MA 16600 Extra Credit
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How Euler Astonished the World with the Amazing Fact $e^{iz} = \cos z + i \sin z$ and Other Amazing Facts On the Journey to It

Once upon a time (1748) there was a mathematician named Euler, who at the age of 35 used infinite series to find some amazing facts. The goal of this activity is to show how he arrived at the Amazing Fact in the title above, but more beautiful gems will be shown along the way. In particular, …

Amazing fact #1

The infinite series $\dfrac{1}{0!} + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \cdots = e = 2.71828\,1828\,459\,045\ldots$

The series $\dfrac{1}{0!} + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots$ converges quickly. We can write a sequence of partial sums $S_1, S_2, S_3, \ldots$

$S_1 = 1$

$S_2 = 1 + 1 = 2$

$$S_3 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} = \frac{5}{2} = 2.5$$

$$S_4 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = \frac{8}{3} = 2.\overline{6}$$

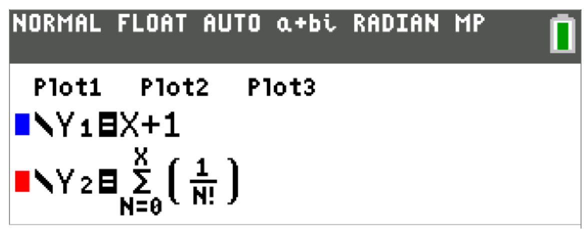
$$S_5 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = \frac{65}{24} = 2.708\overline{3}$$

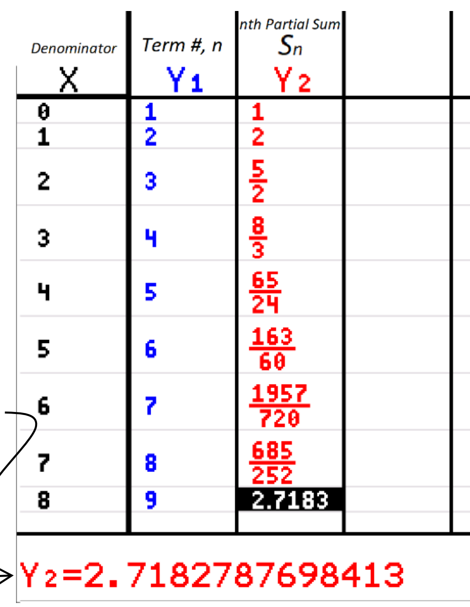
$$S_6 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = \frac{163}{60} = 2.71\overline{6}$$

$$S_7 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = \frac{1957}{720} = 2.7180\overline{5}$$

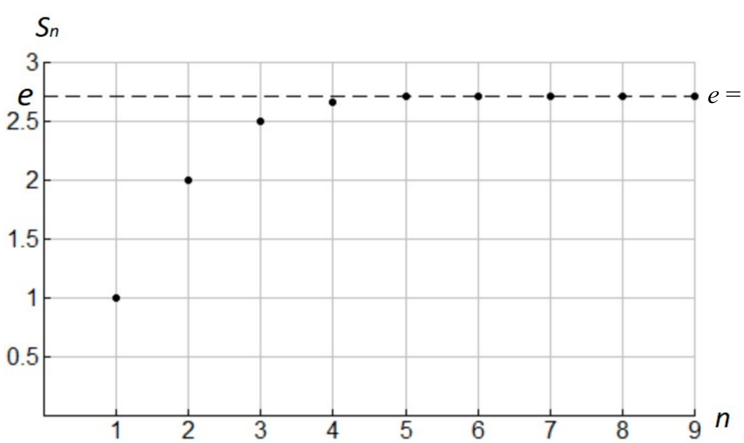
$$S_8 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} = \frac{685}{252} \approx 2.71825$$

$$S_9 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} = \frac{109601}{40320} \approx 2.7182787698413$$





