Practice Questions from 11.1-11.4

1. Answer the following for the power series $\sum c_n(x-a)^n$. Complete the blanks.

   a. The power series $\sum c_n(x-a)^n$ is centered at the value $x =$ <u>a</u>.

   b. Suppose the interval of convergence is **all real numbers**. Then the radius of convergence is $R =$ <u>∞</u>.

   c. Suppose the interval of convergence is only **the value $x = a$**. Then the radius of convergence is $R =$ <u>0</u>.

   d. Suppose the interval of convergence is $|x - a| < b$, i.e. $a - b < x < a + b$.

   Then the radius of convergence is $R =$ <u>b</u>.

2. The interval of convergence of $\sum_{n=1}^{\infty}\left(\frac{x-4}{2}\right)^n$ is $\boxed{2} < x < \boxed{6}$. Show work below.

   Hint: It is a geometric series.

$$-1 < \frac{x-4}{2} < 1$$

$$-2 < x-4 < 2$$

$$4-2 < x < 2+4$$

$$2 < x < 6$$

3. Suppose we have the following

   $f(34) = 60$,

   $f'(34) = 43$,

   $f''(34) = 22$, and

   $f'''(34) = 30$.

   a. Write the third-order Taylor polynomial approximation to f at $x = 34$ in expanded form. Simplify please.

$$t(x) = \frac{f(34)}{0!} + \frac{f'(34)}{1!}(x-34) + \frac{f''(34)}{2!}(x-34)^2 + \frac{f'''(34)}{3!}(x-34)^3$$

$$= 60 + 43(x-34) + \frac{22}{2!}(x-34)^2 + \frac{30}{3!}(x-34)^3$$

$$= 60 + 43(x-34) + 11(x-34)^2 + 5(x-34)^3$$

   b. **True or False:** The polynomial $t(x)$ is the tangent cubic to f at the value $x = 34$. ________

   True!

4 . Report the interval of convergence of $\sum_{n=0}^{\infty} n!x^{5n}$. Select one. Use ratio test to find which values of x make it converge.

A. $-1<x<1$ B. $-\frac{1}{\sqrt{5}}<x<\frac{1}{\sqrt{5}}$ C. $-\sqrt[5]{5}<x<\sqrt[5]{5}$ D. $x=0$ E. $-\frac{1}{5}<x<\frac{1}{5}$ F. $-\infty<x<\infty$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{5(n+1)}}{n!\;x^{5n}}\right| = \lim_{n\to\infty}\left(\frac{\cancel{n!}(n+1)}{\cancel{n!}}\cdot\left|\frac{\cancel{x^{5n}}x^5}{\cancel{x^{5n}}}\right|\right) = \lim_{n\to\infty}(n+1)\cdot|x^5| = \infty > 1$$ for all x.

5. The interval of convergence of $\sum_{n=1}^{\infty}\frac{x^{3n}}{n!}$ is $\boxed{-\infty}<x<\boxed{\infty}$. Show work below.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{3n+3}}{(n+1)!}\cdot\frac{n!}{x^{3n}}\right| = \lim_{n\to\infty}\left(\frac{\cancel{n!}}{\cancel{n!}(n+1)}\left|\frac{\cancel{x^{3n}}x^3}{\cancel{x^{3n}}}\right|\right) = \lim_{n\to\infty}\frac{1}{n+1}\cdot|x^3| = 0 < 1$$ for all x.

6. Consider $\sum_{n=1}^{\infty}\frac{(3x)^n}{n}$

a. The radius of convergence is $R =$ 1/3. Show work below.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(n+1)}\cdot\frac{n}{(3x)^n}\right| = \lim_{n\to\infty}\frac{n}{n+1}\cdot|3x| = \lim_{n\to\infty}|3x| < 1$$ for $-1<3x<1$, $-\frac{1}{3}<x<\frac{1}{3}$ before checking endpoints

b. If x is equal to the **left endpoint** of the interval of convergence, the series $\sum_{n=1}^{\infty}\frac{(3x)^n}{n}$ will ______. {converge, diverge}

If $x=-\frac{1}{3}$, we have $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

c. If x is equal to the **right endpoint** of the interval of convergence, the series $\sum_{n=1}^{\infty}\frac{(3x)^n}{n}$ will ______. {converge, diverge}

