KEY

1. a. Plot and **label** the complex numbers.

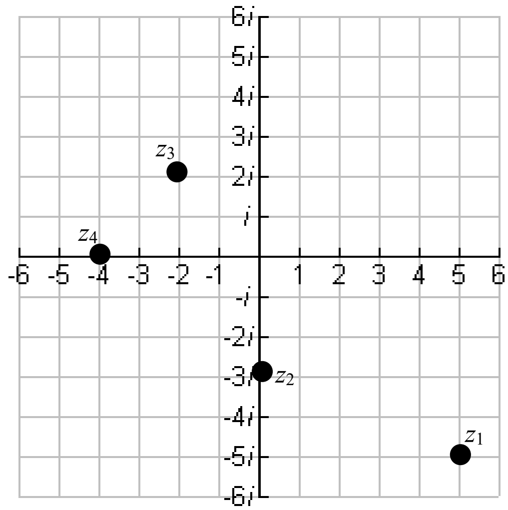
$z_1 = 5 - 5i$

$z_2 = -3i$

$z_3 = 2\sqrt{2}e^{i3\pi/4}$

$z_4 = 4e^{7\pi i}$

$re^{(i\theta)} = r(\cos\theta + i\sin\theta)$



b. Write z_1 and z_2 in exponential form $re^{i\theta}$, where r and θ are exact real numbers (and θ is in radians).
Hint: Part (a) may help.
(There are many correct answers for θ; however, report exact **radians** please.)

$z_1 = 5 - 5i$

$r = 5\sqrt{2}$

$\theta = \frac{7\pi}{4}$ (Also $-\frac{\pi}{4}$) (Radians!)

$5\sqrt{2}e^{i7\pi/4}$

$z_2 = -3i$

$r = 3$

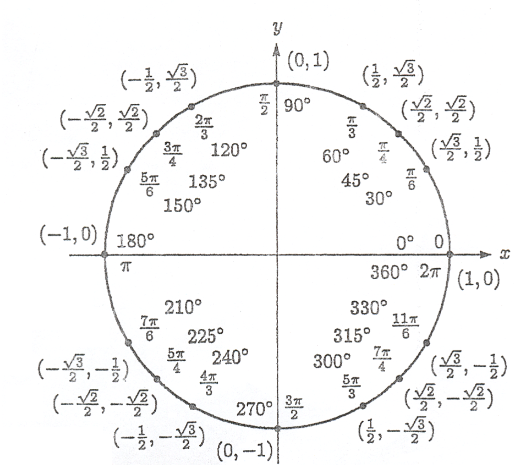
$\theta = \frac{3\pi}{2}$ (Also $-\frac{\pi}{2}$) (Radians!)

Exponential form of z_2 is $3e^{i3\pi/2}$

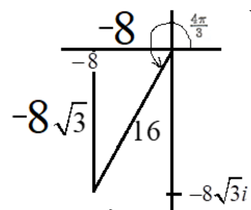
c. Write z_3 and z_4 in rectangular form $a + bi$, where a and b are real numbers.

$z_3 = 2\sqrt{2}e^{i3\pi/4} = \boxed{-2} + \boxed{2}\, i$

$z_4 = 4e^{7\pi i} = \boxed{-4} + \boxed{0}\, i$



2. a. Write $(2e^{i\pi/3})^4$ in the exponential form $re^{i\theta}$, where θ is exact and in radians.



$r = 16$

$\theta = \frac{4\pi}{3}$ (Radians!)

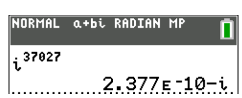
Exponential form $re^{i\theta}$ of $(2e^{i\pi/3})^4$ is $16e^{i4\pi/3}$

b. Write $(2e^{i\pi/3})^4$ in rectangular form $a + bi$, using exact values.

$\left(2e^{(i\frac{\pi}{3})}\right)^4 = \boxed{-8} + \boxed{-8\sqrt{3}} \cdot i$

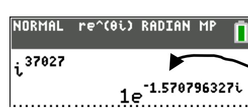
3. Consider the complex number i^{37027}.

a. A student uses a calculator to try to write the number in rectangular form $a + bi$, where a and b are real numbers. See the screen below.
What should the exact answer really be?
Report the exact answer in rectangular form $a + bi$:



$i^{37027} = \boxed{0} + \boxed{-1} \cdot i$

b. When trying to write the number in polar form $re^{i\theta}$ where r and θ are real numbers, a student sees the screen below. What is the exact radian measure of the angle θ on the screen? (It involves π.)



