The three cases for end behavior of rational functions

Assume a and b are any constants, and m, n and p are positive integers.

Then as $x \to \pm\infty$, $f(x) = \dfrac{ax^m + \text{ remaining terms of lower degree}}{bx^n + \text{ remaining terms of lower degree}}$ has the same **end behavior** as $y = \dfrac{ax^m}{bx^n}$.

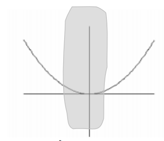
The *short run* behavior is covered up to emphasize that only the **end** behavior is being mirrored.

If the degree m of the numerator is **larger than** the degree n of the denominator, then

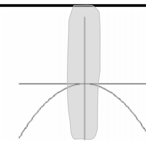
$$\text{as } x \to \pm\infty,\quad f(x) = \frac{ax^m + \text{remaining terms}}{bx^n + \text{remaining terms}} \to y = \frac{ax^m}{bx^n} \to y = \frac{ax^p}{b}$$

Assume $p = m - n$

$p = 2, 4, 6, \ldots$

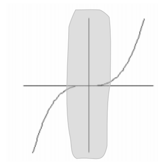


as $x \to \pm\infty$, $y \to \infty$

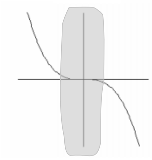


as $x \to \pm\infty$, $y \to -\infty$

$p = 3, 5, 7, \ldots$

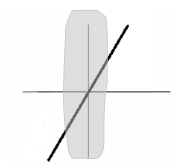


as $x \to -\infty$, $y \to -\infty$
as $x \to \infty$, $y \to \infty$

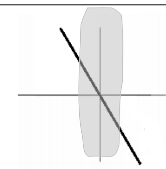


as $x \to -\infty$, $y \to \infty$
as $x \to \infty$, $y \to -\infty$

$p = 1$



as $x \to -\infty$, $y \to -\infty$
as $x \to \infty$, $y \to \infty$



as $x \to -\infty$, $y \to \infty$
as $x \to \infty$, $y \to -\infty$

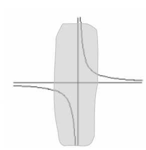
For this case $f(x)$ has **no** horizontal asymptote.

If the degree m of the numerator is **smaller than** the degree n of the denominator, then

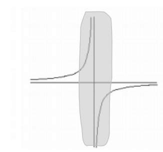
$$\text{as } x \to \pm\infty,\quad f(x) = \frac{ax^m + \text{remaining terms}}{bx^n + \text{remaining terms}} \to y = \frac{ax^m}{bx^n} \to y = \frac{a}{bx^p}$$

Assume $p = n - m$

$p = 2, 4, 6, 8, \ldots$

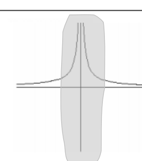


as $x \to \pm\infty$, $y \to 0$

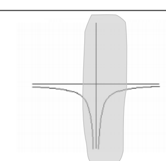


as $x \to \pm\infty$, $y \to 0$

$p = 1, 3, 5, 7, \ldots$



as $x \to \pm\infty$, $y \to 0$



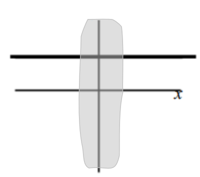
as $x \to \pm\infty$, $y \to 0$

For this case $f(x)$ has a horizontal asymptote. It is the line $y = 0$.

If the degree m of the numerator is **equal to** the degree n of the denominator, then

$$\text{as } x \to \pm\infty,\quad f(x) = \frac{ax^m + \text{remaining terms}}{bx^n + \text{remaining terms}} \to y = \frac{ax^m}{bx^n} \to \frac{ax^p}{bx^p} = \frac{a}{b}$$

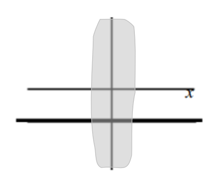
Assume $p = m = n$



$\frac{a}{b}$

as $x \to \pm\infty$, $y \to \frac{a}{b}$

$\frac{a}{b} > 0$



$\frac{a}{b}$

as $x \to \pm\infty$, $y \to \frac{a}{b}$

$\frac{a}{b} < 0$

For this case $f(x)$ has a horizontal asymptote. It is the line $y = \dfrac{a}{b}$.