Ellipses in the Inversive Plane

Adam Coffman
Indiana University Purdue University Fort Wayne
Fort Wayne, Indiana 46805-1499
CoffmanA@ipfw.edu

Marc Frantz
Indiana University
Bloomington, Indiana 47405
mfrantz@indiana.edu

In inversive geometry, which deals with the space $\mathbb{C} \cup \{\infty\}$ and the group of Möbius transformations, the properties of circles are well-known. In particular, the image of a line or circle under a Möbius transformation is another line or circle. Some interesting questions arise when considering the actions of Möbius transformations on ellipses. Of course the circle could be considered a special case, but do any of its inversive properties generalize to non-circular ellipses? More specifically, the following Questions refer to $\mathcal{E}$, an ellipse which is not a circle, contained in the subset $\mathbb{C}$ of the extended complex plane, $\mathbb{C} \cup \{\infty\}$, and $\mathcal{E}' \subseteq \mathbb{C}$, another ellipse.

Questions:

I. What is the image of $\mathcal{E}$ under a Möbius transformation?

II. Which Möbius transformations are symmetries of $\mathcal{E}$?

III. Which Möbius transformations T, if any, have the property $T(\mathcal{E}) \subseteq \mathcal{E}'$?

From this point, we use the term "ellipse" to mean only "non-circular ellipse," and the term "circle" refers to both circles and extended lines (which include the point ∞). The Questions could be generalized to real conics in general, and we briefly survey a few inversive properties of parabolas and hyperbolas. However, the geometry of the ellipse turns out to be a little more subtle, and it seems that the hyperbolas have already had their share of attention in the literature, which is one reason for an expository article focusing (!) on the ellipse. So, while the reader ponders these Questions about ellipses, we summarize some facts about Möbius transformations.

Möbius transformations are maps $T : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, of the form

$$T(Z) = \frac{aZ + b}{cZ + d}, \tag{1}$$

or

$$T(Z) = \frac{a\bar{Z} + b}{c\bar{Z} + d}, \tag{2}$$

with complex coefficients that satisfy $ad - bc \neq 0$, and the usual conventions for ∞ input and output. Such transformations are all continuous, one-to-one, and onto, and they form a group under composition. Transformations of type (2) are the "indirect" transformations, which reverse orientation, and those of type (1) are the "direct" Möbius transformations, which form the subgroup of "linear fractional transformations." The transformations with $c = 0$ fix the point ∞, and form the subgroup of "similarities." There are direct and indirect similarities, and the direct similarities form a subgroup, consisting of functions of the form $T(Z) = aZ + b$, with $a \neq 0$.

In the group of similarities, there are four elements which preserve an ellipse $\mathcal{E}$: the identity, the half-turn around the center of the ellipse, and the reflections in the major and minor axes. Question **II** asks whether there are any other Möbius transformations which are symmetries of the ellipse.

It is clear that a similarity transformation of $\mathbb{C}$ takes a conic curve (a circle, ellipse, hyperbola, or parabola) into another conic curve of the same eccentricity $e \geq 0$. A hyperbola has the same group of similarity symmetries as an ellipse, and the parabola has only the identity and one reflection.

If the extended complex plane is considered as the Riemann sphere, the closure of a hyperbola or a parabola includes the point ∞, so we just make a convention that ∞ is an element of every hyperbola and parabola (and, as usual in Möbius geometry, every straight line). This way, the image of any conic curve under a Möbius transformation T is a closed curve in the Riemann sphere. The images of a hyperbola are seen to have a point of transverse self-intersection. The images of a parabola have a singular "cusp" point. The ellipse is a "simple" closed curve, in the technical sense of having no self-intersection.

Among the transformations that do not fix ∞, the "inversions" are of the form

$$T_{w,r}(Z) = \frac{r^2}{\bar{Z} - \bar{w}} + w,$$

which fix the points on a circle with center $w \in \mathbb{C}$ and radius $r > 0$. Some references on inversive geometry ([7], [19], [28]), the literature on "special plane curves" ([1], [9], [10], [13], [15], [16], [29], [32], [33], [36], [37]), and web sites on graphics and mathematics ([8], [14], [21]) show that many famously named curves are inverses of conics, including the limaçons of E. Pascal.

