1. $$\frac{\cos^2\theta}{\sin\theta} + \frac{1}{\csc\theta} = \frac{\cos^2\theta}{\sin\theta} + \sin\theta$$

Write everything in terms of sine and cosine.
Sine and cosecant are reciprocals of each other: $\frac{1}{\csc\theta} = \sin\theta$

$$= \frac{\cos^2\theta}{\sin\theta} + \sin\theta\cdot\left(\frac{\sin\theta}{\sin\theta}\right)$$

To add fractions, create common denominators.

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta}$$

Combine numerators.

$$= \frac{1}{\sin\theta}$$

Use the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$.

$$= \csc\theta$$

By definition, $\frac{1}{\sin\theta} = \csc\theta$.

2. $$\frac{\sin^3\theta}{\cos\theta} + \sin\theta\cos\theta = \frac{\sin^3\theta}{\cos\theta} + \sin\theta\cos\theta\left(\frac{\cos\theta}{\cos\theta}\right)$$

To add fractions,
create common denominators.

$$= \frac{\sin^3\theta}{\cos\theta} + \frac{\sin\theta\cos^2\theta}{\cos\theta}$$

Multiply.

$$= \frac{\sin\theta\left(\sin^2\theta + \cos^2\theta\right)}{\cos\theta} \quad (\sin^2\theta + \cos^2\theta \to 1)$$

Factor out $\sin\theta$ from both terms.

$$= \frac{\sin\theta}{\cos\theta}$$

Use the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$.

$$= \tan\theta$$

By definition, $\frac{\sin\theta}{\cos\theta} = \tan\theta$.

3. $$\frac{1+\cot\theta}{1+\tan\theta} = \frac{\left(1+\frac{\cos\theta}{\sin\theta}\right)}{\left(1+\frac{\sin\theta}{\cos\theta}\right)}\cdot\left(\frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}\right)$$

Write everything in terms of sine and cosine.
Since we have $\sin\theta$ as well as $\cos\theta$ in the denominator,
we can clear denominators by multiplying by a form of 1,
where 1 = $\frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}$

$$= \frac{1\cdot\sin\theta\cos\theta + \frac{\cos\theta}{\sin\theta}\cdot\sin\theta\cos\theta}{1\cdot\sin\theta\cos\theta + \frac{\sin\theta}{\cos\theta}\cdot\sin\theta\cos\theta}$$

Distribute.

$$= \frac{1\cdot\sin\theta\cos\theta + \frac{\cos\theta}{\cancel{\sin\theta}}\cdot\cancel{\sin\theta}\cos\theta}{1\cdot\sin\theta\cos\theta + \frac{\sin\theta}{\cancel{\cos\theta}}\cdot\sin\theta\cancel{\cos\theta}}$$

Cancel where appropriate.

$$= \frac{\sin\theta\cos\theta + \cos\theta\cos\theta}{\sin\theta\cos\theta + \sin\theta\sin\theta}$$

Notice we can factor out $\cos\theta$ in the numerator
and we can factor out $\sin\theta$ in the denominator

$$= \frac{\cos\theta\left(\sin\theta + \cos\theta\right)}{\sin\theta\left(\cos\theta + \sin\theta\right)}$$

Cancel.

$$= \cot\theta$$

By definition, $\frac{\cos\theta}{\sin\theta} = \cot\theta$.

Alternatively: $\dfrac{1+\cot\theta}{1+\tan\theta} = \dfrac{\left(1+\frac{\cos\theta}{\sin\theta}\right)}{\left(1+\frac{\sin\theta}{\cos\theta}\right)} = \dfrac{\left(\frac{\sin\theta}{\sin\theta}+\frac{\cos\theta}{\sin\theta}\right)}{\left(\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\right)} = \dfrac{\left(\frac{\sin\theta+\cos\theta}{\sin\theta}\right)}{\left(\frac{\cos\theta+\sin\theta}{\cos\theta}\right)}$ Create common denominators and add fractions.

$$= \frac{\left(\frac{\sin\theta+\cos\theta}{\sin\theta}\right)}{1}\cdot\frac{1}{\left(\frac{\cos\theta+\sin\theta}{\cos\theta}\right)} = \left(\frac{\sin\theta+\cos\theta}{\sin\theta}\right)\cdot\left(\frac{\cos\theta}{\cos\theta+\sin\theta}\right)$$

$= \dfrac{\cancel{\sin\theta+\cos\theta}}{\sin\theta}\cdot\dfrac{\cos\theta}{\cancel{\cos\theta+\sin\theta}}$ Cancel.

$$= \frac{\cos\theta}{\sin\theta}$$

$= \cot\theta$ // By definition, $\frac{\cos\theta}{\sin\theta} = \cot\theta$.

4. $\sin\theta\sec\theta = \sin\theta\cdot\dfrac{1}{\cos\theta} = \tan\theta$ //

5. $\dfrac{\sec\theta}{\csc\theta} = \dfrac{\sec\theta}{1}\cdot\dfrac{1}{\csc\theta}$ Write as multiplication.

$= \dfrac{1}{\cos\theta}\cdot\sin\theta$ Write in terms of sine and cosine.

