MA 16600 Extra Credit Euler's Identity $e^{iz} = \cos z + i \sin z$

Due the date of your final exam.

(+10i pts) These bonus points will be added to your Final Exam score.

Name ________________________________

Section 9:00 10:00/10:30

Once upon a time (1748) there was a mathematician named Euler, who used the above definition and infinite series to find some amazing facts. We have seen how the sum of an infinite decreasing series can approach (or converge) to a specific value.

Amazing fact #1

The infinite series $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e$

See for yourself by completing the boxes.

(.5i) 1.a. $\frac{1}{0!} + \frac{1}{1!} = \boxed{}$

b. $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} = \boxed{}$ (Report as a decimal)

c. $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = \boxed{}$ (Report to five decimal places)

d. $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = \boxed{}$ (Report to five decimal places)

If we keep adding terms, we get closer and closer to $e \approx 2.718281828182845904523536028747135266\,2$

Amazing fact #2

If w is any complex number, the infinite series $1 + \frac{w^1}{1!} + \frac{w^2}{2!} + \frac{w^3}{3!} + \frac{w^4}{4!} + \cdots = e^w$

Notice this equation is true for $w = 0$ since $e^0 = 1$ and $1 + 0 + 0 + 0 + \ldots = 1$.

Notice this holds for $w = 1$ by **Amazing Fact #1**.

(.5i) 2. If $w = i$ in **Amazing Fact #2**,

then $e^i = \frac{1}{0!} + \frac{i^1}{1!} + \frac{i^2}{2!} + \frac{i^3}{3!} + \frac{i^4}{4!} + \frac{i^5}{5!} + \frac{i^6}{6!} + \frac{i^7}{7!} + \frac{i^8}{8!} + \frac{i^9}{9!} + \frac{i^{10}}{10!} + \frac{i^{11}}{11!} + \frac{i^{12}}{12!} + \cdots\cdots$

We can simplify this just a little using powers of i, where $i^2 = -1$, $i^3 = -i$, etc.

Rewrite the expansion with simplified numerators of $\pm i$ or ± 1, by carefully completing the boxes below.

$e^i = \frac{1}{0!} + \frac{i}{1!} - \frac{1}{2!} - \frac{i}{3!} + \frac{1}{4!} + \frac{\boxed{}}{5!} + \frac{\boxed{}}{6!} + \frac{\boxed{}}{7!} + \frac{\boxed{}}{8!} + \frac{\boxed{}}{9!} + \frac{\boxed{}}{10!} + \frac{\boxed{}}{11!} + \frac{\boxed{}}{12!} + \cdots\cdots$

(i) 3. Suppose $w = iz$ in **Amazing Fact #2**, where z is any complex number.
Write the first <u>thirteen</u> terms of the infinite series expansion of e raised to the power of iz.
Powers of i should be simplified. Double check you only report thirteen terms.

$e^{iz} = \boxed{}$

(OVER PLEASE)

Amazing fact #3

If w is any complex number in radians, the infinite series

$$\frac{w^1}{1!} - \frac{w^3}{3!} + \frac{w^5}{5!} - \frac{w^7}{7!} + \frac{w^9}{9!} - \frac{w^{11}}{11!} + \frac{w^{13}}{13!} - \frac{w^{15}}{15!} + \cdots = \sin w$$

Notice this is true for $w = 0$ since $\sin 0 = 0 - 0 + 0 - 0 + \ldots = 0$

Amazing fact #4

If w is any complex number in radians, the infinite series

$$\frac{1}{0!} - \frac{w^2}{2!} + \frac{w^4}{4!} - \frac{w^6}{6!} + \frac{w^8}{8!} - \frac{w^{10}}{10!} + \frac{w^{12}}{12!} - \frac{w^{14}}{14!} + \cdots = \cos w$$

($3i$) 4. Complete the boxes to carefully write each of these using sigma notation.

$$e^w = \sum_{m=0}^{\infty} \square \qquad \sin w = \sum_{m=0}^{\infty} \square \qquad \cos w = \sum_{m=0}^{\infty} \square$$

Amazing facts #5 and 6

($2i$) 5. Complete the boxes with simplified numerators of $\pm i$ or ± 1 to find the value of $\sin i$ and $\cos i$. Powers of i should be simplified.

$$\sin i = \frac{\square}{1!} + \frac{\square}{3!} + \frac{\square}{5!} + \frac{\square}{7!} + \frac{\square}{9!} + \frac{\square}{11!} + \cdots\cdots$$

$$\cos i = \frac{1}{0!} + \frac{\square}{2!} + \frac{\square}{4!} + \frac{\square}{6!} + \frac{\square}{8!} + \frac{\square}{10!} + \cdots\cdots$$

Amazing fact #7

(i) 6. From **Amazing Fact #2**, we found the value of e raised to the power of iz, where z is any complex number. Now write the sum in two rows of terms, separating even and odd powers. Complete the boxes with simplified numerators of $\pm i$ or ± 1:

$$e^{iz} = 1 \quad - \frac{z^2}{2!} \quad + \frac{z^4}{4!} \quad + \frac{\square \cdot z^6}{6!} \quad + \frac{\square \cdot z^8}{8!} \quad + \frac{\square \cdot z^{10}}{10!}$$

$$+ \frac{iz}{1!} \quad - \frac{iz^3}{3!} \quad + \frac{\square \cdot z^5}{5!} \quad + \frac{\square \cdot z^7}{7!} \quad + \frac{\square \cdot z^9}{9!} \quad + \frac{\square \cdot z^{11}}{11!} \quad + \cdots\cdots$$

TIP: Check if you substitute z = 1 you get Amazing Fact #2

($.5i$) 7. What does the *first* row of terms of even powers equal? (Your expression involves z.)

($.5i$) 8. What does the *second* row of terms of odd powers equal? (Your expression involves z.)

(i) 9. Use your answer to the previous two questions to write e^{iz} in terms of the trig functions.