Curve	Eccentricity	Center of Inversion	Image
hyperbola	$e > 1$	center	hyperbolic lemniscate of Booth = self-intersecting hippopede
hyperbola	$e > 1$	focus	self-intersecting limaçon
hyperbola	$e = 2$	focus	trisectrix
hyperbola	$e = 2$	vertex	trisectrix of Maclaurin
hyperbola	$e = \sqrt{2}$	vertex	right strophoid
hyperbola	$e = \sqrt{2}$	center	lemniscate of Bernoulli
parabola	$e = 1$	focus	cusped limaçon = cardioid
parabola	$e = 1$	vertex	cissoid of Diocles
ellipse	$0 < e < 1$	center	elliptic lemniscate of Booth = simple hippopede
ellipse	$0 < e < 1$	focus	simple limaçon

Figure 1: Two conics and their inverses with respect to a circle. Hyperbola and lemniscate (left); parabola and cardioid (right).

Figure 2: Ellipse $\mathcal{E}$ with a non-convex central inversion (left), and with a convex central inversion (right).

The images of an ellipse

According to the above list, the ellipse seems to have fewer famous images under inversion than the hyperbola. The simple limaçons give one answer to Question **I**, in the case where the Möbius transformation is an inversion $T_{w,r}$ with w at a focus.

Another special case of inversion of conics is the family of hippopedes of Proclus ([25]), which are also known as the lemniscates of J. Booth ([2], [3], apparently given this name by G. Loria [16]). The elliptic lemniscate seems to be a little less famous than some other special plane curves, at least in the English-language sources. However, it has many interesting applications, for example, in mechanical linkages ([8], [37]) and fluid physics ([27]). It also appears in solid geometry as an intersection of a plane with a spindle torus, or with Fresnel's elasticity surface ([30]).

References [13] and [36] rather vaguely refer to the central inversion of an ellipse as an "oval." When the eccentricity of $\mathcal{E}$ is large, its image is a non-convex curve, as in Figure 2.

The reader might also want try some graphical experimentation — the web site [21] has an interactive demonstration, where the user can manipulate an ellipse and a circle of inversion, to see various images of the ellipse in the plane. When the eccentricity is small, $\mathcal{E}$ becomes nearly circular, and one might expect its image under a central inversion is also nearly circular. Figure 2, and experiments with [21], show an oval-shaped image, but it is hard to tell just by looking whether the oval is or is not exactly an ellipse. However, we claim that it is not, and that this fact is a Corollary of the following:

Figure 3: Images of a particular ellipse $\mathcal{E}$ and circle $\mathcal{C}$, along with their three common points, under the central inversion T.

Theorem 1. *If $\mathcal{E}$ is an ellipse centered at $0 \in \mathbb{C}$, and $T(Z) = 1/Z$, then the image $T(\mathcal{E})$ is not an ellipse.*

Before stating a proof of the Theorem, we set up a convenient coordinate system, and then mention a few interesting but inconclusive approaches to the problem.

Given $\mathcal{E}$ centered at 0, it can be rotated by some similarity $R(Z) = e^{i\theta} \cdot Z$ into "standard position," so the major diameter lies on the real axis. Since $T = R \circ T \circ R$, we can, without loss of generality, assume $\mathcal{E}$ is already in this form, so in terms of the real coordinates (X, Y) (the components of the complex coordinate $Z = X + iY$):

$$\mathcal{E} = \{(X,Y) : \frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1\}, \tag{3}$$

with $0 < B < A$. In terms of (X, Y), the transformation T is given by

$$(X,Y) \mapsto \left(\frac{X}{X^2+Y^2}, \frac{-Y}{X^2+Y^2}\right). \tag{4}$$

One elementary proof of the claim, following [34], works only for ellipses with large eccentricity. If $B < A/\sqrt{2}$, it is easy to check that the circle with center $(X, Y) = (A/2, 0)$ and radius $A/2$ meets $\mathcal{E}$ in exactly three points, one of which is $(A, 0)$. Since the circle also passes through the origin, the image of the circle under T is a straight (extended) line, which meets $T(\mathcal{E})$ at three (distinct, finite) points, but this is impossible if $T(\mathcal{E})$ is an ellipse.

