SOME NONLINEAR DIFFERENTIAL INEQUALITIES AND AN APPLICATION TO HÖLDER CONTINUOUS ALMOST COMPLEX STRUCTURES

ADAM COFFMAN AND YIFEI PAN

Abstract. We consider some second order quasilinear partial differential inequalities for real valued functions on the unit ball and find conditions under which there is a lower bound for the supremum of nonnegative solutions that do not vanish at the origin. As a consequence, for complex valued functions $f(z)$ satisfying $\partial f/\partial\bar{z} = |f|^\alpha$, $0 < \alpha < 1$, and $f(0) \neq 0$, there is also a lower bound for $\sup|f|$ on the unit disk. For each α, we construct a manifold with an α-Hölder continuous almost complex structure where the Kobayashi-Royden pseudonorm is not upper semicontinuous.

1. Introduction

We begin with an analysis of a second order quasilinear partial differential inequality for real valued functions of n real variables,

$$\Delta u - B|u|^\varepsilon \geq 0, \tag{1}$$

where $B > 0$ and $\varepsilon \in [0, 1)$ are constants. In Section 2, we use a Comparison Principle argument to show that (1) has "no small solutions," in the sense that there is a uniform lower bound $M > 0$ for the supremum of solutions u which are nonnegative on the unit ball and nonzero at the origin.

We also consider a generalization of (1):

$$u\Delta u - B|u|^{1+\varepsilon} - C|\vec{\nabla} u|^2 \geq 0, \tag{2}$$

and find conditions under which there is a similar property of no small solutions, in Theorem 2.4.

As an application of the results on the inequality (1), we show failure of upper semicontinuity of the Kobayashi-Royden pseudonorm for a family of 4-dimensional manifolds with almost complex structures of regularity $\mathcal{C}^{0,\alpha}$, $0 < \alpha < 1$. This generalizes the $\alpha = \frac{1}{2}$ example of [IPR]; it is known

Date: February 3, 2011.

2000 *Mathematics Subject Classification.* Primary 35R45; Secondary 32F45, 32Q60, 32Q65, 35B05.

([IR]) that the Kobayashi-Royden pseudonorm is upper semicontinuous for almost complex structures with regularity $\mathcal{C}^{1,\alpha}$.

Our construction of the almost complex manifolds in Section 4 is very similar to that of [IPR]; we give the details for the convenience of the reader, and to show how the argument breaks down as $\alpha \to 1^-$, due to a shrinking radius of the domain. We also take the opportunity in Section 3 to state some Lemmas which allow for a more quantitative description than that of [IPR].

One of the steps in [IPR] is a Maximum Principle argument applied to a complex valued function $h(z)$ satisfying the equation $\partial h/\partial\bar{z} = |h|^{1/2}$, to get the property of no small solutions. The main difference between our paper and [IPR] is the use of a Comparison Principle in Section 2 instead of the Maximum Principle, and we arrive at this result:

Theorem 1.1. *For any $\alpha \in (0,1)$, suppose $h(z)$ is a continuous complex valued function on the closed unit disk, and on the set $\{z : |z| < 1, h(z) \neq 0\}$, h has continuous partial derivatives and satisfies*

$$\frac{\partial h}{\partial \bar{z}} = |h|^{\alpha}. \tag{3}$$

If $h(0) \neq 0$ then $\sup |h| > S_\alpha$, where the constant $S_\alpha > 0$ is defined by:

$$S_\alpha = \left(\frac{2(1-\alpha)}{2-\alpha}\right)^{1/(1-\alpha)}. \tag{4}$$

2. Some differential inequalities

Let D_R denote the open ball in $\mathbb{R}^n$ centered at $\vec{0}$ with radius $R > 0$, and let $\overline{D}_R$ denote the closed ball.

Lemma 2.1. *Given constants $B > 0$ and $0 \le \varepsilon < 1$, let*

$$M = \left(\frac{B(1-\varepsilon)^2}{2(2\varepsilon + n(1-\varepsilon))}\right)^{\frac{1}{1-\varepsilon}} > 0.$$

Suppose the function $u : \overline{D}_1 \to \mathbb{R}$ satisfies:

- *u is continuous on $\overline{D}_1$,*
- *$u(\vec{x}) \ge 0$ for $\vec{x} \in D_1$,*
- *on the open set $\omega = \{\vec{x} \in D_1 : u(\vec{x}) \neq 0\}$, $u \in \mathcal{C}^2(\omega)$,*
- *for $\vec{x} \in \omega$:*

$$\Delta u(\vec{x}) - B(u(\vec{x}))^{\varepsilon} \ge 0. \tag{5}$$

If $u(\vec{0}) \neq 0$, then $\sup_{\vec{x} \in D_1} u(\vec{x}) > M$.

