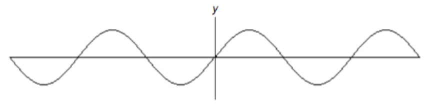
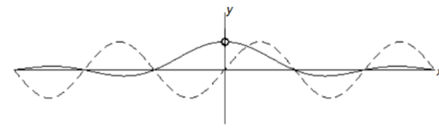
(+4) **How Euler Astonished the World by Showing $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$ at the Age of 28**

1. The functions $y = \sin x$ and $y = \frac{\sin x}{x}$ share infinitely many zeros. List those they share shown in the graph.

$x = \pm\square, \pm\square, \pm\square, \ldots$





2. Write the first six terms of the Taylor polynomial for sine. Use factorial notation.

$\sin x = \square - \square + \square - \square + \square - \square + \ldots.$

3. Euler divided each term of the above Taylor polynomial by x. Complete the boxes to see what he found.

$\frac{\sin x}{x} = \square - \square + \square - \square + \square - \square + \ldots.$

4. Euler used what you wrote in #1 to write the infinite degree polynomial in #3 in *factored* form as a product. Complete the boxes with ***positive*** real numbers. Combine each of the pairs of factors of the same color. Use $(A-B)(A+B) = A^2 - B^2$.

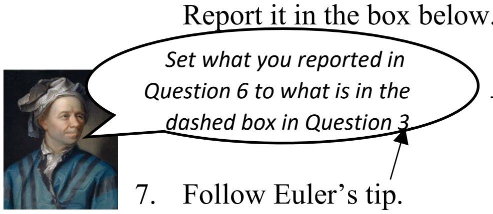
$\frac{\sin x}{x} = \left(1-\frac{x}{\square}\right)\left(1+\frac{x}{\square}\right)\left(1-\frac{x}{\square}\right)\left(1+\frac{x}{\square}\right)\left(1-\frac{x}{\square}\right)\left(1+\frac{x}{\square}\right)\cdots$

$= \left(1-\frac{x^2}{\square}\right)\left(1-\frac{x^2}{\square}\right)\left(1-\frac{x^2}{\square}\right)\cdots$

5. Euler expanded the infinite product you reported in #4 and collected like terms to compare it with #3. TIP: Notice the pattern of the coefficients of the x^2 terms for several partial products of this form.
$(1-A^2x^2)(1-B^2x^2) = 1 - \boldsymbol{(A^2+B^2)}x^2 + A^2B^2x^4.$
$(1-A^2x^2)(1-B^2x^2)(1-C^2x^2) = 1 - \boldsymbol{(A^2+B^2+C^2)}x^2 + (A^2B^2+A^2C^2+B^2C^2)x^4 - (A^2B^2C^2)x^6.$
$(1-A^2x^2)(1-B^2x^2)(1-C^2x^2)(1-D^2x^2) = 1 - \boldsymbol{(A^2+B^2+C^2+D^2)}x^2$
$+ (A^2B^2 + A^2C^2 + A^2D^2 + B^2C^2 + B^2D^2 + C^2D^2)x^4$
$- (A^2B^2C^2 + A^2B^2D^2 + A^2C^2D^2 + B^2C^2D^2)x^6$
$+ (A^2B^2C^2D^2)x^8.$
For the infinite product in Question 4, $\frac{\sin x}{x} = (1-A^2x^2)(1-B^2x^2)(1-C^2x^2)(1-D^2x^2)\cdots$, we would have

$A = \square$, $B = \square$, $C = \square$, $D = \square$, etc.

6. Euler noticed some golden treasure with the ***coefficient of the x^2 term*** of the *expanded* form of $\frac{\sin x}{x}$. Report it in the box below. (It involves π and is itself ***an infinite*** series.)



$\frac{\sin x}{x} = 1 - \square x^2 + \ldots = 1 - \square + \ldots$

7. Follow Euler's tip.

8. We have spent much of the course exploring infinite *sums*. In Question 5 we have an infinite *product.* There are similarities.

a. It has been convenient to use the capital Greek letter sigma (equivalent to our letter *S*) for a sum. Write the expanded form of $\frac{\sin x}{x}$ using Sigma notation.

$$\frac{\sin x}{x} = \frac{x^0}{1!} - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \ldots. = \sum_{k=\square}^{\infty} \square$$

b. It is also convenient use the capital Greek letter Pi (equivalent to our letter *P*) for a product. Write the factored form of $\frac{\sin x}{x}$ using Pi notation.

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)\cdots = \prod_{k=\square}^{\square} \square$$

c. Use Pi notation to report the leading coefficient of the *expanded* form of $\frac{\sin x}{x}$ for the *n*th partial product.

$$\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)\cdots\left(1 - \frac{x^2}{n^2\pi^2}\right)$$

$$= 1 - \sum_{k=1}^{n} \frac{x^2}{(k\pi)^2} + \ldots + \prod_{k=\square}^{\square} \square$$