(i) **1.** For what value of n does S_n first exceed 2.71828 1828 4? $n =$ ☐



$e = 2.71828182845904523536028747135266249775724709369995957496696762772407663035354759457138217852 5\ldots$

Amazing fact #2

If w is any complex number, the infinite series $1 + \frac{w^1}{1!} + \frac{w^2}{2!} + \frac{w^3}{3!} + \frac{w^4}{4!} + \cdots = e^w$

Notice this equation is true for $w = 0$ since $e^0 = 1$ and $1 + 0 + 0 + 0 + \ldots = 1$.
Notice this holds for $w = 1$ by **Amazing Fact #1**.

Amazing Fact #3

(i) **2.** Suppose $w = -1$ in **Amazing Fact #2**. Write the first <u>thirteen</u> terms of the infinite series expansion of e raised to the power of -1. Any powers of -1, i, or $-i$ should be simplified. Check you only report thirteen terms.

$$e^{-1} = \boxed{}$$

Amazing Fact #4

(i) **3.** If $w = i$ in **Amazing Fact #2**,

then $e^i = \frac{1}{0!} + \frac{i^1}{1!} + \frac{i^2}{2!} + \frac{i^3}{3!} + \frac{i^4}{4!} + \frac{i^5}{5!} + \frac{i^6}{6!} + \frac{i^7}{7!} + \frac{i^8}{8!} + \frac{i^9}{9!} + \frac{i^{10}}{10!} + \frac{i^{11}}{11!} + \frac{i^{12}}{12!} + \cdots\cdots$

We can simplify this just a little using powers of i, where $i^2 = -1$, $i^3 = -i$, etc.
Rewrite the expansion with simplified numerators of $\pm i$ or ± 1, by carefully completing the boxes below.

$$e^i = \frac{1}{0!} + \frac{i}{1!} - \frac{1}{2!} - \frac{i}{3!} + \frac{1}{4!} + \frac{\square}{5!} + \frac{\square}{6!} + \frac{\square}{7!} + \frac{\square}{8!} + \frac{\square}{9!} + \frac{\square}{10!} + \frac{\square}{11!} + \frac{\square}{12!} + \cdots\cdots$$

(i) **4.** Suppose $w = iz$ in **Amazing Fact #2**, where z is any complex number.
Write the first <u>thirteen</u> terms of the infinite series expansion of e raised to the power of iz.
Any powers of -1, i, and $-i$ should be simplified. Double check you only report thirteen terms.

$$e^{iz} = \boxed{}$$

Amazing fact #5

If w is any complex number in radians, the infinite series

$$\frac{1}{0!} - \frac{w^2}{2!} + \frac{w^4}{4!} - \frac{w^6}{6!} + \frac{w^8}{8!} - \frac{w^{10}}{10!} + \frac{w^{12}}{12!} - \frac{w^{14}}{14!} + \cdots = \cos w$$

Notice this is true for $w = 0$ since $\cos 0 = 1 - 0 + 0 - 0 + \ldots = 1$

Amazing fact #6

If w is any complex number in radians, the infinite series

$$\frac{w^1}{1!} - \frac{w^3}{3!} + \frac{w^5}{5!} - \frac{w^7}{7!} + \frac{w^9}{9!} - \frac{w^{11}}{11!} + \frac{w^{13}}{13!} - \frac{w^{15}}{15!} + \cdots = \sin w$$

Notice this is true for $w = 0$ since $\sin 0 = 0 - 0 + 0 - 0 + \ldots = 0$

Amazing facts #7 and #8

($2i$) **5.** Complete the boxes with simplified numerators of $\pm i$ or ± 1 to find the value of $\sin i$ and $\cos i$.
Any powers of -1, i, and $-i$ should be simplified.

