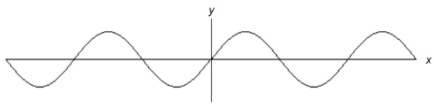
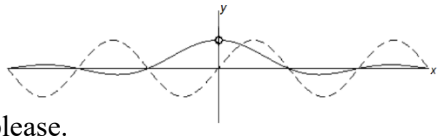
Thanks, Slick Lenny, for the Result $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$

Name _____
(+3 Rhino Participation Bonus Points)

(+2) **1. a.** The functions $y = \sin x$ and $y = \frac{\sin x}{x}$ share infinitely many zeros. List those they share shown in the graph.

$x = \pm\square, \pm\square, \pm\square, \ldots$





b. Write the first six terms of the Taylor polynomial for sine. Use factorial notation please.

$$\sin x = \square\cdot x^{\square} - \square\cdot x^{\square} + \square\cdot x^{\square} - \square\cdot x^{\square} + \square\cdot x^{\square} - \square\cdot x^{\square} + \ldots$$

c. Something cool happens next.



Divide each term by x. Assume x ≠ 0 and everything will be chill.

$$\frac{\sin x}{x} = \square\cdot x^{\square} - \square\, x^{\square} + \square\cdot x^{\square} - \square\cdot x^{\square} + \square\cdot x^{\square} - \square\cdot x^{\square} + \ldots$$

d. Lenny used what you wrote in **1a** to write the infinite degree polynomial in **1c** in *factored* form as a product. Complete the boxes with ***positive*** real numbers. Combine each of the pairs of factors of the same color.
Use $(A-B)(A+B) = A^2 - B^2$.

$$\frac{\sin x}{x} = \left(1-\frac{x}{\square}\right)\left(1+\frac{x}{\square}\right)\left(1-\frac{x}{\square}\right)\left(1+\frac{x}{\square}\right)\left(1-\frac{x}{\square}\right)\left(1+\frac{x}{\square}\right)\cdots$$



Slick trick!

$$= \left(1-\frac{x^2}{\square^2}\right)\left(1-\frac{x^2}{\square^2}\right)\left(1-\frac{x^2}{\square^2}\right)\cdots$$

e. Lenny expanded the infinite product you reported above and collected like terms to compare it with **1c**.
TIP: Notice the pattern of the coefficients of the x^2 terms for several partial products of this form.
$(1-A^2x^2)(1-B^2x^2) = 1 - (A^2+B^2)x^2 + (A^2B^2)x^4$.
$(1-A^2x^2)(1-B^2x^2)(1-C^2x^2) = 1 - (A^2+B^2+C^2)x^2 + (A^2B^2 + A^2C^2 + B^2C^2)x^4 - (A^2B^2C^2)x^6$. In general,
From **1d**,

$$\frac{\sin x}{x} = \left(1-\square^2x^2\right)\left(1-\square^2x^2\right)\left(1-\square^2x^2\cdots\right)\cdots = 1 - \left(\square^2 + \square^2 + \square^2 + \cdots\right)x^2 + \cdots$$

f. Euler noticed some golden treasure with the ***coefficient of the x^2 term*** of the *expanded* form of $\frac{\sin x}{x}$.
Report it below. (It involves π and is itself ***an infinite*** series.)

$$\frac{\sin x}{x} = 1 - \left(\sum_{k=1}^{\infty}\square\right)x^2 + \ldots = 1 - \square\, x^2 + \ldots$$

g. Follow Euler's tip to conclude the result in the title of this handout. Show work.



*Set the infinite series in Question **f** to the coefficient of the x^2 term in the dashed box in Question **c**. Cool!*

(OVER PLEASE)

Pi Is Good.
More Pi Is Better.

*In addition to the notation for **f(x)**, **$f^{-1}(x)$**, **e**, **i**, and the notation we use for the **six trig functions**, you can also thank me for popularizing the use of the Greek letter **sigma** $\sum$ for the summation of a series.*



2. It has been convenient to use the capital Greek letter Sigma (equivalent to our letter *S*) for a sum. Write the expanded form of $\frac{\sin x}{x}$ using Sigma notation.

$$\frac{\sin x}{x} = \frac{x^0}{1!} - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \ldots. = \sum_{k=\square}^{\infty} \square$$

(+.5) **3.** It is also convenient use the capital Greek letter Pi (equivalent to our letter *P*) for a product. Write the factored form of $\frac{\sin x}{x}$ using Pi notation.

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)\cdots = \prod_{k=\square}^{\infty}\left(\square\right)$$

(+.5) **4** Recall how, in **1e**, we saw the expanded form of these partial products.

$(1-A^2x^2)(1-B^2x^2) = 1 - (A^2+B^2)x^2 + (A^2B^2)x^4$. Here $A=\frac{1}{\pi}$ and $B=\frac{1}{2\pi}$.

$(1-A^2x^2)(1-B^2x^2)(1-C^2x^2) = 1 - (A^2+B^2+C^2)x^2 + (A^2B^2+A^2C^2+B^2C^2)x^4 - (A^2B^2C^2)x^6$. Here $A=\frac{1}{\pi}$, $B=\frac{1}{2\pi}$, $C=\frac{1}{3\pi}$.

$$(1-A^2x^2)(1-B^2x^2)(1-C^2x^2)(1-D^2x^2) = 1 - (A^2+B^2+C^2+D^2)x^2 + (A^2B^2+A^2C^2+A^2D^2+B^2C^2+B^2D^2+C^2D^2)x^4 - (A^2B^2C^2+A^2B^2D^2+A^2C^2D^2+B^2C^2D^2)x^6 + (A^2B^2C^2D^2)x^8.$$

Here $A=\frac{1}{\pi}$, $B=\frac{1}{2\pi}$, $C=\frac{1}{3\pi}$, and $D=\frac{1}{4\pi}$.

$(1-A^2x^2)(1-B^2x^2)(1-C^2x^2)(1-D^2x^2)(1-E^2x^2) = 1 - (A^2+B^2+C^2+D^2+E^2)x^2 + \ldots + (A^2B^2C^2D^2E^2)x^{10}$.

Here $A=\frac{1}{\pi}$, $B=\frac{1}{2\pi}$, $C=\frac{1}{3\pi}$, $D=\frac{1}{4\pi}$, and $E=\frac{1}{5\pi}$. Complete the boxes for the **nth** partial product.

$$\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)\cdots\left(1 - \frac{x^2}{n^2\pi^2}\right) = 1 - \sum_{k=\square}^{n} \square + \ldots + \prod_{k=\square}^{n} \square$$

What is Thanksgiving without some π and then some more $\prod$?