Another attempt at a proof would be to find an implicit equation which holds for points on $T(\mathcal{E})$; one such equation can be found by applying the transformation (4) to the variables in Equation (3) (replacing X with $X/(X^2 + Y^2)$ and Y with $-Y/(X^2 + Y^2)$), and then clearing denominators to get:

$$(X^2+Y^2)^2 - \left(\frac{X^2}{A^2} + \frac{Y^2}{B^2}\right) = 0. \tag{5}$$

This certainly appears to be the equation of a quartic curve, but the solution set always contains an isolated point $(X, Y) = (0, 0) = T(\infty)$. So, we see that the locus defined by (5) contains at least one point not in the image $T(\mathcal{E})$. A further potential complication becomes evident when $A = B$, in which case the polynomial is not irreducible — it factors into two quadratics, and its zero locus is the union of a circle with radius $\frac{1}{A}$, and the singleton set $\{(0, 0)\}$. So, merely showing that the points on $T(\mathcal{E})$ satisfy some equation with degree four is not conclusive, since the geometric hypothesis of non-circularity ($A \neq B$) must somehow be connected to the algebraic property of irreducibility. Skeptical mathematicians would not consider the irreducibility of (5) proved by the failure of a computer algebra system to factor it, especially since the irreducibility apparently depends on the values of A and B. Even though it can be shown that the quartic (5) is irreducible, and is the lowest degree polynomial whose zero locus contains $T(\mathcal{E})$, when $A \neq B$, some further algebraic or geometric argument would be required to show that this set does not contain any ellipse.

Proof of Theorem 1. The proof of the Theorem uses the reflection and rotation symmetry of the ellipse $\mathcal{E}$ given by Equation (3), and the fact that T respects this symmetry. If R is any of the similarities $Z \mapsto \bar{Z}$, $Z \mapsto -\bar{Z}$, or $Z \mapsto -Z$, then $T = R \circ T \circ R$, so whatever shape $T(\mathcal{E})$ is, it also is invariant under a half-turn centered at the origin, and has the same reflection symmetry as $\mathcal{E}$. The curve $T(\mathcal{E})$ passes through $Z = X + iY = \pm\frac{1}{A} + i0$ and $0 \pm i\frac{1}{B}$, so if $T(\mathcal{E})$ is an ellipse, then it must have a quadric implicit equation of the form

$$g(X + iY) = A^2X^2 + B^2Y^2 - 1 = 0.$$

$\mathcal{E}$ can be represented parametrically, by a map

$$P : \mathbb{R} \cup \{\infty\} \to \mathbb{C} : u \mapsto \frac{2Au + iB(1 - u^2)}{1 + u^2}, \tag{6}$$

so $T(\mathcal{E})$ has a parametric equation of the form

$$T \circ P : u \mapsto \frac{1 + u^2}{2Au + iB(1 - u^2)}.$$

The element ∞ of the "extended real line" is mapped by P to $-iB$, and by $T \circ P$ to i/B, and the fraction $(T \circ P)(u)$ is in $\mathbb{C}$ for all u since the image of P does not pass through 0. The real and imaginary parts of the parametrization of $T(\mathcal{E})$ are $X + iY = (T \circ P)(u)$, with:

$$\begin{aligned} X &= \frac{2uA(1+u^2)}{B^2u^4 + (4A^2 - 2B^2)u^2 + B^2}, \\ Y &= \frac{B(u^4-1)}{B^2u^4 + (4A^2 - 2B^2)u^2 + B^2}. \end{aligned}$$

The image of the parametrization $T \circ P$ must satisfy the implicit equation $g = 0$ for all $u \in \mathbb{R} \cup \{\infty\}$, but for $u \in \mathbb{R}$, the composition $g((T \circ P)(u))$ is equal to:

$$\frac{4u^2(u-1)^2(u+1)^2(B-A)^2(B+A)^2}{(B^2u^4 + (4A^2 - 2B^2)u^2 + B^2)^2},$$

which, if $A \neq B$, has non-zero values for some u. In this case, there are only four points (at $u = 0, 1, -1$, and as $u \to \infty$) where $T(\mathcal{E})$ meets the ellipse $g(X + iY) = 0$.