$$\cos i = \frac{\square}{0!} + \frac{\square}{2!} + \frac{\square}{4!} + \frac{\square}{6!} + \frac{\square}{8!} + \frac{\square}{10!} + \cdots\cdots$$

$$\sin i = \frac{\square}{1!} + \frac{\square}{3!} + \frac{\square}{5!} + \frac{\square}{7!} + \frac{\square}{9!} + \frac{\square}{11!} + \cdots\cdot = i\left(\frac{\square}{1!} + \frac{\square}{3!} + \frac{\square}{5!} + \frac{\square}{7!} + \frac{\square}{9!} + \frac{\square}{11!} + \cdots\right)$$

Amazing fact #9

(i) **6.** Find the value of $\cos i + i\sin i$ and write it as a power of e.
Complete the boxes with simplified numerators of $\pm i$ or ± 1.
Any powers of -1, i, and $-i$ should be simplified.
Use **Amazing Facts #3, #7**, and **#8**.

$$\cos i + i\sin i = \frac{\square}{0!} + \frac{\square}{2!} + \frac{\square}{4!} + \frac{\square}{6!} + \frac{\square}{8!} + \frac{\square}{10!} + \cdots$$

$$+ \frac{\square}{1!} + \frac{\square}{3!} + \frac{\square}{5!} + \frac{\square}{7!} + \frac{\square}{9!} + \frac{\square}{11!} + \cdots$$

$$= \frac{\square}{0!} + \frac{\square}{1!} + \frac{\square}{2!} + \frac{\square}{3!} + \frac{\square}{4!} + \frac{\square}{5!} + \frac{\square}{6!} + \frac{\square}{7!} + \frac{\square}{8!} + \frac{\square}{9!} + \frac{\square}{10!} + \cdots = e^{\square}$$

(i) **7.** From **Amazing Fact #2**, we found the value of e raised to the power of iz, where z is any complex number.
Now write the sum in two rows of terms, separating even and odd powers.
Complete the boxes with simplified numerators of $\pm i$ or ± 1:
Any powers of -1, i, and $-i$ should be simplified.

$$e^{iz} = 1 \quad - \frac{z^2}{2!} \quad + \frac{z^4}{4!} \quad + \frac{\square \cdot z^6}{6!} \quad + \frac{\square \cdot z^8}{8!} \quad + \frac{\square \cdot z^{10}}{10!} \quad + \cdots$$

$$+ \frac{iz}{1!} \quad - \frac{iz^3}{3!} \quad + \frac{\square \cdot z^5}{5!} \quad + \frac{\square \cdot z^7}{7!} \quad + \frac{\square \cdot z^9}{9!} \quad + \frac{\square \cdot z^{11}}{11!}$$

TIP: Check that if you substitute $z = -i$ you get the series for $e^{i\cdot(-i)} = e^1$, or **Amazing Fact #1,**
if you substitute $z = i$ you get the series for $e^{i\cdot(i)} = e^{-1}$, or **Amazing Fact #3,** and
if you substitute $z = 1$ you get the series for $e^{i(1)}$, or **Amazing Fact #4.**

(.5i) **8.** What function does the *first* row of terms of even powers in Question **7** equal? Your yellow sheet may help.
Hint: Look back at your previous amazing facts. This is a function that involves z.
(Do not express this function in sigma notation.)

$$1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \boxed{}$$

(.5i) **9.** What function does the *second* row of terms of odd powers in Question **7** equal? Your yellow sheet may help.
Hint: Look back at your previous amazing facts. This is a function that involves z.
(Do not express this function in sigma notation.)

$$\frac{iz}{1!} - \frac{iz^3}{3!} + \frac{\square \cdot z^5}{5!} + \cdots = \boxed{}$$

Amazing Fact #10

(i) **10.** Use your answer to the previous two questions to write e^{iz} in terms of the trigonometric functions.
Hint: Recall the goal of this activity stated on the first page. (Do not express this function in sigma notation.)

$$1 + \frac{iz}{1!} - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \cdots\cdots = e^{iz} = \boxed{}$$