^5}{5} + \dots$

$$= -5\left(-\frac{1}{5}\right) + \frac{1}{2}\cdot\left(5\cdot -\frac{1}{5}\right)^2 - \frac{1}{3}\cdot\left(5\cdot -\frac{1}{5}\right)^3 + \frac{1}{4}\cdot\left(5\cdot -\frac{1}{5}\right)^4 - \frac{1}{5}\cdot\left(5\cdot -\frac{1}{5}\right)^5 + \dots$$

$$= 1 + \frac{1}{2}\cdot(-1)^2 - \frac{1}{3}\cdot(-1)^3 + \frac{1}{4}\cdot(-1)^4 - \frac{1}{5}\cdot(-1)^5 + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \infty$$

This is the famous harmonic series.

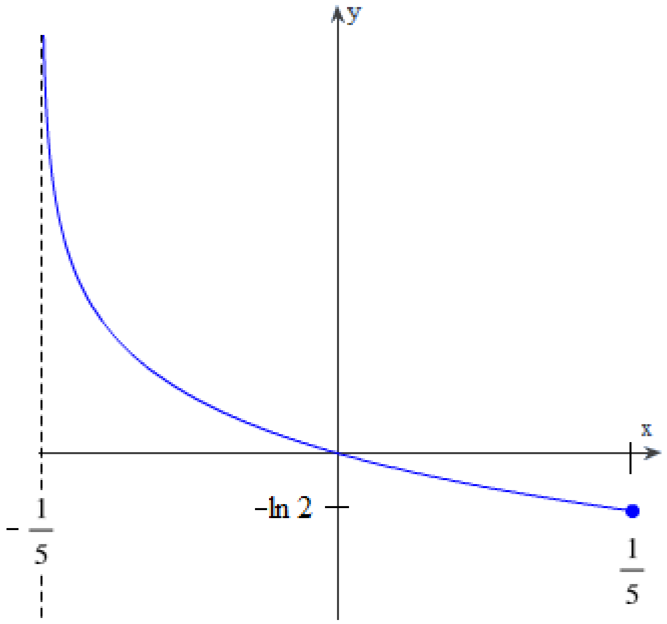
ii. When $x = R$, i.e., $x = \frac{1}{5}$, the series $-5x + \frac{(5x)^2}{2} - \frac{(5x)^3}{3} + \frac{(5x)^4}{4} - \frac{(5x)^5}{5} + \dots$

$$= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots = -\ln 2$$

This is the famous alternating harmonic series.

e. Report the interval of convergence. $(-1, 1]$ or $-1 < x \leq 1$.

f. Sketch a graph on the interval of convergence.



Practice Questions over 12.1

1. Eliminate the parameter, t, to obtain an equation of the form $y = f(x)$.

a. $x = \sqrt[3]{t-1}$, $y = \cos t$ (There is no new domain restriction.)
$t = x^3 + 1$ so the curve is $y = \cos(x^3 + 1)$

b. $x = \sqrt[3]{t-1}$, $y = \cos \sqrt[3]{t-1}$ (There is no new domain restriction.)
Replace $\sqrt[3]{t-1}$ with x so the curve is $y = \cos x$.

c. $x = -\sqrt{t}$, $y = -3\sqrt{t} + 6e^{\sqrt{t}}$ (Specify the domain restriction.)
Replace $-\sqrt{t}$ with x so the curve is

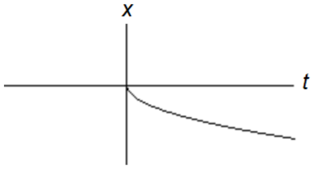
$y = -3\sqrt{t} + 6e^{\sqrt{t}}$

$y = 3(-\sqrt{t}) + 6e^{-(-\sqrt{t})}$

$y = 3x + 6e^{-x}$.

We have $y = 3x + 6e^{-x}$.

Restrict the domain to $x \le 0$ since the range of the graph of $x = -\sqrt{t}$ is $x \le 0$.



d. $x = -\sqrt{t}$, $y = 2t + 1$ (Specify the domain restriction.)
Square both sides of $x = -\sqrt{t}$ and we have $x^2 = t$, but restrict the domain to $x \le 0$.
Replace t with x^2 so the curve is $y = 2x^2 + 1, x \le 0$.

e. $x = e^{-t}$, $y = 7e^{-3t}$ (Specify the domain restriction.)
Replace e^{-t} with x so the curve is $y = 7e^{-3t} = 7(e^{-t})^3 = 7x^3$ with $x > 0$.
Restrict the domain to $x > 0$ since the range of the graph of $x = e^{-t}$ is $x > 0$.

f. $x = 4\sin t$, $y = 3 + 4\cos t$

Use $\cos^2 t + \sin^2 t = 1$ with $\sin t = \dfrac{x}{4}$ and $\cos t = \dfrac{y-3}{4}$.