Proof. Define a comparison function

$$v(\vec{x}) = M|\vec{x}|^{\frac{2}{1-\varepsilon}},$$

so $v \in \mathcal{C}^2(\mathbb{R}^n)$ since $0 \leq \varepsilon < 1$. By construction of M, it can be checked that v is a solution of this nonlinear Poisson equation on the domain $\mathbb{R}^n$:

$$\Delta v(\vec{x}) - B|v(\vec{x})|^{\varepsilon} \equiv 0.$$

Suppose, toward a contradiction, that $u(\vec{x}) \leq M$ for all $\vec{x} \in D_1$. For a point $\vec{x}_0$ on the boundary of $\omega \subseteq \mathbb{R}^n$, either $|\vec{x}_0| = 1$, in which case by continuity, $u(\vec{x}_0) \leq M = v(\vec{x}_0)$, or $0 < |\vec{x}_0| < 1$ and $u(\vec{x}_0) = 0$, so $u(\vec{x}_0) \leq v(\vec{x}_0)$. Since $u \leq v$ on the boundary of ω, the Comparison Principle ([GT] Theorem 10.1) applies to the subsolution u and the solution v on the domain ω. The relevant hypothesis for the Comparison Principle in this case is that the second term expression of (5), $-BX^{\varepsilon}$, is weakly decreasing, which uses $B > 0$ and $\varepsilon \geq 0$. (To satisfy this technical condition for all $X \in \mathbb{R}$, we define a function $c : \mathbb{R} \to \mathbb{R}$ by $c(X) = -BX^{\varepsilon}$ for $X \geq 0$, and $c(X) = 0$ for $X \leq 0$. Then c is weakly decreasing in X, v satisfies $\Delta v(\vec{x}) + c(v(\vec{x})) \equiv 0$ and u satisfies $\Delta u(\vec{x}) + c(u(\vec{x})) \geq 0$.)

The conclusion of the Comparison Principle is that $u \leq v$ on ω, however $\vec{0} \in \omega$ and $u(\vec{0}) > v(\vec{0})$, a contradiction. ∎

Of course, the constant function $u \equiv 0$ satisfies the inequality (5), and so does the radial comparison function v, so the initial condition $u(\vec{0}) \neq 0$ is necessary.

Example 2.2. In the $n = 1$ case, $M = \left(\frac{B(1-\varepsilon)^2}{2(1+\varepsilon)}\right)^{\frac{1}{1-\varepsilon}}$. For points c_1, $c_2 \in \mathbb{R}$, $c_1 < c_2$, define a function

$$u(x) = \begin{cases} M(x-c_2)^{\frac{2}{1-\varepsilon}} & \text{if } x \geq c_2 \\ 0 & \text{if } c_1 \leq x \leq c_2 \\ M(c_1-x)^{\frac{2}{1-\varepsilon}} & \text{if } x \leq c_1 \end{cases}.$$

Then $u \in \mathcal{C}^2(\mathbb{R})$, and it is nonnegative and satisfies $u'' = B|u|^{\varepsilon}$ (the case of equality in the $n = 1$ version of (5)). For $c_1 < 0 < c_2$, this gives an infinite collection of solutions of the ODE $u'' = B|u|^{\varepsilon}$ which are identically zero in a neighborhood of 0, so the ODE does not have a unique continuation property. For $c_1 > 0$ or $c_2 < 0$, the function u satisfies $u(0) \neq 0$ and the other hypotheses of Lemma 2.1, and its supremum on $(-1, 1)$ exceeds M even though it can be identically zero on an interval not containing 0.

Example 2.3. In the case $n = 2$, $B = 1$, $\varepsilon = 0$, (5) becomes the linear inequality $\Delta u \geq 1$ and the number $M = \frac{1}{4}$ agrees with Lemma 2 of [IPR], which was proved there using a Maximum Principle argument.

By applying Lemma 2.1 to the Laplacian of a power of u, we get the following generalization.

Theorem 2.4. *Given constants $B > 0$, $C \in \mathbb{R}$, and $\varepsilon < 1$, let*

$$M = \begin{cases} \left(\frac{B(1-\varepsilon)^2}{2(2(\varepsilon - C)+n(1-\varepsilon))}\right)^{\frac{1}{1-\varepsilon}} & \text{if } C \leq \varepsilon \\ \left(\frac{B(1-\varepsilon)}{2n}\right)^{\frac{1}{1-\varepsilon}} & \text{if } C \geq \varepsilon \end{cases} .$$

Suppose the function $u : \overline{D}_1 \to \mathbb{R}$ satisfies:

- *u is continuous on $\overline{D}_1$,*
- *$u(\vec{x}) \geq 0$ for $\vec{x} \in D_1$,*
- *on the open set $\omega = \{\vec{x} \in D_1 : u(\vec{x}) \neq 0\}$, $u \in \mathcal{C}^2(\omega)$,*
- *for $\vec{x} \in \omega$:*

$$u(\vec{x})\Delta u(\vec{x}) \geq B|u(\vec{x})|^{1+\varepsilon} + C|\vec{\nabla} u(\vec{x})|^2.$$

If $u(\vec{0}) \neq 0$, then $\sup_{\vec{x} \in D_1} u(\vec{x}) > M$.