If $x=\frac{1}{3}$, we have $\sum_{n=1}^{\infty}\frac{1}{n}$

d. State the **reasons** which justify your claims about the endpoints in parts **b** and **c**.

part b: Alternating Harmonic Series (or AST)
part c: Harmonic Series

So the interval of convergence is $-\frac{1}{3}\le x<\frac{1}{3}$

7. Consider $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{x^{n+11}}{n^2}\right)$

a. The radius of convergence is $R =$ 1. Show work below.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+12}}{(n+1)^2}\cdot\frac{n^2}{x^{n+11}}\right| = \lim_{n\to\infty}\left(\frac{n^2}{(n+1)^2}\left|\frac{\cancel{x^n}x^{12}}{\cancel{x^n}\cdot x^{11}}\right|\right)$$

$$= \lim_{n\to\infty}\frac{n^2}{(n+1)^2}\cdot\lim_{n\to\infty}|x| = 1\cdot\lim_{n\to\infty}|x| < 1 \text{ if } |x|<1 \text{ or } -1<x<1$$ before checking endpoints

b. If x is equal to the **left endpoint** of the interval of convergence, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{x^{n+11}}{n^2}\right)$ will ______. {converge, diverge}

If $x=-1$, we have $\sum_{n=1}^{\infty}(-1)^n(-1)^1\cdot\left(\frac{(-1)^n(-1)^{11}}{n^2}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2}$

c. If x is equal to the **right endpoint** of the interval of convergence, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{x^{n+11}}{n^2}\right)$ will ______. {converge, diverge}

If $x=1$, we have $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n^2}$

d. State the **reasons** which justify your claims about the endpoints in parts **b** and **c**.

part b: p-series $p=2$
part c: A.S.T

So the interval of convergence is $-1\le x\le 1$

Fun Facts:

> For all x we have $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
>
> For $-1 < x < 1$ we have $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$
>
> For $-1 < x \le 1$ we have $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ For $-1 \le x \le 1$ we have $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

8. Complete: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \boxed{\ln 2}$. The name of this series is called the Alternating Harmonic series.

TIP: Use a Fun Fact above.

Write in the box an **exact** number or DNE or ∞ or $-\infty$.

Be specific please.

9. Complete: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots = \boxed{\infty}$. The name of this series is called the Harmonic series.

Write in the box an **exact** number or DNE or ∞ or $-\infty$.

Be specific please.

10. a. In sigma notation the series $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots = \sum_{\boxed{n=1}}^{\infty} \boxed{\frac{-x^n}{n}}$

b. Use one of the Fun Facts above to determine what function $f(x)$ the series $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$ approximates.

$f(x) = \boxed{\ln(1-x)}$. The radius of convergence is $R = \boxed{1}$

Simplified please.

c. Write the first four terms of the series in expanded form if $x = -1$. $\boxed{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}} + \ldots$

The left endpoint $x = -1$ ______ in the interval of convergence. Explain your answer.
{is, is not}

Reason: Question 8. It is Alt. Harm. Series

Simplified please.

d. Write the first four terms of the series in expanded form if $x = 1$. $\boxed{-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4}} - \ldots$

The right endpoint $x = 1$ ______ in the interval of convergence. Explain your answer.
{is, is not}

Reason: Question 9. It is the opposite of the harmonic series.

11. Consider function $f(x) = 10\tan^{-1}(2w)$. Write the first **four** terms of the series.

Simplified please.

$10\tan^{-1}(2w) = \boxed{20w - \frac{80w^3}{3} + \frac{320w^5}{5} - \frac{1280w^7}{7}} + \ldots$

use $\tan^{-1}(\square) = \square - \frac{\square^3}{3} + \frac{\square^5}{5} - \frac{\square^7}{7}$ ($\square = 2w$)
so
$\tan^{-1}(2w) = 2w - \frac{8w^3}{3} + \frac{32w^5}{5} - \frac{128w^7}{7}$
and now multiply each term by 10

12. Consider function $f(w) = \sin(w^2)$. Write the first **four** terms of the series.

Simplified please.