The curve is $\dfrac{(y-3)^2}{4^2} + \dfrac{x^2}{4^2} = 1$ but we can also write $(y-3)^2 + x^2 = 4^2$.

Or $\dfrac{x^2}{4^2} + \dfrac{(y-3)^2}{4^2} = 1$ or $x^2 + (y-3)^2 = 4^2$. This is a circle with center (0, −3) and radius 4.

g. $x = 3\sin t$, $y = 3 - 6\cos t$

Use $\sin^2 t + \cos^2 t = 1$ with $\sin t = \dfrac{x}{3}$ and $\cos t = \dfrac{y-3}{-6}$.

The curve is $\dfrac{x^2}{3^2} + \left(\dfrac{y-3}{-6}\right)^2 = 1$ or $\dfrac{x^2}{3^2} + \dfrac{(y-3)^2}{6^2} = 1$

h. $x = 3\cos t$, $y = 3 - 9\cos^2 t$

Replace $\cos t$ with $\dfrac{x}{3}$ and $\cos^2 t$ with $\dfrac{x^2}{9}$ so the curve is $y = 3 - 9 \cdot \dfrac{x^2}{9} = 3 - x^2, -3 \le x \le 3$.

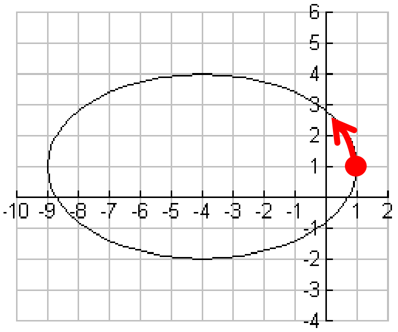
2. Write a set of parametric equations $x = f(t)$, $y = g(t)$ for the curve

a. $x = 4y^5 - 3y^2 + 2\cos y - e^{7y}$
$x = 4t^5 - 3t^2 + 2\cos t - e^{7t}$ and $y = t$

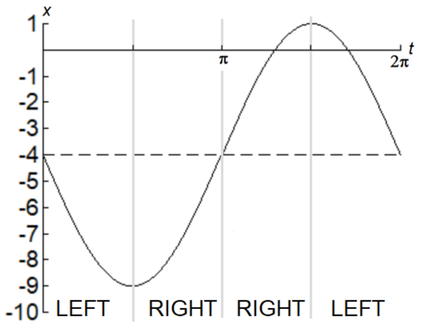
b. The circle $(x-1)^2 + (y+2)^2 = 49$
The center is $(1, -2)$ and radius is 7.
Parametric equations: $x = 7\cos t + 1$
$y = 7\sin t - 2$
Other options are possible.

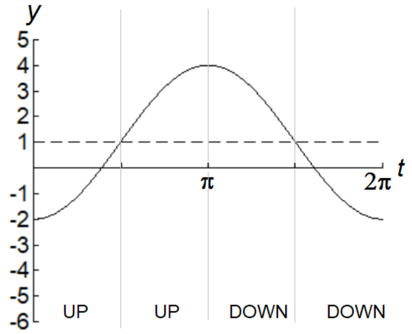
c. The ellipse $\dfrac{(x-5)^2}{9} + \dfrac{(y+2)^2}{4} = 1$
Parametric equations:
$x = 3\cos t + 5$
$y = 2\sin t - 2$

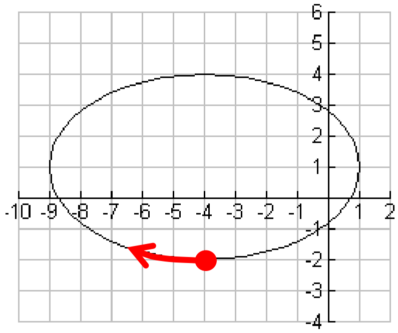
d. The ellipse shown with the initial value of $t = 0$ $x = 1$, $y = 1$ traveling counterclockwise.
$x = 5\cos t - 4$
$y = 3\sin t + 1$



e. The same ellipse shown to the right with the initial value of $t = 0$, $x = -4$, $y = -2$ traveling clockwise. This means we have the following motion for x and y:

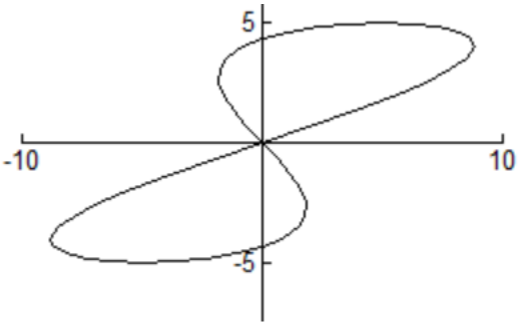






Parametric equations: $x = -5\sin t - 4$
$y = -3\cos t + 1$

3. The graph of the parametric equations $x = 5\sin t - 5\sin 2t$ and $y = 5\sin t$ is shown for $0 \le t \le 2\pi$.



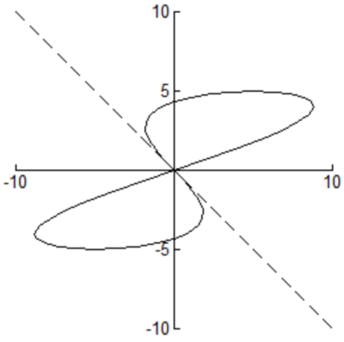
$x = 5\sin t - 5\sin 2t$

$$\frac{dx}{dt} = 5\cos t - 5(\cos 2t)\cdot\frac{d}{dt}(2t)$$
$$= 5\cos t - 10\cos 2t$$

$y = 5\sin t$

$$\frac{dy}{dt} = 5\cos t$$

a. Evaluate $\frac{dy}{dx}$ at the origin when $t = 0$. $\frac{dy}{dx} = \boxed{-1}$

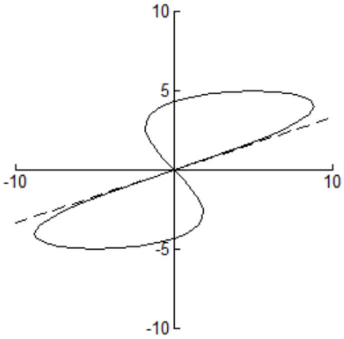


If $t = 0$, then $\frac{dx}{dt} = 5\cos(t) - 10\cos(2t) = 5\cos(0) - 10\cos(0) = 5(1) - 10(1) = -5$

If $t = 0$, then $\frac{dy}{dt} = 5\cos(0) = 5$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{5}{-5} = -1$$

b. Evaluate $\frac{dy}{dx}$ at the origin when $t = \pi$. $\frac{dy}{dx} = \boxed{\frac{1}{3}}$



If $t = \pi$, then $\frac{dx}{dt} = 5\cos(t) - 10\cos(2t) = 5\cos(\pi) - 10\cos(2\pi) = -5 - 10 = -15$

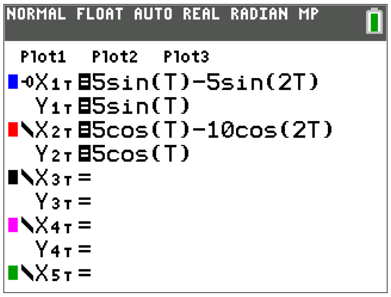
If $t = \pi$, then $\frac{dy}{dt} = 5\cos(\pi) = -5$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-5}{-15} = \frac{1}{3}$$

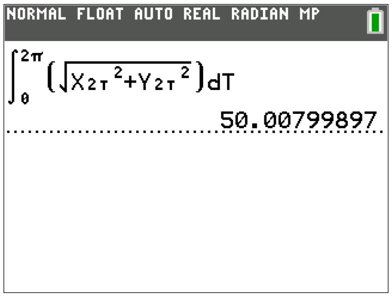
c. The arc length from $t = 0$ to $t = 2\pi$ of this curve is given by $\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

Complete the boxes to set up the integral to find the arc length. You need not simplify.
Then use FNINT to find the arc length rounded to the nearest whole number.

$$\int_0^{2\pi} \sqrt{\boxed{(5\cos t - 10\cos 2t)^2 + (5\cos t)^2}}\,dt \approx \boxed{50}$$



```
NORMAL FLOAT AUTO REAL RADIAN MP
 Plot1  Plot2  Plot3
-0X1T=5sin(T)-5sin(2T)
  Y1T=5sin(T)
 \X2T=5cos(T)-10cos(2T)
  Y2T=5cos(T)
 \X3T=
  Y3T=
 \X4T=
  Y4T=
 \X5T=
```



```
NORMAL FLOAT AUTO REAL RADIAN MP
∫0^2π (√(X2T²+Y2T²))dT
                    50.00799897
```