Proof. Let $\mu = \min\{\varepsilon, C\}$, so $\mu \leq \varepsilon < 1$, and on the set ω,

$$u(\vec{x})\Delta u(\vec{x}) \geq B|u(\vec{x})|^{1+\varepsilon} + \mu|\vec{\nabla} u(\vec{x})|^2.$$

Consider the function $u^{1-\mu}$ on $\overline{D}_1$, so $u^{1-\mu} \in \mathcal{C}^0(\overline{D}_1) \cap \mathcal{C}^2(\omega)$, and on the set ω,

$$\begin{aligned} \Delta(u^{1-\mu}) &= (1-\mu)u^{-\mu-1}(u\Delta u - \mu|\vec{\nabla} u|^2) \\ &\geq (1-\mu)u^{-\mu-1}Bu^{1+\varepsilon} \\ &= (1-\mu)B\left(u^{1-\mu}\right)^{(\varepsilon-\mu)/(1-\mu)} . \end{aligned}$$

Since $(1-\mu)B > 0$, and $\mu \leq \varepsilon < 1 \implies 0 \leq \frac{\varepsilon-\mu}{1-\mu} < 1$, Lemma 2.1 applies to $u^{1-\mu}$. If $(u(\vec{0}))^{1-\mu} \neq 0$, then

$$\begin{aligned} \sup u^{1-\mu} &> \left(\frac{(1-\mu)B(1-\frac{\varepsilon-\mu}{1-\mu})^2}{2(2\frac{\varepsilon-\mu}{1-\mu} + n(1-\frac{\varepsilon-\mu}{1-\mu}))}\right)^{\frac{1}{1-\frac{\varepsilon-\mu}{1-\mu}}} \\ \implies \sup u &> \left(\frac{B(1-\varepsilon)^2}{2(2(\varepsilon-\mu)+n(1-\varepsilon))}\right)^{\frac{1}{1-\varepsilon}} . \end{aligned}$$

∎

Functions satisfying a differential inequality of the form (1) or (2) also satisfy a Strong Maximum Principle; the only condition is $B > 0$.

Theorem 2.5. *Given any open set $\Omega \subseteq \mathbb{R}^n$, and any constants $B > 0$, $C, \varepsilon \in \mathbb{R}$, suppose the function $u : \Omega \to \mathbb{R}$ satisfies:*

- *u is continuous on Ω,*
- *on the set $\omega = \{\vec{x} \in \Omega : u(\vec{x}) > 0\}$, $u \in \mathcal{C}^2(\omega)$,*
- *on the set ω, u satisfies*

$$u\Delta u - B|u|^{1+\varepsilon} - C|\vec{\nabla} u|^2 \geq 0.$$

If $u(\vec{x}_0) > 0$ for some $\vec{x}_0 \in \Omega$, then u does not attain a maximum value on Ω.

Proof. Note that the constant function $u \equiv 0$ is the only locally constant solution of the inequality for $B > 0$. If $B = 0$ then obviously any constant function would be a solution.

Given a function u satisfying the hypotheses, ω is a nonempty open subset of Ω. Suppose, toward a contradiction, that there is some $\vec{x}_1 \in \Omega$ with $u(\vec{x}) \leq u(\vec{x}_1)$ for all $x \in \Omega$. In particular, $u(\vec{x}_1) \geq u(\vec{x}_0) > 0$, so $\vec{x}_1 \in \omega$. Let ω_1 be the connected component of ω containing $\vec{x}_1$.

For $\vec{x} \in \omega_1$, u satisfies the linear, uniformly elliptic inequality

$$\Delta u(\vec{x}) + (-B(u(\vec{x}))^{\varepsilon-1})u(\vec{x}) + (-C\frac{\vec{\nabla} u(\vec{x})}{u(\vec{x})}) \cdot \vec{\nabla} u(\vec{x}) \geq 0,$$

where the coefficients (defined in terms of the given u) are locally bounded functions of $\vec{x}$, and $(-B(u(\vec{x}))^{\varepsilon-1})$ is negative for all $\vec{x} \in \omega$. It follows from the Strong Maximum Principle ([GT] Theorem 3.5) that since u attains a maximum value at $\vec{x}_1$, then u is constant on ω_1. Since the only constant solution is 0, it follows that $u(\vec{x}_1) = 0$, a contradiction. ∎

The next Lemma shows how an inequality like (5) with $n = 2$ can arise from a first order PDE for a complex valued function.