$= \dfrac{\sin\theta}{\cos\theta} = \tan\theta$ // Use fractional multiplication.

6. $\csc\theta\tan\theta = \dfrac{1}{\sin\theta}\cdot\dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\cos\theta} = \sec\theta$ //

7. $\tan\theta\sin\theta + \cos\theta = \dfrac{\sin\theta}{\cos\theta}\sin\theta + \cos\theta$ Write everything in terms of sine and cosine.

$$= \frac{\sin^2\theta}{\cos\theta} + \frac{\cos\theta\cos\theta}{\cos\theta}$$

$= \dfrac{\sin^2\theta+\cos^2\theta}{\cos\theta}$ Multiply.

$$= \frac{\cancel{\sin^2\theta+\cos^2\theta}^{\,1}}{\cos\theta}$$

$= \dfrac{1}{\cos\theta}$ Use the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$.

$= \sec\theta$ // By definition, $\frac{1}{\cos\theta} = \sec\theta$.

8. $\dfrac{\sin\theta}{1-\cos\theta} - \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{\sin\theta}{(1-\cos\theta)}\cdot\left(\dfrac{1+\cos\theta}{1+\cos\theta}\right) - \dfrac{\sin\theta}{(1+\cos\theta)}\cdot\left(\dfrac{1-\cos\theta}{1-\cos\theta}\right)$ To add fractions, create common denominators.

$= \dfrac{\sin\theta(1+\cos) - \sin\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$ Combine numerators.

$= \dfrac{\sin\theta(1+\cos\theta) - \sin\theta(1-\cos\theta)}{\sin^2\theta}$ Use $(1+\cos\theta)(1-\cos\theta) = 1-\cos^2\theta = \sin^2\theta$

$= \dfrac{\sin\theta+\sin\theta\cos\theta-\sin\theta+\sin\theta\cos\theta}{\sin\theta\sin\theta}$ Distribute. Write $\sin^2\theta = \sin\theta\sin\theta$

$= \dfrac{2\cancel{\sin\theta}\cos\theta}{\cancel{\sin\theta}\sin\theta} = \dfrac{2\cos\theta}{\sin\theta} = 2\cot\theta$ // Cancel By definition, $\frac{\cos\theta}{\sin\theta} = \cot\theta$.

9. $$\frac{1-\cos\theta}{\sec\theta-1} = \frac{(1-\cos\theta)}{\left(\frac{1}{\cos\theta}-1\right)}\cdot\frac{\cos\theta}{\cos\theta}$$

Write everything in terms of cosine. Since we have $\cos\theta$ in the denominator, we can clear denominators by multiplying by a form of 1, where 1 = $\frac{\cos\theta}{\cos\theta}$

$$= \frac{(1-\cos\theta)\cos\theta}{\left(\frac{1}{\cos\theta}\cos\theta-1\cdot\cos\theta\right)}$$

Distribute.

$$= \frac{(1-\cos\theta)\cdot\cos\theta}{(1-\cos\theta)}$$

Cancel.

$$= \cos\theta \;/\!/\!/$$

Alternatively: $$\frac{1-\cos\theta}{\sec\theta-1} = \frac{(1-\cos\theta)}{\left(\frac{1}{\cos\theta}-1\right)} = \frac{(1-\cos\theta)}{\left(\frac{1}{\cos\theta}-\frac{\cos\theta}{\cos\theta}\right)}$$

Create common denominators and add fractions.

$$= \frac{(1-\cos\theta)}{\left(\frac{1-\cos\theta}{\cos\theta}\right)}$$

$$= \frac{(1-\cos\theta)}{1}\cdot\frac{1}{\left(\frac{1-\cos\theta}{\cos\theta}\right)}$$

$$= \left(\frac{1-\cos\theta}{1}\right)\cdot\left(\frac{\cos\theta}{1-\cos\theta}\right)$$

Cancel.

$$= \cos\theta \;/\!/\!/$$

10. $$\sin\theta + \frac{\cot^2\theta}{\csc\theta} = \sin\theta + \frac{1}{\csc\theta}\cdot\cot^2\theta$$

$$= \sin\theta + \sin\theta\cdot\cot^2\theta$$

$$= \sin\theta + \sin\theta\cdot\left(\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= \sin\theta + \frac{\cos^2\theta}{\sin\theta}$$

$$= \sin\theta\cdot\left(\frac{\sin\theta}{\sin\theta}\right) + \frac{\cos^2\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta+\cos^2\theta}{\sin\theta}$$

$$= \frac{1}{\sin\theta}$$

$$= \csc\theta \;/\!/\!/$$

11. $$\frac{1}{1-\cos\theta}+\frac{1}{1+\cos\theta} = \frac{1}{(1-\cos\theta)}\cdot\left(\frac{1+\cos\theta}{1+\cos\theta}\right) + \frac{1}{(1+\cos\theta)}\cdot\left(\frac{1-\cos\theta}{1-\cos\theta}\right)$$

$$= \frac{(1+\cos\theta) + (1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{2}{\sin^2\theta}$$

$$= 2\csc^2\theta \;/\!/\!/$$