For example, it is easy to check that the point $A/\sqrt{2} + iB/\sqrt{2}$ is on $\mathcal{E}$, but the point $T((A+iB)/\sqrt{2}) = \sqrt{2}(A - iB)/(A^2 + B^2)$ is not on the ellipse $A^2X^2 + B^2Y^2 = 1$ unless $A = B$. ∎

The last paragraph of the proof is actually enough for the claim of the Theorem. The parametric equations for $\mathcal{E}$ were able to show a little more about $T(\mathcal{E})$, and they appear again in the last section of this article.

Corollary 2. *If $\mathcal{E}$ is an ellipse in $\mathbb{C}$ with center w, and r is any positive number, then the image $T_{w,r}(\mathcal{E})$ is not an ellipse.*

Proof. The inversion $T_{w,r}$ is equal to a composition $S_1 \circ T \circ S_2$, where S_2 is the indirect similarity $Z \mapsto \overline{Z - w}$, S_1 is the direct similarity $Z \mapsto r^2 \cdot Z + w$, and T is the reciprocal function of the Theorem. If the image of $T_{w,r}(\mathcal{E})$ were an ellipse, then the image $S_2(\mathcal{E})$ would be an ellipse centered at 0, whose image under T must also be an ellipse, because it is mapped by the similarity S_1 to an ellipse. This contradicts Theorem 1, so $T_{w,r}(\mathcal{E})$ is not an ellipse. ∎

So, central inversions have been ruled out as possible symmetries of the ellipse $\mathcal{E}$ (Question **II**), or as maps from $\mathcal{E}$ to $\mathcal{E}'$ (Question **III**), and Corollary 2 could be restated as "no hippopede is an ellipse." Theorem 1 and Corollary 2 do not address all possible images of the ellipse under inversions. The question of when such an image is convex was solved by [34]. In general, the image of $\mathcal{E}$ under a Möbius transformation T is not an ellipse, limaçon, or elliptic lemniscate, but it is always some real quartic curve (there is some nonzero real polynomial p of degree at most 4 so that if $X+iY \in T(\mathcal{E})$, then $p(X,Y)=0$).

Instead of attempting any complete answer to Question **I**, we just introduce another bit of terminology.

Definition 3. A "real biquadratic curve" is a plane curve that satisfies an implicit equation of the form

$$c_{22}Z^2\bar{Z}^2 + c_{21}Z^2\bar{Z} + c_{12}Z\bar{Z}^2 + c_{20}Z^2 + c_{11}Z\bar{Z} + c_{02}\bar{Z}^2 + c_{10}Z + c_{01}\bar{Z} + c_{00} = 0,$$

where the complex coefficients satisfy $c_{jk} = \overline{c_{kj}}$ (so, in particular, c_{jj} is real, and the LHS is real-valued for all $Z \in \mathbb{C}$).

The biquadratic curves, considered by [11], [22], [20], [23], [24], and [31], form a subclass of the quartic curves, including the Ovals of Descartes:

$$m|Z-C| + n|Z-D| = 1, \text{ for } m,n \in \mathbb{R},\ \ C,D \in \mathbb{C}$$

and Cassini:

$$|Z-C| \cdot |Z-D| = k^2,\ \ k \in \mathbb{R},$$

and some real cubic curves. The real conics are real biquadratics with $c_{22} = c_{21} = c_{12} = 0$. For the ellipse, Equation (3) can be written in terms of complex coordinates, $Z = X+iY$, $\bar{Z} = X - iY$:

$$\left(\frac{1}{4A^2} - \frac{1}{4B^2}\right)Z^2 + \left(\frac{1}{4A^2} - \frac{1}{4B^2}\right)\bar{Z}^2 + \left(\frac{1}{2A^2} + \frac{1}{2B^2}\right)Z\bar{Z} - 1 = 0. \quad (7)$$

Applying a Möbius transformation to Equation (7) gives a real biquadratic, and in general, the images of real conics under Möbius transformations form a proper subclass of the real biquadratics. References [11], [20], [23], and [24] use the term "nodal biquadratic" to refer to inversive images of ellipses and hyperbolas, and "cuspidal biquadratic" to refer to inversive images of parabolas.