Lemma 2.6. *Consider constants α, $\gamma \in \mathbb{R}$ with $0 < \alpha < 1$. Let $\omega \subseteq \mathbb{C}$ be an open set, and suppose $h : \omega \to \mathbb{C}$ satisfies:*

- *$h \in \mathcal{C}^1(\omega)$,*
- *$h(z) \neq 0$ for all $z \in \omega$,*
- *$\dfrac{\partial h}{\partial \bar{z}} = |h|^\alpha$ on ω.*

Then, the following inequality is satisfied on ω:

$$\Delta(|h|^{(1-\alpha)\gamma}) \geq (4(1-\alpha)\gamma - (2-\alpha)^2)|h|^{(1-\alpha)(\gamma-2)}. \tag{6}$$

Remark. The special case $\alpha = \frac{1}{2}$, $\gamma = \frac{3}{2}$ is Lemma 1 of [IPR]; its Proof there is a long calculation in polar coordinates, which can be generalized to some other values of α by an analogous argument. However, using $z, \bar{z}$ coordinates allows for a shorter calculation.

Proof of Lemma 2.6. We first want to show that h is smooth on ω, applying the regularity and bootstrapping technique of PDE to the equation $\partial h/\partial\bar z = |h|^\alpha$. We recall the following fact (for a more general statement, see Theorem 15.6.2 of [AIM]): for a nonnegative integer ℓ, and $0<\beta<1$, if $\varphi\in\mathcal{C}^{\ell,\beta}_{loc}(\omega)$ and $g:\omega\to\mathbb{C}$ has first derivatives in $L^2_{loc}(\omega)$ and is a solution of $\partial g/\partial\bar z=\varphi$, then $g\in\mathcal{C}^{\ell+1,\beta}_{loc}(\omega)$. In our case, $\varphi=|h|^\alpha\in\mathcal{C}^1(\omega)\subseteq\mathcal{C}^{0,\beta}_{loc}(\omega)$ (since $h\in\mathcal{C}^1(\omega)$ and is nonvanishing), and $g=h$ has continuous first derivatives, so we can conclude that $g=h\in\mathcal{C}^{1,\beta}_{loc}(\omega)$. Repeating gives that $h\in\mathcal{C}^{2,\beta}_{loc}(\omega)$, etc.

Since the conclusion is a local statement, it is enough to express ω as a union of open subsets ω_k and establish the conclusion on each subset. For each $z_k\in\omega$, there is a sufficiently small disk ω_k containing z_k, where real exponentiation of $h(z)$ is well-defined on ω_k, by choosing a single-valued branch of log to define $h^r=\exp(r\log(h))$.

The condition $\dfrac{\partial h}{\partial\bar z}=|h|^\alpha$ can be re-written

$$h_{\bar z}=(\bar h)_z=|h|^\alpha=h^{\alpha/2}\bar h^{\alpha/2}.$$

This leads to

$$\begin{aligned}
h_{z\bar z} &= (h_{\bar z})_z=(h^{\alpha/2}\bar h^{\alpha/2})_z\\
&= \frac{\alpha}{2}\left(h^{(\alpha/2)-1}\bar h^{\alpha/2}h_z+h^\alpha\bar h^{\alpha-1}\right)\\
&= \overline{((\bar h)_{z\bar z})},
\end{aligned}$$

which is used in a line of the next step. For an arbitrary exponent $m\in\mathbb{R}$,

$$\begin{aligned}
(|h|^m)_{z\bar z} &= (h^{m/2}\bar h^{m/2})_{z\bar z}\\
&= \frac{\partial}{\partial z}\left(\frac{m}{2}h^{\frac m2-1}h_{\bar z}\bar h^{\frac m2}+h^{\frac m2}\frac{m}{2}\bar h^{\frac m2-1}(\bar h)_{\bar z}\right)\\
&= \frac{m}{2}\frac{\partial}{\partial z}\left(h^{\frac m2-1+\frac\alpha2}\bar h^{\frac m2+\frac\alpha2}+h^{\frac m2}\bar h^{\frac m2-1}(\bar h)_{\bar z}\right)\\
&= \frac{m}{2}\left[\left(\frac m2+\frac\alpha2-1\right)h^{\frac m2+\frac\alpha2-2}h_z\bar h^{\frac m2+\frac\alpha2}\right.\\
&\quad +h^{\frac m2+\frac\alpha2-1}\left(\frac m2+\frac\alpha2\right)\bar h^{\frac m2+\frac\alpha2-1}(\bar h)_z\\
&\quad +\frac m2 h^{\frac m2-1}h_z\bar h^{\frac m2-1}(\bar h)_{\bar z}\\
&\quad \left.+h^{\frac m2}\left(\frac m2-1\right)\bar h^{\frac m2-2}(\bar h)_z(\bar h)_{\bar z}+h^{\frac m2}\bar h^{\frac m2-1}(\bar h)_{z\bar z}\right].\\
&= \frac m2\left[\mathrm{Re}\left((m+\alpha-2)|h|^{m+\alpha-4}\bar h^2h_z\right)+\left(\frac m2+\alpha\right)|h|^{m+2\alpha-2}\right.\\
&\quad \left.+\frac m2|h|^{m-2}|h_z|^2\right].
\end{aligned}$$

