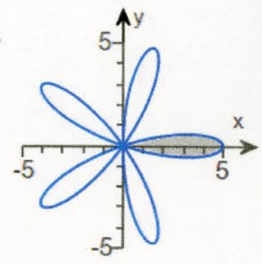
MA 16600 Practice Questions over 12.3 and 12.4

1. Recall the area from $\theta = \alpha$ to $\theta = \beta$ inside a polar graph is $\int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$

Find the exact area of the region inside one leaf of the 5-leaved rose $r = 5\cos 5\theta$. You can use the FNINT command, but provide an exact area.



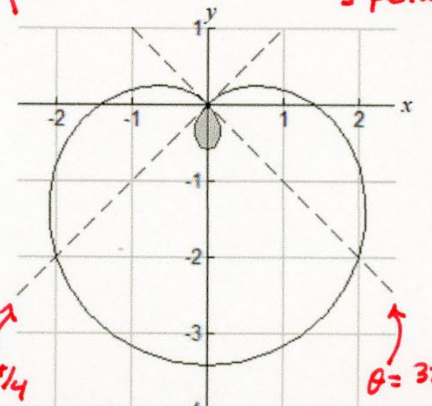
$$\int_{\boxed{-\pi/10}}^{\boxed{\pi/10}} \boxed{\frac{1}{2}(5\cos 5\theta)^2} \, d\theta = \boxed{\frac{5}{4}\pi} \quad \text{or} \quad \boxed{1.25\pi}$$

The complete curve is traced from $\theta = 0$ to $\theta = \pi$ so we could also find

$\frac{1}{5}\int_0^\pi \frac{1}{2}(5\cos 5\theta)^2 \, d\theta = 1.25\pi$ or $\frac{5}{4}\pi$. (We divide by 5 since there are 5 petals.)

2. Set up the integral to calculate the area of the region inside the inner loop of the limaçon $r = \sqrt{2} - 2\sin\theta$. Use the FNINT command to find the area and approximate it the area to two decimal places.

To find the integration limits, find where $r = \sqrt{2} - 2\sin\theta = 0$ where $0 \le \theta < 2\pi$, since this will be where the inner loop starts and ends.
TIP: The dashed lines in the above graph are the polar equations $\theta = \alpha$ and $\theta = \beta$, where α and β are the lower and upper limits of integration. You can enter these values in your polar grapher as θmin and θmax to check that you have sketched only the inner loop.



$\theta = \pi/4$ $\qquad$ $\theta = 3\pi/4$

$$\int_{\boxed{\pi/4}}^{\boxed{3\pi/4}} \boxed{\frac{1}{2}(\sqrt{2} - 2\sin\theta)^2} \, d\theta \approx \boxed{0.14}$$

The grid suggests the lines have slopes ± 1.
Solve the equation $r = \sqrt{2} - 2\sin\theta = 0$
$2\sin\theta = \sqrt{2}$
$\sin\theta = \sqrt{2}/2$
$\theta = \pi/4, 3\pi/4$

b. (+1 Bonus) What is the exact area, in terms of π? Show work for credit.

$$\frac{1}{2}\int_{\pi/4}^{3\pi/4} (2 - 4\sqrt{2}\sin\theta + 4\sin^2\theta)\,d\theta = \int_{\pi/4}^{3\pi/4} (1 - 2\sqrt{2}\sin\theta + 2\sin^2\theta)\,d\theta$$

$$= \int_{\pi/4}^{3\pi/4} d\theta - 2\sqrt{2}\int_{\pi/4}^{3\pi/4} \sin\theta\,d\theta + 2\int_{\pi/4}^{3\pi/4} \frac{1}{2}(1 - \cos 2\theta)\,d\theta = \theta\Big|_{\pi/4}^{3\pi/4} + 2\sqrt{2}\cos\theta\Big|_{\pi/4}^{3\pi/4} + \int_{\pi/4}^{3\pi/4} (1 - \cos 2\theta)\,d\theta$$

$$= \left(\frac{3\pi}{4} - \frac{\pi}{4}\right) + 2\sqrt{2}\left(\cos\frac{3\pi}{4} - \cos\frac{\pi}{4}\right) + \int_{\pi/4}^{3\pi/4} d\theta - \frac{1}{2}\int_{\pi/4}^{3\pi/4} \cos 2\theta \; 2d\theta$$

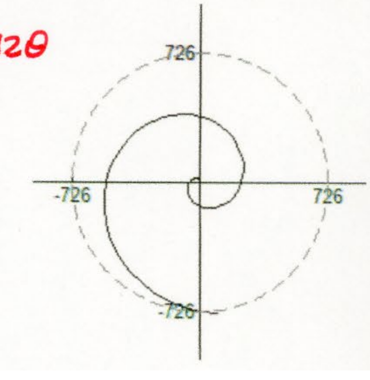
$$= \left(\pi/2\right) + 2\sqrt{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) + \left(\pi/2\right) - \frac{1}{2}\sin 2\theta\Big|_{\pi/4}^{3\pi/4}$$

$$= \pi + \underbrace{2\sqrt{2}(-\sqrt{2})}_{-4} - \frac{1}{2}\left(\underbrace{\sin\frac{3\pi}{2}}_{-1} - \underbrace{\sin\frac{\pi}{2}}_{1}\right) = \pi - 4 - \frac{1}{2}(-1-1) = \pi - 4 + 1 = \boxed{\pi - 3}$$

3. The arc length from $\theta = 0$ to $\theta = 11$ of a polar spiral $r = 6\theta^2$ is given by $\int_0^{11} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$. Calculate the arc length correct to the nearest whole number.