$\theta = -\frac{\pi}{2}$

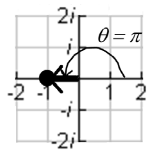
c. Report the location of i^{37027} in the complex plane.

A. the positive real axis
B. the positive imaginary axis
C. the negative real axis
(D) the negative imaginary axis

4. If the complex number z is represented by a vector, describe how to construct the vector u which is the complex number z multiplied by the number $re^{i\theta}$, i.e, $u = z \cdot re^{i\theta}$.

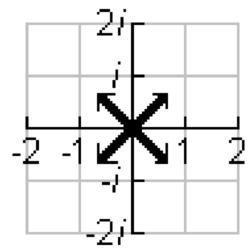
Sketch u so that its length is r times the length of z and its angle is rotated by the value θ.

5. Report your answers in polar form $r\text{cis}\theta$ in radians, exponential form $re^{i\theta}$ in radians, and in rectangular form $a + bi$ Report all the fourth roots of the number -1 and sketch them on the complex plane.

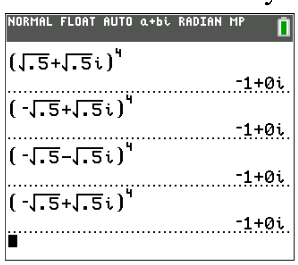


Step 1: Represent -1 in the form $re^{(i\theta)} = r\text{cis}\theta = r(\cos\theta + i\sin\theta)$.
Sketching the number can help you determine $r = 1$ and $\theta = \pi$.
We could write $-1 = 1cis\ \pi = 1e^{(i\pi)}$.

Step 2: To find a fourth root, we raise the complex number to the fourth power, i.e., $(e^{(i\pi)})^{1/4}$ is one of these. But there are four of these roots.

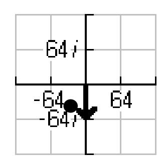


Step 3: Sketch $(e^{(i\pi)})^{1/4} = e^{(i\pi/4)} = cis\frac{\pi}{4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.
The four roots are evenly spaced $\frac{2\pi}{4} = \frac{\pi}{2}$ apart,
so we have $cis\frac{\pi}{4}$, $cis\frac{3\pi}{4}$, $cis\frac{5\pi}{4}$, and $cis\frac{7\pi}{4}$, or
we could write $e^{(i\pi/4)}$, $e^{(i3\pi/4)}$, $e^{(i5\pi/4)}$, and $e^{(i7\pi/4)}$, or,
by symmetry: $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, and $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$,
You can check this by raising each to the 4th power: $(e^{(i\pi/4)})^4 = e^{(i\pi)} = -1$,
$(e^{(i3\pi/4)})^4 = e^{(i3\pi)} = -1$,
$(e^{(i5\pi/4)})^4 = e^{(i5\pi)} = -1$, and
$(e^{(i7\pi/4)})^4 = e^{(i7\pi)} = -1$.



6. Report all the sixth roots of the number $-64i$. Report your answers in polar form $r\text{cis}\theta$ in degrees.

Step 1: Represent $-64i$ in the form $re^{i\theta} = r\text{cis}\theta$. Since $re^{(i\theta)} = r\text{cis}\theta = r(\cos\theta + i\sin\theta)$,
we have $r = 64$ and $\theta = 270^\circ$. We could write $-64i = 64\,cis\,270^\circ = 64e^{i270^\circ}$.
Alternatively, you could also use $\theta = -90^\circ$.



Step 2: To find a sixth root, we raise the complex number to the one sixth power, i.e., $(64e^{i270^\circ})^{1/6} = 2e^{i45^\circ}$ is one of these. But there are six of these roots.