The "node" of an image of the hyperbola is, of course, the double point, and it is interesting that the ellipse was put into the same class by the previously mentioned papers on the algebraic properties of biquadratics. A more geometrically oriented article, [22], singles out the hyperbola and parabola as special cases of biquadratics, but not the ellipse, possibly because of its lack of a flashy singularity. (Some other articles on inverses of conics, which put more emphasis on the hyperbola than the ellipse, are [18] and [26].) The last section of this article shows that the point at infinity, which inverted to the isolated point observed in (5), is a "node" of the ellipse, where a certain self-intersection can be seen using complex projective coordinates. Some old-fashioned terminology ([7], [10]) classifies nodes, calling a point with two real tangent lines a "crunode," and a point which is isolated in the real plane, but with two tangent lines in complex coordinates, an "acnode."

To begin to address Questions **II** and **III**, we first consider their analogues for the hyperbola and the parabola. If a hyperbola in $\mathbb{C}$ is mapped to some other hyperbola in $\mathbb{C}$ by a Möbius transformation, then its double point at infinity must be a fixed point, since the only double point in the image is also at infinity. Similarly, if the image of a parabola is a smooth curve in $\mathbb{C}$, then, since Möbius transformations take smooth (or singular) points of a curve to smooth (or singular) points, the cusp at ∞ must be fixed. We can conclude that only similarities can map a parabola onto another parabola, or a hyperbola onto another hyperbola, which must therefore have the same eccentricity.

Ellipses, on the other hand, have no "distinguished" points in a differential-topological sense: they are smoothly embedded in $\mathbb{C}$. The family of ellipses also has no "conformal" invariants, in the sense that any ellipse is equivalent to the unit circle under some conformal transformation. However, the next sections show that some points associated to the ellipse (the vertices and the node) are distinguished with respect to inversive geometry. This leads to complete answers to Questions **II** and **III**.

The vertices of an ellipse

The following Theorem is the answer to Question **III**, and it is given several different proofs. These different arguments represent different ways of thinking about both ellipses and inversive geometry, from the points of view of synthetic, analytic, and algebraic geometry. After a few Lemmas, the first proof involves only some elementary properties of curves and circles in inversive geometry.

Theorem 4. *Given ellipses $\mathcal{E}$, $\mathcal{E}' \subseteq \mathbb{C}$, if the Möbius transformation T maps $\mathcal{E}$ into $\mathcal{E}'$, then T is a similarity.*

In particular, such a transformation exists if and only if $\mathcal{E}$ and $\mathcal{E}'$ have the same eccentricity. Before proving the Theorem, we state a few Lemmas which are quite plausible, but given quick and elementary proofs anyway.

Lemma 5. *A circle $\mathcal{C}$ and a conic $\mathcal{L}$ which does not contain $\mathcal{C}$ can have at most four points of intersection, and at most two points of tangency.*

Proof. We recall that "circle" includes straight lines, but we take for granted that lines have the claimed properties. The circle, with finite radius $r > 0$ and center (U, V), admits a parametric equation of the form

$$P : \mathbb{R} \to \mathbb{R}^2 : t \mapsto \left(\frac{2rt}{1+t^2} + U, \frac{r(1-t^2)}{1+t^2} + V \right). \tag{8}$$

The image $P(\mathbb{R})$ covers the whole circle except one point, $(U, V - r)$. If this point happens to be an element of $\mathcal{L}$, there is some rotation transformation R which fixes the circle, so that $R((U, V - r)) \notin \mathcal{L}$. $R \circ P$ is still given by quadratic rational functions of t, so if necessary we can replace P by $R \circ P$ to get a parametrization of $\mathcal{C}$ which contains all the points of intersection. If $q(X, Y) = 0$ is a quadratic implicit equation for $\mathcal{L}$, the composition $(q \circ P)(t)$ is zero exactly at the points where $P(t)$ meets $\mathcal{L}$. Expanding $q \circ P$ gives a rational function $\frac{N(t)}{D(t)}$ whose denominator is never zero, and whose numerator has degree at most 4. Since $\lim_{t\to\infty} P(t)$ exists and is not in $\mathcal{L}$,

$$\lim_{t\to\infty} q(P(t)) = q(\lim_{t\to\infty} P(t)) \neq 0,$$

so N is not identically zero, and there are at most four points of intersection. A point of intersection $P(t_0)$ is a point of tangency exactly when $\frac{dP}{dt}(t_0)$ is orthogonal to the gradient, $(\nabla q)(P(t_0))$. By the chain rule,