With the aim of applying Lemma 2.1 to the function $|h|^m$, we consider the expression (8), with real constants B, ε, and $m \neq 0$. In line (9), we assign

$$\varepsilon = \frac{1}{m}(m + 2\alpha - 2) \tag{7}$$

to be able to combine like terms, and in line (10), we choose $B = 4m - (2-\alpha)^2$ to complete the square.

$$\begin{aligned}
(8) \qquad & \Delta(|h|^m) - B(|h|^m)^\varepsilon \\
= \;& 4(|h|^m)_{z\bar{z}} - B|h|^{m\varepsilon} \\
= \;& 2m\Big[\mathrm{Re}\left((m+\alpha-2)|h|^{m+\alpha-4}\bar{h}^2 h_z\right) + \Big(\frac{m}{2}+\alpha\Big)|h|^{m+2\alpha-2} \\
& + \frac{m}{2}|h|^{m-2}|h_z|^2\Big] - B|h|^{m\varepsilon} \\
(9) \qquad = \;& (m(m+2\alpha) - B)|h|^{m+2\alpha-2} \\
& + \mathrm{Re}\left(2m(m+\alpha-2)|h|^{m+\alpha-4}\bar{h}^2 h_z\right) + m^2|h|^{m-2}|h_z|^2 \\
\geq \;& |h|^{m-2}\left((m^2 + 2\alpha m - B)|h|^{2\alpha}\right. \\
& \left. -2|m||m+\alpha-2||h|^\alpha|h_z| + m^2|h_z|^2\right) \\
(10) \qquad = \;& |h|^{m-2}\left(|m+\alpha-2||h|^\alpha - |m||h_z|\right)^2 \geq 0.
\end{aligned}$$

Considering the form of (7), it is convenient to choose $m = (1-\alpha)\gamma$ for some constant $\gamma \neq 0$. The claim of the Lemma follows; the $\gamma = 0$ case can be checked separately. ∎

The parameter γ can be chosen arbitrarily large; to apply Lemma 2.1 to get the "no small solutions" result of Theorem 1.1, we need the RHS coefficient in (6) to be positive, so $\gamma > \frac{(2-\alpha)^2}{4(1-\alpha)}$, and also the RHS exponent $(1-\alpha)(\gamma-2)$ to be nonnegative, so $\gamma \geq 2$. In contrast, the $\alpha = \frac{1}{2}$, $\gamma = \frac{3}{2}$ case appearing in Lemma 1 of [IPR] has RHS exponent $-\frac{1}{4}$. The approach of Theorem 2 of [IPR] is to use the negative exponent together with the result of Example 2.3 to show that assuming h has a small solution leads to a contradiction. As claimed, their method can be generalized to apply to other nonpositive exponents, but $\frac{(2-\alpha)^2}{4(1-\alpha)} < \gamma \leq 2$ holds only for $\alpha < 2(\sqrt{2}-1) \approx 0.8284$.

Proof of Theorem 1.1. Given a continuous $h : \overline{D}_1 \to \mathbb{C}$ satisfying the hypotheses of Theorem 1.1, on the set $\omega = \{z \in D_1 : h(z) \neq 0\}$, $h \in \mathcal{C}^1(\omega)$, and the conclusion of Lemma 2.6 can be re-written:

$$\Delta(|h|^{(1-\alpha)\gamma}) \geq (4(1-\alpha)\gamma - (2-\alpha)^2)(|h|^{(1-\alpha)\gamma})^{1-\frac{2}{\gamma}}. \tag{11}$$

