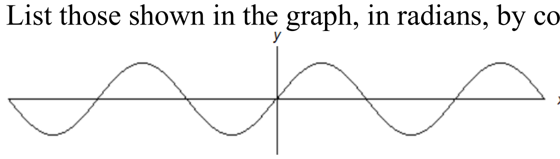
How Euler Astonished the World by Showing $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ at the Age of 28

1. The sine function has infinitely many zeros. List those shown in the graph, in radians, by completing the boxes.

   $x = 0,\ \pm\boxed{},\ \pm\boxed{},\ \pm\boxed{},\ \ldots$



2. Write the first six terms of the Taylor polynomial for sine. Use factorial notation.

   $$\sin x = \boxed{} - \boxed{} + \boxed{} - \boxed{} + \boxed{} - \boxed{} + \ \ldots.$$

3. Euler divided each term of the above Taylor polynomial by x. Complete the boxes to see what he found.

   $$\frac{\sin x}{x} = \boxed{} - \boxed{} + \boxed{} - \boxed{} + \boxed{} - \boxed{} + \ \ldots.$$

4. Euler wrote the infinite degree polynomial in Question 3 in *factored* form as a product. Complete the boxes with ***positive*** real numbers. Combine each of the pairs of factors of the same color, using $(A-B)(A+B) = A^2 - B^2$.

$$\frac{\sin x}{x} = \underbrace{\left(1 - \frac{x}{\boxed{}}\right)\left(1 + \frac{x}{\boxed{}}\right)}\underbrace{\left(1 - \frac{x}{\boxed{}}\right)\left(1 + \frac{x}{\boxed{}}\right)}\underbrace{\left(1 - \frac{x}{\boxed{}}\right)\left(1 + \frac{x}{\boxed{}}\right)}\cdots$$

$$= \left(1 - \frac{x^2}{\boxed{}}\right)\left(1 - \frac{x^2}{\boxed{}}\right)\left(1 - \frac{x^2}{\boxed{}}\right)\cdots$$

5. Euler expanded the infinite product you reported in Question 4 and collected like terms.
   TIP: Notice the pattern of the coefficients of the x^2 terms for several partial products of this form.

   $(1-A^2x^2)(1-B^2x^2) = 1 - (\mathbf{A^2 + B^2})x^2 + A^2B^2x^4$.

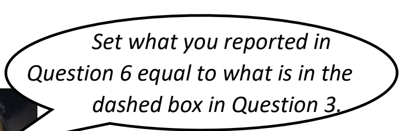
   $(1-A^2x^2)(1-B^2x^2)(1-C^2x^2) = 1 - (\mathbf{A^2 + B^2 + C^2})x^2 + (A^2B^2 + A^2C^2 + B^2C^2)x^4 - (A^2B^2C^2)x^6$.

   $(1-A^2x^2)(1-B^2x^2)(1-C^2x^2)(1-D^2x^2) = 1 - (\mathbf{A^2 + B^2 + C^2 + D^2})x^2$
   $+ (A^2B^2 + A^2C^2 + A^2D^2 + B^2C^2 + B^2D^2 + C^2D^2)x^4$
   $- (A^2B^2C^2 + A^2B^2D^2 + A^2C^2D^2 + B^2C^2D^2)x^6$
   $+ (A^2B^2C^2D^2)x^8$.

   For the infinite product in Question 4, $\frac{\sin x}{x} = (1-A^2x^2)(1-B^2x^2)(1-C^2x^2)(1-D^2x^2)\ldots$ we have

   $A = \boxed{}$, $B = \boxed{}$, $C = \boxed{}$, $D = \boxed{}$, etc.

6. Report the coefficient of the x^2 term of the *expanded* form of $\frac{\sin x}{x}$. (It involves π.)



$$\frac{\sin x}{x} = 1 - \boxed{}\,x^2 + \ldots = 1 - \boxed{} + \ldots$$

7. Follow Euler's tip.

Rhino Participation Bonus Due Monday, November 14

(+0.25) **a.** It has been convenient to use the capital Greek letter sigma (equivalent to our letter *S*) for a sum. Write the expanded form of $\frac{\sin x}{x}$ using sigma notation.

$$\frac{\sin x}{x} = \frac{x^0}{1!} - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \ldots. = \sum_{k=\boxed{\quad}}^{\infty} \boxed{\qquad\qquad}$$

(+0.25) **b.** It is also convenient use the capital Greek letter pi (equivalent to our letter *P*) for a product. Write the factored form of $\frac{\sin x}{x}$ using pi notation.

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)\cdots = \prod_{k=\boxed{\quad}}^{\boxed{\quad}} \boxed{\qquad\qquad}$$

(+0.5) **c.** Use pi notation to report the leading coefficient of the *expanded* form of $\frac{\sin x}{x}$ for the *n*th partial product.

$$\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)\cdots\left(1 - \frac{x^2}{n^2\pi^2}\right)$$

$$= 1 - \sum_{k=1}^{n} \frac{x^2}{(k\pi)^2} + \ldots + \prod_{k=\boxed{\quad}}^{\boxed{\quad}} \boxed{\qquad\qquad\qquad}$$