$$\left(\frac{d}{dt}(q \circ P)\right)(t_0) = (\nabla q)(P(t_0)) \cdot \left(\frac{dP}{dt}(t_0)\right),$$

so the curve is tangent to $\mathcal{L}$ when t_0 is a root of both $(q \circ P)(t)$ and $\frac{d}{dt}(q \circ P)(t) = \frac{N'(t)}{D(t)} - \frac{N(t)D'(t)}{(D(t))^2}$, which implies t_0 is a root of $N(t)$ and $N'(t)$, and therefore a double root of the quartic $N(t)$; there can be at most two such double roots. ∎

The next Lemma shows that, in Theorem 4, it is not necessary to assume that $\mathcal{E}'$ is non-circular.

Lemma 6. *Given any Möbius transformation T, the image $T(\mathcal{E})$ is not contained in any circle.*

Proof. Supposing that there is a circle $\mathcal{C}$ containing $T(\mathcal{E})$, the Möbius transformation T^{-1} takes $\mathcal{C}$ to another circle, $T^{-1}(\mathcal{C})$, which contains $\mathcal{E}$. $\mathcal{E}$ and $T^{-1}(\mathcal{C})$ meet at infinitely many points, so by the previous Lemma, $\mathcal{E}$ contains, and therefore is equal to, $T^{-1}(\mathcal{C})$, contradicting the assumption that $\mathcal{E}$ is not a circle. ∎

Figure 4: Bitangent circles (dashed) with a common point of tangency.

Lemma 7. *If a Möbius transformation T satisfies $T(\mathcal{E}) \subseteq \mathcal{E}'$, then $T(\mathcal{E}) = \mathcal{E}'$.*

Proof. There are many quick proofs of the fact that T restricted to $\mathcal{E}$ is "onto" $\mathcal{E}'$, using some topology. One argument is that if $T : \mathcal{E} \to \mathcal{E}'$ were not onto, then $T : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ would be a homeomorphism which takes the disconnected complement of $\mathcal{E}$ to the connected complement of $T(\mathcal{E})$. Another proof uses the Borsuk-Ulam theorem for topological 1-spheres ([17], §5.9): any continuous map from $\mathcal{E}$ to a proper subset of $\mathcal{E}'$ cannot be one-to-one. Since the restriction $T : \mathcal{E} \to \mathcal{E}'$ is one-to-one and continuous, it must be onto. ∎

First Proof of Theorem 4. If $T(\mathcal{E}) \subseteq \mathcal{E}'$, and R is a reflection symmetry of $\mathcal{E}'$, then $R \circ T$ also takes $\mathcal{E}$ into $\mathcal{E}'$. So, without loss of generality we can assume T is a direct transformation of the form (1), and we want to show that $c = 0$.

An ellipse has four points lying on its axes of reflective symmetry. We call these points "vertices," and any similarity transformation of the plane takes the vertices of an ellipse to the vertices of the image ellipse.

Since the similarities form a subgroup of the group of linear fractional transformations, we can also assume that $\mathcal{E}$ is in standard position, Equation (3). Its vertices are the points $(\pm A, 0)$ and $(0, \pm B)$, or in complex coordinates, $\pm A$ and $\pm iB$.

Let $\mathcal{C}$ be a circle tangent to an ellipse $\mathcal{L}$ at a point Z_0. We say that $\mathcal{C}$ is *bitangent* at Z_0 if there is some point $Z_1 \neq Z_0$ so that $\mathcal{C}$ is tangent to $\mathcal{L}$ at Z_1. From Figure 4, it is easy to see that at any point Z_0 which is not a vertex, there are at least two bitangent circles: one interior, centered on the major axis, and one exterior, centered on the minor axis.

At a vertex of an ellipse, there is only one bitangent circle: if the other point of tangency were any point besides the opposite vertex, then by the reflection symmetry of the ellipse and circle, there would be a third point of tangency, contradicting Lemma 5.