The hypotheses of Lemma 2.1 are satisfied with $n = 2$, $u(x,y) = |h(x+iy)|^{(1-\alpha)\gamma}$, and $u(\vec{0}) \neq 0$, when the RHS of (11) has a positive coefficient (so $\gamma > \frac{(2-\alpha)^2}{4(1-\alpha)}$) and the quantity $\varepsilon = 1 - \frac{2}{\gamma}$ is in $[0,1)$ (for $\gamma \geq 2$). The conclusion of Lemma 2.1 is:

$$\begin{aligned}\sup_{z\in D_1} |h(z)|^{(1-\alpha)\gamma} &> M = \left(\frac{1}{4}\cdot(4(1-\alpha)\gamma-(2-\alpha)^2)\cdot\left(\frac{2}{\gamma}\right)^2\right)^{\gamma/2} \\ \implies \sup_{z\in D_1}|h(z)| &> \left(\frac{4(1-\alpha)\gamma-(2-\alpha)^2}{\gamma^2}\right)^{\frac{1}{2(1-\alpha)}}.\end{aligned}$$

We can optimize this lower bound, using elementary calculus to show that the maximum value of $\frac{4(1-\alpha)\gamma-(2-\alpha)^2}{\gamma^2}$ is achieved at the critical point $\gamma = \frac{(2-\alpha)^2}{2(1-\alpha)} > \max\left\{2, \frac{(2-\alpha)^2}{4(1-\alpha)}\right\}$, and the lower bound for the sup is S_α as appearing in (4). ∎

Note that S_α is decreasing for $0 < \alpha < 1$, with $S_{1/2} = \frac{4}{9}$, $S_{2/3} = \frac{1}{8}$, and $S_\alpha \to 0$ as $\alpha \to 1^-$. This Theorem is used in the Proof of Theorem 4.3.

Example 2.7. As noted by [IPR], a one-dimensional analogue of Equation (3) in Theorem 1.1 is the well-known (for example, [BR] §I.9) ODE $u'(x) = B|u(x)|^\alpha$ for $0 < \alpha < 1$ and $B > 0$, which can be solved explicitly. By an elementary separation of variables calculation, the solution on an interval where $u \neq 0$ is $|u(x)| = (\pm(1-\alpha)(Bx+C))^{\frac{1}{1-\alpha}}$. The general solution on the domain $\mathbb{R}$ is, for $c_1 < c_2$,

$$u(x) = \begin{cases} (1-\alpha)^{\frac{1}{1-\alpha}}(B(x-c_2))^{\frac{1}{1-\alpha}} & \text{if } x \geq c_2 \\ 0 & \text{if } c_1 \leq x \leq c_2 \\ -(1-\alpha)^{\frac{1}{1-\alpha}}(B(c_1-x))^{\frac{1}{1-\alpha}} & \text{if } x \leq c_1 \end{cases}.$$

So $u \in \mathcal{C}^1(\mathbb{R})$, and if $u(0) \neq 0$, then $\sup_{-1<x<1} |u(x)| > ((1-\alpha)B)^{\frac{1}{1-\alpha}}$.

3. Lemmas for holomorphic maps

We continue with the D_R notation for the open disk in the complex plane centered at the origin. The following quantitative Lemmas on inverses of holomorphic functions $D_R \to \mathbb{C}$ are used in a step of the Proof of Theorem 4.3 where we put a map $D_r \to \mathbb{C}^2$ into a normal form, (14).

Lemma 3.1 ([G] Exercise I.1.). *Suppose $f : D_1 \to D_1$ is holomorphic, with $f(0) = 0$, $|f'(0)| = \delta > 0$. For any $\eta \in (0,\delta)$, let $s = \left(\frac{\delta-\eta}{1-\eta\delta}\right)\eta$; then the restricted function $f : D_\eta \to D_1$ takes on each value $w \in D_s$ exactly once.* ∎

The hypotheses imply $\delta \leq 1$ by the Schwarz Lemma.

Lemma 3.2. *For a holomorphic map $Z_1 : D_r \to D_2$ with $Z_1(0) = 0$, $Z_1'(0) = 1$, if $r > \frac{4\sqrt{2}}{3}$ then there exists a continuous function $\phi : \overline{D}_1 \to D_r$ which is holomorphic on D_1 and which satisfies $(Z_1 \circ \phi)(z) = z$ for all $z \in \overline{D}_1$.*

Remark. It follows from the Schwarz Lemma that $r \leq 2$, and it follows from the fact that ϕ is an inverse of Z_1 that $\phi(0) = 0$ and $\phi'(0) = 1$.