Step 3: The six roots are evenly spaced $\frac{360^\circ}{6} = 60^\circ$ apart,
so we have $2\,cis\,45^\circ$, $2\,cis\,(45^\circ + 60^\circ) = cis\,(105^\circ)$, $cis\,(105^\circ + 60^\circ) = cis\,(165^\circ)$,
$cis\,(165^\circ + 60^\circ) = cis\,(225^\circ)$, $cis\,(225^\circ + 60^\circ) = cis\,(285^\circ)$, $cis\,(285^\circ + 60^\circ) = cis\,(345^\circ)$.
You can check this by raising each to the 6th power:

$(2\,cis\,45^\circ)^6 = \left(2e^{i45^\circ}\right)^6 = 2^6 e^{i45^\circ \cdot 6} = 64e^{i270^\circ} = -64i$,

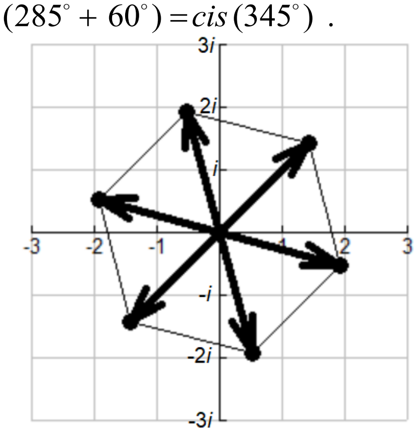
$(2\,cis\,105^\circ)^6 = 2^6 e^{i105^\circ \cdot 6} = 64e^{i630^\circ} = 64e^{i(270+360^\circ)} = -64i$,

$(2\,cis\,165^\circ)^6 = 2^6 e^{i165^\circ \cdot 6} = 64e^{i990^\circ} = 64e^{i(270+2\cdot 360^\circ)} = -64i$,

$(2\,cis\,225^\circ)^6 = 2^6 e^{i225^\circ \cdot 6} = 64e^{i1350^\circ} = 64e^{i(270+3\cdot 360^\circ)} = -64i$,

$(2\,cis\,285^\circ)^6 = 2^6 e^{i285^\circ \cdot 6} = 64e^{i1710^\circ} = 64e^{i(270+4\cdot 360^\circ)} = -64i$, and

$(2\,cis\,345^\circ)^6 = 2^6 e^{i345^\circ \cdot 6} = 64e^{i2070^\circ} = 64e^{i(270+5\cdot 360^\circ)} = -64i$.



Had you decided to use $\theta = -90^\circ$, the rectangular form $a + bi$ and plots would be the same, but the exponential form would be $(64e^{-90^\circ i})^{1/6} = 2e^{-15^\circ i}$ and then

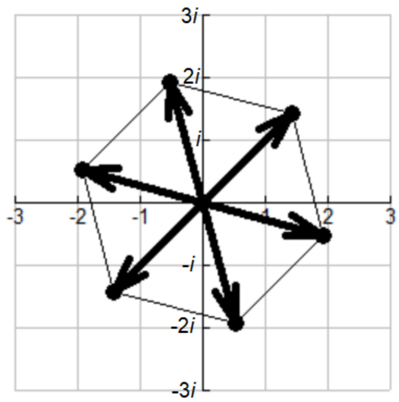
$2e^{-75^\circ i}$,

$2e^{-135^\circ i}$,

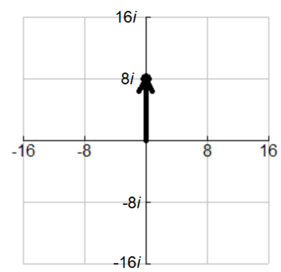
$2e^{-195^\circ i}$,

$2e^{-255^\circ i}$, and

$2e^{-315^\circ i}$.



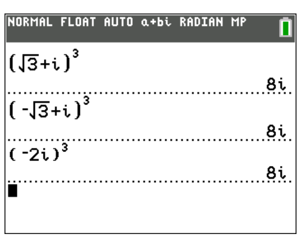
7. Report all the third roots of the number $8i$ and sketch them on the complex plane. Report your answers in polar form $r\text{cis}\theta$ in radians, exponential form $re^{i\theta}$ in radians, and in rectangular form $a + bi$.



