KEY Practice Questions from Section 10.6-10.8

1. Suppose $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \frac{7}{6} + \ldots$

a. What is a_n? $a_n = \boxed{\frac{n+1}{n}}$

b. The series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ will ______________________________.
{converge absolutely, converge conditionally, diverge }

c. Give a reason for your claim in part **b**. Since $\lim_{n\to\infty}\frac{n+1}{n} = 1$ then $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{n+1}{n}$ diverges by the AST.

Alternatively, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{n+1}{n}$ has $\lim_{n\to\infty}(-1)^{n+1}\cdot\frac{n+1}{n} \neq 0$. (It diverges by oscillation to 1 and −1.)

Thus $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{n+1}{n}$ diverges by the nth Term Test for Divergence.

2. Each alternating series below converges by the Alternating Series Test (AST).
Determine if the convergence is conditional or absolute.

a. $\sum_{n=1}^{\infty}\frac{(-1)^n 7n}{4n^3-3}$ will converge __________________ because
{ absolutely, conditionally }

the series $\sum_{n=1}^{\infty}\frac{7n}{4n^3-3}$ will ____________ by the ________________________ with $b_n = \boxed{\frac{n}{n^3} = \frac{1}{n^2}}$.
{converge, diverge} {Comparison Test, Limit Comparison Test}

Provide the details of your claim below. Use $a_n = \frac{7n}{4n^3-3}$ and $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{7n}{4n^3-3}\cdot\frac{n^2}{} = \frac{7}{4}$$

Thus $\sum_{n=1}^{\infty}\frac{7n}{4n^3-3}$ converges by the LCT. Since $\sum_{n=1}^{\infty}\frac{7n}{4n^3-3}$ converges, $\sum_{n=1}^{\infty}\frac{(-1)^n 7n}{4n^3-3}$ converges absolutely.

b. $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = \frac{1}{15} - \frac{1}{20} + \frac{1}{25} - \frac{1}{30} + \frac{1}{35} - \frac{1}{40} + \ldots$ will converge __________________ because
{ absolutely, conditionally }

the series $\sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}\boxed{\frac{1}{5n+10}}$ will ____________ by the ________________________ with $b_n = \boxed{\frac{1}{n}}$.
{converge, diverge} {Comparison Test, Limit Comparison Test}

Provide the details of your claim below. Use $a_n = \frac{1}{5n+10}$ and $b_n = \frac{1}{n}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{5n+10}\cdot\frac{n}{} = \frac{1}{5}$$

Thus $\sum_{n=1}^{\infty}\frac{1}{5n+10}$ diverges by the LCT, $\sum_{n=1}^{\infty}\frac{(-1)^n}{5n+1}$ converges conditionally.

(Since $\lim_{n\to\infty}\frac{1}{5n+10} = 0$ and $a_n = \frac{1}{5n+10}$ decreases, $\sum_{n=1}^{\infty}\frac{(-1)^n}{5n+1}$ converges by the AST.

3. Report the two conditions for an alternating series $\sum_{n=1}^{\infty}b_n = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$ to converge, where a_n is positive for all n.

i. **$a_n \to 0$ as $n \to \infty$ (or $\lim_{n\to\infty} a_n = 0$), i.e. the nth Term Test for Divergence must not disqualify it.**

ii. **a_n is non-increasing**

4. Give an example of any kind of divergent alternating series. Hint: Think about your answer to Question 3.

$\sum_{k=0}^{\infty}(-1)^k = \sum_{k=0}^{\infty}(-1)^k \cdot 1$ is one example. $\sum_{k=0}^{\infty}(-1)^k \cdot 2$ is another. $\sum_{k=0}^{\infty}(-2)^k = \sum_{k=0}^{\infty}(-1)^k \cdot 2^k$ is another.

$\sum_{k=0}^{\infty}(-1)^k \cdot k$ is another. All of these examples are of the form $\sum_{k=0}^{\infty}(-1)^k \cdot a_k$ where $\lim_{n\to\infty} a_k \neq 0$ and, because of the ***n*th Term Test for Divergence**, each of these series must diverge.

5. Give an example of a divergent alternating series with the property that its *n*th term approaches 0.
There are many correct answers. Hint: think of your answer to Question **3**.
You may write it in long form (expanded form) or use sigma notation, but use correct notation.