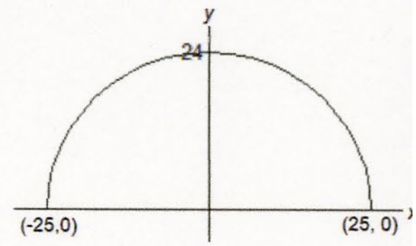
$\frac{dr}{d\theta} = 12\theta$

You can use the FNINT command. Round to the nearest whole number.



$$\int_0^{11} \sqrt{\boxed{(6\theta^2)^2 + (12\theta)^2}} \, d\theta \approx \boxed{2779}$$

4. The ceiling of a building is an ellipse with the dimensions shown (dimension feet). A person standing at c units from the y-axis that marks the center of the ellipse is able to hear the whispers of those standing the same distance on the other side of the y-axis. Assume both the person listening and those who are speaking are in the same vertical plane.



a. Write the formula of an ellipse with RUN = 25 and RISE = 24.
Your formula should be for a full ellipse, not the semi-ellipse shown.

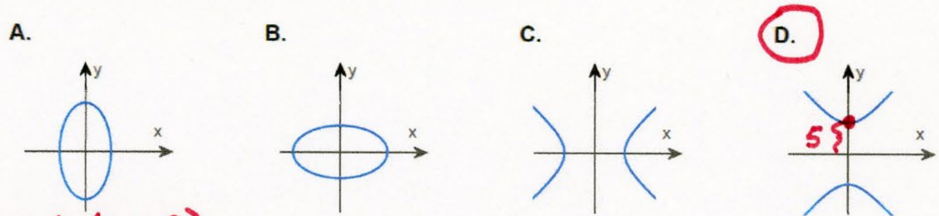
$$\frac{x^2}{25^2} + \frac{y^2}{24^2} = 1$$

b. What is c? Report a positive value. $c^2 = 25^2 - 24^2 = 49$ so $c = \boxed{7}$

5. Consider the conic section. $\dfrac{y^2}{25} - \dfrac{x^2}{36} = 1$. RUN = 6 RISE = 5

a. Select which of these looks most like the graph?

This a hyperbola with vertices on the vertical axis of symmetry.

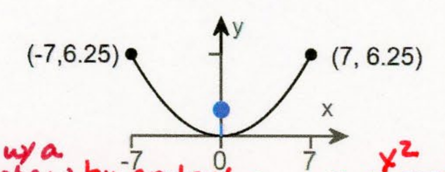


b. Report the vertices. (0, 5) and (0, −5)

Report the focal points as **exact** values: $(0, \pm\sqrt{61})$ $c^2 = 25 + 36 = 61$ so $c = \sqrt{61}$

If the conic section is a hyperbola, report the asymptotes. Otherwise leave blank. $y = \pm\frac{5}{6}x$ slope = $\frac{\text{RISE}}{\text{RUN}}$

6. A satellite dish is in the shape of a parabolic surface. Signals coming from a satellite strike the surface of the dish and are reflected to the focus, where the receiver is located. The satellite dish has a diameter of 14 feet and a depth of 6.25 feet.



a. Report the equation of the parabola. $7.84y = x^2$ ← check w/ a grapher by entering $y = \frac{x^2}{7.84}$

$4py = x^2 \Rightarrow 4p(6.25) = 7^2$

$25p = 49 \Rightarrow p = 1.96 \Rightarrow 4p = 7.84$

b. How far from the base of the dish should the receiver be placed?

1.96 ft

7. The vertices of a hyperbola centered at the origin are at the points (6,0) and (−6,0).
Its asymptotes are $y = \pm\frac{1}{2}x$. Which of these is its equation?

RUN = 6

A. $\dfrac{x^2}{4} - \dfrac{y^2}{2} = 1$ B. $\dfrac{x^2}{4} - \dfrac{y^2}{1} = 1$ C. $\dfrac{x^2}{2} - \dfrac{y^2}{1} = 1$ D. $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$ E. $\dfrac{x^2}{6} - \dfrac{y^2}{3} = 1$

F. $\dfrac{x^2}{36} - \dfrac{y^2}{18} = 1$ G. None of these

$\frac{1}{2} = \frac{\text{RISE}}{\text{RUN}}$ and RUN = 6

so $\frac{1}{2} = \frac{\text{RISE}}{6} \Rightarrow \text{RISE} = 3$

$$\frac{x^2}{\text{RUN}^2} - \frac{y^2}{\text{RISE}^2} = 1$$

$$\frac{x^2}{36} - \frac{y^2}{9} = 1$$