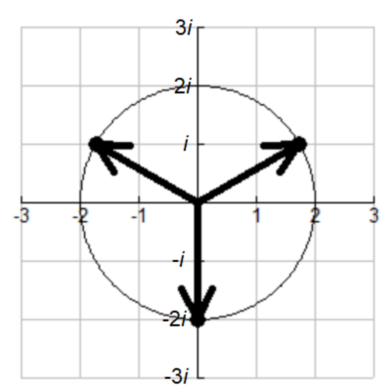
Step 1: Represent $8i$ in the form $re^{i\theta} = r\text{cis}\theta$. Since $re^{(i\theta)} = r\text{cis}\theta = r(\cos\theta + i\sin\theta)$, we have $r = 8$ and $\theta = \frac{\pi}{2}$. We could write $84i = 8\,cis\,\frac{\pi}{2} = 8e^{\pi i/2}$.
Alternatively, you could also use $\theta = -\frac{3\pi}{2}$.

Step 2: To find a third root, we raise the complex number to the one third power, i.e., $(8e^{\pi i/2})^{1/3} = 2e^{\pi i/6}$ is one of these. But there are three of these roots.

Step 3: The three roots are evenly spaced $\dfrac{2\pi}{3} = \dfrac{4\pi}{6}$ apart, or 120°, $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$
so we have $2\,cis\,\frac{\pi}{6}$, $2\,cis\,\frac{5\pi}{6}$, and $2\,cis\,\frac{3\pi}{2}$, or
we could write $2e^{(i\pi/6)}$, $2e^{(i5\pi/6)}$, and $2e^{(i3\pi/2)}$, or,
$2\cdot\frac{\sqrt{3}}{2} + 2\cdot\frac{1}{2}i = \sqrt{3} + i$,
$2\cdot(-\frac{\sqrt{3}}{2}) + 2\cdot\frac{1}{2}i = -\sqrt{3} + i$, and
$-2i$.
You can check this by raising each to the 3rd power: $(2e^{(i\pi/6)})^3 = 8e^{(i\pi/2)} = 8i$,
$(2e^{(i5\pi/6)})^3 = 8e^{(i5\pi/2)} = 8i$,
$(2e^{(i3\pi/2)})^3 = 8e^{(i9\pi/2)} = 8i$.



Had you decided to use $\theta = -\frac{3\pi}{2}$,
the rectangular form $a + bi$ and plots would be the same,
but the exponential form would be
$(8e^{-3\pi i/2})^{1/3} = 2e^{-\pi i/2}$ and then $2e^{-7\pi i/6}$ and $2e^{-11\pi i/6}$.



8. Consider the complex geometric series $f(z) = \sum_{k=0}^{\infty} 50z^k = 50 + 50z + 50z^2 + 50z^3 + \ldots$ which converges on $|z| < 1$. Report the value of $f\left(\frac{3i}{4}\right) = \sum_{k=0}^{\infty} 50\left(\frac{3i}{4}\right)^k$.

a. We separate even powers of $\frac{3i}{4}$ and odd powers of $\frac{3i}{4}$.

$$f\left(\frac{3i}{4}\right) = \sum_{k=0}^{\infty} 50\left(\frac{3i}{4}\right)^k = 50\left(1 + \left(\frac{3i}{4}\right)^2 + \left(\frac{3i}{4}\right)^4 + \left(\frac{3i}{4}\right)^6 + \ldots\right) + 50\left(\left(\frac{3i}{4}\right)^1 + \left(\frac{3i}{4}\right)^3 + \left(\frac{3i}{4}\right)^5 + \left(\frac{3i}{4}\right)^7 + \ldots\right)$$

First simplify powers of ***i***.
Then combine real parts in the first row and imaginary parts in the second row.
Then factor out 50 in the first row and $50 \cdot \frac{3i}{4}$ in the second row. Enter **real** numbers in each box.