Proof of Lemma 3.2. Define a new holomorphic function $f : D_1 \to D_1$ by

$$f(z) = \frac{1}{2} \cdot Z_1(r \cdot z),$$

so $f(0) = 0$, $f'(0) = \frac{r}{2}$, and Lemma 3.1 applies with $\delta = \frac{r}{2}$. If we choose $\eta = \frac{3r}{8}$, then $s = \frac{3r^2}{64-12r^2}$, and the assumption $r > \frac{4\sqrt{2}}{3}$ implies $s > \frac{1}{2}$. It follows from Lemma 3.1 that there exists a function $\psi : D_s \to D_\eta$ such that $(f \circ \psi)(z) = z$ for all $z \in \overline{D}_{1/2} \subseteq D_s$; this inverse function ψ is holomorphic on $D_{1/2}$. The claimed function $\phi : \overline{D}_1 \to D_{r\eta} \subseteq D_r$ is defined by $\phi(z) = r \cdot \psi(\frac{1}{2} \cdot z)$, so for $z \in \overline{D}_1$,

$$Z_1(\phi(z)) = Z_1(r \cdot \psi(\frac{1}{2} \cdot z)) = 2 \cdot f(\psi(\frac{1}{2} \cdot z)) = 2 \cdot \frac{1}{2} \cdot z = z.$$

∎

4. J-holomorphic disks

For $S > 0$, consider the bidisk $\Omega_S = D_2 \times D_S \subseteq \mathbb{C}^2$, as an open subset of $\mathbb{R}^4$, with coordinates $\vec{x} = (x_1, y_1, x_2, y_2) = (z_1, z_2)$ and the trivial tangent bundle $T\Omega_S \subseteq T\mathbb{R}^4$. Consider an almost complex structure J on Ω_S given by a complex structure operator on $T_{\vec{x}}\Omega_S$ of the following form:

$$J(\vec{x}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 \\ \lambda & 0 & 1 & 0 \end{pmatrix}, \tag{12}$$

where $\lambda : \Omega_S \to \mathbb{R}$ is any function.

A differentiable map $Z : D_r \to \Omega_S$ is a J-holomorphic disk if $dZ \circ J_{std} = J \circ dZ$, where J_{std} is the standard complex structure on $D_r \subseteq \mathbb{C}$. Let $z = x + iy$ be the coordinate on D_r. For J of the form (12), if $Z(z)$ is defined by complex valued component functions,

$$Z : D_r \to \Omega_S : Z(z) = (Z_1(z), Z_2(z)), \tag{13}$$

then the J-holomorphic property implies that $Z_1 : D_r \to D_2$ is holomorphic in the standard way.

Example 4.1. If the function $\lambda(z_1, z_2)$ satisfies $\lambda(z_1, 0) = 0$ for all $z_1 \in D_2$, then the map $Z : D_2 \to \Omega_S : Z(z) = (z, 0)$ is a J-holomorphic disk.

Definition 4.2. The Kobayashi-Royden pseudonorm on Ω_S is a function $T\Omega_S \to \mathbb{R} : (\vec{x}, \vec{v}) \mapsto \|(\vec{x}, \vec{v})\|_K$, defined on tangent vectors $\vec{v} \in T_{\vec{x}}\Omega_S$ to be the number

$$\mathrm{glb}\left\{\frac{1}{r} : \exists \text{ a } J\text{-holomorphic } Z : D_r \to \Omega_S,\ Z(0) = \vec{x},\ dZ(0)(\frac{\partial}{\partial x}) = \vec{v}\right\}.$$

Under the assumption that $\lambda \in \mathcal{C}^{0,\alpha}(\Omega_S)$, $0 < \alpha < 1$, it is shown by [IR] and [NW] that there is a nonempty set of J-holomorphic disks through $\vec{x}$ with tangent vector $\vec{v}$ as in the Definition, so the pseudonorm is a well-defined function. Further, each such disk satisfies $Z \in \mathcal{C}^1(D_r)$.

At this point we pick $\alpha \in (0, 1)$ and set $\lambda(z_1, z_2) = -2|z_2|^\alpha$. Let $S = S_\alpha > 0$ be the constant defined by formula (4) from Theorem 1.1. Then, (Ω_S, J) is an almost complex manifold with the following property:

Theorem 4.3. *If* $0 \neq b \in D_S$ *then* $\|((0, b), (1, 0))\|_K \geq \frac{3}{4\sqrt{2}}$.

Remark. Since $\frac{3}{4\sqrt{2}} \approx 0.53$, and $\|((0, 0), (1, 0))\|_K \leq \frac{1}{2}$ by Example 4.1, the Theorem shows that the Kobayashi-Royden pseudonorm is not upper semicontinuous on $T\Omega_S$.

Proof. Consider a J-holomorphic map $Z : D_r \to \Omega_S$ of the form (13), and suppose $Z(0) = (0, b) \in \Omega_S$ and $dZ(0)(\frac{\partial}{\partial x}) = (1, 0)$. Then the holomorphic function $Z_1 : D_r \to D_2$ satisfies $Z_1(0) = 0$, $Z_1'(0) = 1$, and $Z_2 \in \mathcal{C}^1(D_r)$ satisfies $Z_2(0) = b$.