It is easier to write this in long form: $1 - \frac{1}{10} + \frac{1}{2} - \frac{1}{100} + \frac{1}{3} - \frac{1}{1000} + \frac{1}{4} - \frac{1}{10^4} + \frac{1}{5} - \frac{1}{10^5} + \ldots$

If we wanted to use sigma notation (which is unnecessary and more challenging), we could write

$$\sum_{k=1}^{\infty} b_k, \text{ where } b_k = \begin{cases} \frac{1}{k} & , k = 1, 3, 5, 7, \ldots \\ \frac{-1}{10^{k/2}} = \frac{-1}{\sqrt{10^k}} = (-1)\cdot\left(\frac{1}{\sqrt{10}}\right)^k & , k = 2, 4, 6, 8, \ldots \end{cases}$$

The series diverges because it is the sum of a harmonic series $\sum_{k=1}^{\infty}\frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = \infty$ and a geometric series $-\frac{1}{10} - \frac{1}{100} - \frac{1}{1000} - \frac{1}{10^4} - \frac{1}{10^5} - \ldots$ with first term $a = -\frac{1}{10}$ and ratio $r = \frac{1}{10}$ which would converge to the sum $\frac{-1/10}{1 - 1/10} = \frac{-1/10}{9/10} = -\frac{1}{9}$. Therefore, $\sum_{k=1}^{\infty} b_k = \infty - \frac{1}{9} = \infty$.

$\sum (-1)^n a_n$

4. Give an example of any kind of divergent alternating series. Hint: Think about your answer to Question 3. $\sum (-1)^n \cdot 1$

Also use a_n for any expression where $\lim a_n \neq 0$.

5. Give an example of an alternating series with the property that its nth term approaches 0 but it still diverges. There are many correct answers. Hint: Think about your answer to Question 3. You may write it in long form (expanded form) or use sigma notation, but use correct notation. $1 - \frac{1}{10} + \frac{1}{2} - \frac{1}{100} + \frac{1}{3} - \frac{1}{1000} + \frac{1}{4} - \frac{1}{10^4} + \dots$ (Geo + Harmonic)

6. The Ratio Test and Root Test are based on the properties of convergence of
A. a p-series, $p \neq 1$. B. the harmonic series C. the alternating series D. a television series E. the world series F. a geometric series

7. Which of these will help you determine if the series $\sum_{n=0}^{\infty} 2e^n$ converges or diverges? Select all possible answers.

A. limit comparison test with a p-series, $p \neq 1$. B. limit comparison test with the harmonic series C. a geometric series $r = e$
D. alternating series test E. absolute convergence test (i.e., convergence of $\sum |a_n|$ implies convergence of $\sum a_n$)
E. integral test F. ratio test G. nth Term Test for Divergence

You can use root test too since the series $\sum 2e^n = 2\sum e^n$

8. Which of these will help you determine if the series $\sum_{n=0}^{\infty} e^{-2n}$ converges or diverges? Select all possible answers.

A. limit comparison test with a p-series, $p \neq 1$. B. limit comparison test with the harmonic series C. a geometric series $r = e^{-2}$
D. alternating series test E. absolute convergence test (i.e., convergence of $\sum |a_n|$ implies convergence of $\sum a_n$)
E. integral test F. ratio test G. nth Term Test for Divergence

you can use root test too

9. Which of these will help you determine if the series $\sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n^2} \right)$ converges or diverges? Select all possible answers.

it converges absolutely

A. limit comparison test with a p-series, $p \neq 1$. B. limit comparison test with the harmonic series C. a geometric series
D. alternating series test E. absolute convergence test (i.e., convergence of $\sum |a_n|$ implies convergence of $\sum a_n$)
E. ratio test F. nth Term Test for Divergence

10. Which of these will help you determine if the series $\sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{\sqrt{n}} \right)$ converges or diverges? Select all possible answers.

it converges conditionally

A. limit comparison test with a p-series, $p \neq 1$. B. limit comparison test with the harmonic series C. a geometric series
D. alternating series test E. absolute convergence test (i.e., convergence of $\sum |a_n|$ implies convergence of $\sum a_n$)
E. ratio test F. nth Term Test for Divergence

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+3}{(n+1)!} \cdot \frac{n!}{n+2} = \lim_{n\to\infty} \left(\frac{\cancel{n!}\,(n+3)}{\cancel{n!}\,(n+1)(n+2)} \right) = 0 < 1$

11. Which of these will help you determine if the series $\sum_{n=1}^{\infty} \left(\frac{n+2}{n!} \right)$ converges or diverges? Select all possible answers.

A. limit comparison test with a p-series, $p \neq 1$. B. limit comparison test with the harmonic series C. a geometric series
D. alternating series test E. absolute convergence test (i.e., convergence of $\sum |a_n|$ implies convergence of $\sum a_n$)
E. integral test F. ratio test G. nth Term Test for Divergence

12. Use the Ratio Test for each.

a. The series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$ will ______ {converge, diverge} by the Ratio Test because $\lim_{n\to\infty} \boxed{\frac{2}{n+1}} = \boxed{0}$

$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2^n \cdot 2 n!}{(n+1) n! \, 2^n}$

Write in the box a **simplified expression** involving n.