$\sin(w^2) = \boxed{w^2 - \frac{w^6}{3!} + \frac{w^{10}}{5!} - \frac{w^{14}}{7!}} + \ldots$

13. Consider function $f(w) = e^{-w}$. Write the first **four** terms of the series.

Simplified please.

$e^{-w} = \boxed{1 - w + \frac{w^2}{2!} - \frac{w^3}{3!}} + \ldots$

use $e^{\square} = 1 + (\boxed{-w}) + \frac{(\boxed{-w})^2}{2!} + \frac{(\boxed{-w})^3}{3!} + \cdots$ ($\square = -w$)

14. The term-by-term derivative of $f(x) = \sum_{n=0}^{\infty} 5x^n = 5 + 5x + 5x^2 + 5x^3 + 5x^4 + \cdots$ is the power series below.

a. Write the first four nonzero terms of the series for $f'(x)$.

Simplified please

$f'(x) = \boxed{5 + 10x + 15x^2 + 20x^3} + \ldots$

b. The radius of convergence of $f'(x)$ is $R =$ 1 This is a geometric series with $r = x$ and $a = 5$ so it converges if $|x| < 1$

c. If x is equal to the **left endpoint** of the interval of convergence, the series for $f'(x)$ will ________.
{converge, diverge}

$x = -1 \Rightarrow 5 - 10 + 15 + 20 - \cdots$ diverges by oscillation

d. If x is equal to the **right endpoint** of the interval of convergence, the series for $f'(x)$ will ________.
{converge, diverge}

$x = 1 \Rightarrow 5 + 10 + 15 + 20 + \cdots = \infty$

on $(-1, 1)$

e. Write the series for $f'(x)$ in sigma notation.

$f'(x) = \sum_{n=\boxed{1}}^{\infty} \left(\boxed{5nx^{n-1}}\right)$

Take the derivative with respect to x of $5x^n$ to get $5nx^{n-1}$

f. When x is in the interval of convergence, we can write the series for $f'(x)$ as what rational function?

$f'(x) = \dfrac{\boxed{5}}{\boxed{(1-x)^2}}$

$f(x) = 5 + 5x + 5x^2 + 5x^3 + \cdots = \dfrac{5}{1-x}$

By geometric series

Differentiate the right hand side

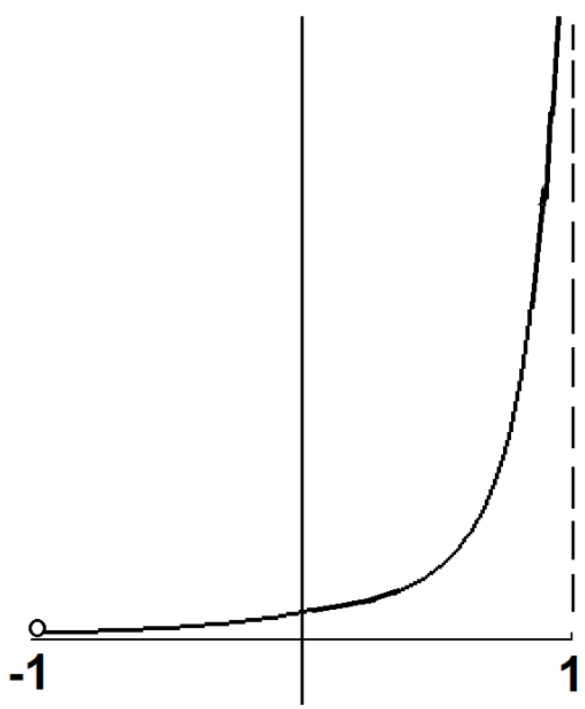
$$\frac{d}{dx} \cdot \frac{5}{(1-x)} = \frac{d}{dx} \cdot 5(1-x)^{-1}$$

$$= 5 \frac{d}{dx}(1-x)^{-1}$$

$$= 5 \cdot -(1-x)^{-2} \cdot \frac{d}{dx}(-x)$$

$$= 5 \cdot (-1) \cdot \frac{1}{(1-x)^2} \cdot (-1)$$

$$= \frac{5}{(1-x)^2}$$



Hole at $x = -1$

Vertical Asymptote at $x = 1$.