Suppose, toward a contradiction, that there exists such a map Z with $b \neq 0$ and $r > \frac{4\sqrt{2}}{3}$. Then Lemma 3.2 applies to Z_1: there is a reparametrization ϕ which puts Z into the following normal form:

$$\begin{aligned} (Z \circ \phi) : \overline{D}_1 &\to \Omega_S \\ z &\mapsto (Z_1(\phi(z)), Z_2(\phi(z))) = (z, f(z)), \end{aligned} \tag{14}$$

where $f = Z_2 \circ \phi : \overline{D}_1 \to D_S$ satisfies $f \in \mathcal{C}^0(\overline{D}_1) \cap \mathcal{C}^1(D_1)$. From the fact that $Z \circ \phi$ is J-holomorphic on D_1, it follows from the form (12) of J that if $f(z) = u(x, y) + iv(x, y)$, then f satisfies this system of nonlinear Cauchy-Riemann equations on D_1:

$$\frac{du}{dy} = -\frac{dv}{dx} \quad \text{and} \quad \frac{du}{dx} + \lambda(z, f(z)) = \frac{dv}{dy} \tag{15}$$

with the initial conditions $f(0) = b$, $u_x(0) = u_y(0) = v_x(0) = 0$ and $v_y(0) = \lambda(0,b) = -2|b|^\alpha$. The system of equations implies

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x}(u+iv) + i\frac{\partial}{\partial y}(u+iv)\right)\\ &= \frac{1}{2}(u_x - v_y + i(v_x + u_y))\\ &= -\frac{1}{2}\lambda(z, f(z)) = |f|^\alpha. \qquad (16)\end{aligned}$$

So, Theorem 1.1 applies, with $f = h$. The conclusion is that

$$\sup_{z\in D_1} |f(z)| > S_\alpha,$$

but this contradicts $|f(z)| < S = S_\alpha$. ∎

The previously mentioned existence theory for J-holomorphic disks shows there are interesting solutions of the equation (16), and therefore also the inequality (11).

Example 4.4. For $0 < \alpha < 1$, (Ω_S, J), $\lambda(z_1, z_2) = -2|z_2|^\alpha$ as above, a map $Z : D_r \to \Omega_S$ of the form $Z(z) = (z, f(z))$ is J-holomorphic if $f(x,y) = u(x,y) + iv(x,y)$ is a solution of (15). Again generalizing the $\alpha = \frac{1}{2}$ case of [IPR], examples of such solutions can be constructed (for small r) by assuming $v \equiv 0$ and u depends only on x, so (15) becomes the ODE $u'(x) - 2|u(x)|^\alpha = 0$. This is the equation from Example 2.7; we can conclude that J-holomorphic disks in Ω_S do not have a unique continuation property.

Acknowledgments. The authors acknowledge the insightful remarks of the referee, whose suggestions led to a shorter proof of Lemma 2.6 and an improvement of Theorem 2.4.

References

[AIM] K. Astala, T. Iwaniec, and G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, PMS **48**, Princeton University Press, Princeton, 2009. MR 2472875 (2010j:30040), Zbl 1182.30001.

[BR] G. Birkhoff and G.-C. Rota, *Ordinary Differential Equations*, Ginn & Co., 1962. MR 0138810 (25 #2253), Zbl 0102.29901.

[G] J. Garnett, *Bounded Analytic Functions*, rev. first ed., GTM **236**, Springer, 2007. MR 2261424 (2007e:30049), Zbl 1106.30001.

[GT] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer CIM, 2001. MR 1814364 (2001k:35004), Zbl 1042.35002.

[IPR] S. Ivashkovich, S. Pinchuk, and J.-P. Rosay, *Upper semi-continuity of the Kobayashi-Royden pseudo-norm, a counterexample for Hölderian almost complex structures*, Ark. Mat. (2) **43** (2005), 395–401. MR 2173959 (2006g:32038), Zbl 1091.32009.

[IR] S. Ivashkovich and J.-P. Rosay, *Schwarz-type lemmas for solutions of $\overline{\partial}$-inequalities and complete hyperbolicity of almost complex manifolds*, Ann. Inst. Fourier (Grenoble) (7) **54** (2004), 2387–2435. MR 2139698 (2006a:32032), Zbl 1072.32007.

[NW] A. Nijenhuis and W. Woolf, *Some integration problems in almost-complex and complex manifolds*, Ann. of Math. (2) **77** (1963), 424–489. MR 0149505 (26 #6992), Zbl 0115.16103.

Department of Mathematical Sciences, Indiana University - Purdue University Fort Wayne, 2101 E. Coliseum Blvd., Fort Wayne, IN, USA 46805-1499

E-mail address: CoffmanA@ipfw.edu

School of Sciences, Nanchang University, Nanchang, P.R.China 330022

E-mail address: Pan@ipfw.edu