Write in the box an **exact** number or DNE or ∞ or $-\infty$.

b. The series $\sum_{n=1}^{\infty} \frac{4^n}{n^{800}}$ will ______ {converge, diverge} by the Ratio Test because $\lim_{n\to\infty} \boxed{\frac{4n^{800}}{(n+1)^{800}}} = \boxed{4}$. $4 > 1$

$\left| \frac{a_{n+1}}{a_n} \right| = \frac{4^{n+1}}{(n+1)^{800}} \cdot \frac{n^{800}}{4^n} = \frac{4 \cdot 4^n \cdot n^{800}}{4^n (n+1)^{800}}$

Write in the box a **simplified expression** involving n.

Write in the box an **exact** number or DNE or ∞ or $-\infty$.

8. Use the Root Test for each.

a. The series $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{8n^4}{7n^4+n+5}\right)^n$ will ________ {converge, diverge} by the Root Test because $\lim_{n\to\infty}$ $\boxed{\frac{8n^4}{7n^4+n+5}}$ = $\boxed{\frac{8}{7}}$

$\frac{8}{7}>1$

$\lim_{n\to\infty}|a_n|^{\frac{1}{n}}=\lim_{n\to\infty}\left(\frac{8n^4}{7n^4+n+5}\right)^{n/n}=$

Write in the box a **simplified expression** involving n.

Write in the box an **exact** number or DNE or ∞ or $-\infty$.

b. The series $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{2n}{3n+2}\right)^n$ will ________ {converge, diverge} by the Root Test because $\lim_{n\to\infty}$ $\boxed{\frac{2n}{3n+2}}$ = $\boxed{\frac{2}{3}}$.

$\frac{2}{3}<1$

$\lim_{n\to\infty}|a_n|^{\frac{1}{n}}=\lim_{n\to\infty}\left(\frac{2n}{3n+2}\right)^{n/n}=$

Write in the box a **simplified expression** involving n.

Write in the box an **exact** number or DNE or ∞ or $-\infty$.

c. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n+1}{n}\right)^{n^2}$ will ________ {converge, diverge} by the Root Test because $\lim_{n\to\infty}$ $\boxed{\left(1+\frac{1}{n}\right)^n}$ = $\boxed{e}$.

$e>1$

$\lim_{n\to\infty}|a_n|^{\frac{1}{n}}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{n^2/n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Write in the box a **simplified expression** involving n.

Write in the box an **exact** number or DNE or ∞ or $-\infty$.

9. Consider the series $\sum_{n=1}^{\infty}\left(1+\frac{a}{n}\right)^{18n}$ for some real number a.

a. The series will ________ {converge, diverge}.

b. Circle the best answer to determine part **a**.
A. It is a p-series. B. It is a geometric series C. Use the Ratio Test D. Use the Root Test
E. Use the nth Term Test for Divergence

c. Explain more fully below how part **b** justifies part **a**. $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{18n}=\lim_{n\to\infty}\left(\left(1+\frac{a}{n}\right)^n\right)^{18}=e^{a\cdot 18}\neq 0$

10. Consider the series $\sum_{n=1}^{\infty}(-2)^n$

a. The series will ________ {converge, diverge}.

b. Which of these will help you determine if the series $\sum_{n=1}^{\infty}(-2)^n$ converges or diverges? Select all possible answers.

A. It is a p-series. B. It is a geometric series C. Use the Ratio Test D. Use the Root Test
E. Use the nth Term Test for Divergence

c. Explain more fully below how part **b** justifies part **a** for each of your choices.

n^{th} term test: $\lim_{n\to\infty}(-2)^n=\text{DNE}\neq 0$

root test: $\lim_{n\to\infty}\left|(-2)^{\cancel{n}}\right|^{\frac{1}{\cancel{n}}}=2>1$

ratio test: $\lim_{n\to\infty}\left|\frac{(-2)^{n+1}}{(-2)^n}\right|=\lim_{n\to\infty}\left|\frac{\cancel{2^n}\cdot 2}{\cancel{2^n}}\right|=2>1$

geometric series: $r=-2>-1$

11. Consider the series $\sum_{n=1}^{\infty}n(-0.5)^n$

a. The series will ________ {converge, diverge}.

b. Justify your claim in part a.

ratio test: $\lim_{n\to\infty}\left|\frac{(n+1)(-.5)^{n+1}}{n(-.5)^n}\right|=\lim_{n\to\infty}\left|(.5)\cdot\frac{n+1}{n}\right|=0.5<1$

Note: $\lim_{n\to\infty}\frac{n+1}{n}=1$