You can write the real numbers as powers of $\frac{3}{4}$.

$$f\left(\frac{3i}{4}\right) = 50\left(1 + \boxed{-\left(\frac{3}{4}\right)^2} + \boxed{\left(\frac{3}{4}\right)^4} + \boxed{-\left(\frac{3}{4}\right)^6} + \ldots\right) + 50 \cdot \frac{3i}{4}\left(1 + \boxed{-\left(\frac{3}{4}\right)^2} + \boxed{\left(\frac{3}{4}\right)^4} + \boxed{-\left(\frac{3}{4}\right)^6} + \ldots\right)$$

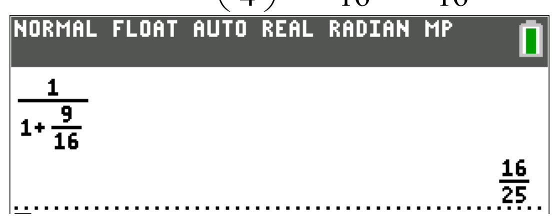
b. The geometric series $1 + \left(\frac{3i}{4}\right)^2 + \left(\frac{3i}{4}\right)^4 + \left(\frac{3i}{4}\right)^6 + \ldots$ has $a = 1$ and $r = \boxed{-\frac{9}{16}}$ and sum equal to $\boxed{\frac{16}{25}}$.

The geometric series here has $a = 1$ and $r = \boxed{-\frac{9}{16}}$ and sum equal to $\boxed{\frac{16}{25}}$.

Here's why: the series $1 - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^4 - \left(\frac{3}{4}\right)^6 + \ldots$ has the first term 1 and ratio $r = \left(\frac{3i}{4}\right)^2 = \frac{9i^2}{16} = -\frac{9}{16}$.

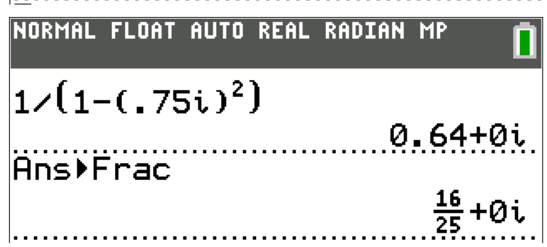
This sum of the series is $\frac{a}{1-r} = \frac{1}{1-(-\frac{9}{16})} = \frac{1}{1+\frac{9}{16}} = \frac{16}{25}$.

You can use the calculator and Frac it or use the stacked fraction:



You can also use the calculator with a complex $r = \left(\frac{3i}{4}\right)^2$

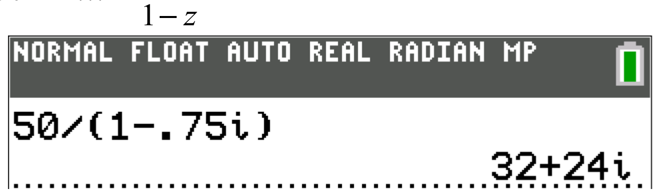
Notice this works even in Real mode.



c. Combining, we have $f\left(\frac{3i}{4}\right) = \sum_{k=0}^{\infty} 50\left(\frac{3i}{4}\right)^k = \boxed{32} + \boxed{24}\,i$ (Insert integers in the boxes.)

Method 1: $f(z) = \sum_{k=0}^{\infty} 50z^k = 50 + 50z + 50z^2 + 50z^3 + \ldots = \frac{50}{1-z}$

so $f\left(\frac{3i}{4}\right) = \sum_{k=0}^{\infty} 50\left(\frac{3i}{4}\right)^k = \frac{50}{1-\frac{3i}{4}}$



Method 2: From part **b**, we have $1-\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^4-\left(\frac{3}{4}\right)^6+\ldots=\frac{16}{25}$

$$f\left(\frac{3i}{4}\right)=50\left(1-\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^4-\left(\frac{3}{4}\right)^6+\ldots\right)+50\cdot\frac{3i}{4}\left(1-\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^4-\left(\frac{3}{4}\right)^6+\ldots\right)$$

$$=50\cdot\left(\frac{16}{25}\right)+50\cdot\frac{3i}{4}\cdot\left(\frac{16}{25}\right)$$

$$=2\cdot 16+\frac{50}{25}\cdot\frac{16}{4}\cdot\frac{3i}{1}$$

+

$$=32+2\cdot 4\cdot 3i$$

$$=32+24i$$

TIP: More problems like Question 8 are in HW 26 Complex Numbers Part 1 and also in HW 26 Complex Numbers Part 1 (Just for Practice. No Grade Will be Recorded.)