Let Z_0 be a vertex of $\mathcal{E}$, but, suppose toward a contradiction that $T(Z_0)$ is not a vertex in $\mathcal{E}'$ (since $\mathcal{E}'$ is not a circle, not every point is a vertex). Then, at $T(Z_0)$, $\mathcal{E}'$ has two bitangent circles, and by Lemma 7, there are distinct points $Z_1, Z_2 \in \mathcal{E}$ so that these circles are tangent to $\mathcal{E}'$ at $T(Z_0)$ and $T(Z_1)$, and $T(Z_0)$ and $T(Z_2)$, respectively. Since the linear fractional transformation T^{-1} preserves circles and the tangency of curves, these two bitangent circles of $\mathcal{E}'$ at $T(Z_0)$ are transformed to two bitangent circles of $\mathcal{E}$ at Z_0, contradicting the assumption that Z_0 is a vertex of $\mathcal{E}$. So, $T(A)$, $T(-A)$, $T(iB)$, and $T(-iB)$ must be the vertices of $\mathcal{E}'$.

The images of the axes of $\mathcal{E}$ must therefore be circles orthogonal to the ellipse $\mathcal{E}'$ at its vertices. Again using the fact that $\mathcal{E}'$ is non-circular, an easy argument shows that they must coincide with the axes of $\mathcal{E}'$; that is, the circles are straight lines.

Now suppose by way of contradiction that $c \neq 0$. By the previous paragraph, the point $-d/c$ which T maps to ∞ must lie on both the major and minor axes of $\mathcal{E}$, so $d = -d/c = 0$. Hence T has the form $T(Z) = a + b/Z$, with $b \neq 0$. Since $Z \mapsto a + bZ$ is a similarity, the image of $\mathcal{E}$ under the map $Z \mapsto 1/Z$ must be an ellipse. However, this contradicts Theorem 1. ∎

Bitangent circles of ellipses are also considered by [19], and [19] and [22] call the vertices the "apses" of the curve. From a calculus point of view, the vertices of an ellipse are the four points where its curvature has a local extreme value. This suggests a more analytic (or differential-geometric) approach to Theorem 4, which might be less elementary because it involves higher derivatives instead of only tangent circles.

Second Proof of Theorem 4. The key point in the first proof was that Möbius transformations must take the vertices of one ellipse to the vertices of another. More generally, the "vertices" of any smooth curve in the plane are points where the curvature has a local extreme value ([30]). We won't prove it here, but it is known ([22], [4]) that a Möbius transformation takes the vertices of a curve to the vertices of its image. The second half of the first proof proceeds without changes. ∎

The following Corollary is just the special case of Theorem 4 where $\mathcal{E} = \mathcal{E}'$. It appears as a statement without proof in [35], and it answers Question **II**.

Corollary 8. *The only Möbius transformations T which are symmetries of an ellipse ($T(\mathcal{E}) \subseteq \mathcal{E}$) are similarities.* ∎

The Corollary could be proved directly, using the methods of either of the above proofs of Theorem 4. If one is willing to assume Corollary 8, then it could be used to prove the more general Theorem 4.

Third Proof of Theorem 4. Let $C_1 \in \mathbb{C}$ be the center of $\mathcal{E}$, and C_2 the center of $\mathcal{E}'$, so that $H(Z) = -Z + 2C_2$ is the half-turn around the center of $\mathcal{E}'$. Assuming, as in the first proof, that T is given by (1), the composition $T^{-1} \circ H \circ T$ is of the form

$$Z \mapsto \frac{(2C_2cd - ad - bc)Z + 2d(C_2d - b)}{2c(a - C_2c)Z + ad + bc - 2C_2cd}.$$

Since $(H \circ T)(Z) \in \mathcal{E}'$ for $Z \in \mathcal{E}$, and $T^{-1}(W) \in \mathcal{E}$ for $W \in \mathcal{E}'$ by Lemma 7, $T^{-1} \circ H \circ T$ is a symmetry of $\mathcal{E}$, so Corollary 8 applies. The coefficient $2c(a - C_2c)$ must be zero, so either $c = 0$ (proving the claim of the Theorem), or $c \neq 0$ and $C_2 = \frac{a}{c} = T(\infty)$. In this second case, the composition becomes $\frac{(ad-bc)Z + 2d(\frac{ad-bc}{c})}{bc-ad} = -Z - 2\frac{d}{c}$, so this symmetry of $\mathcal{E}$ is the half-turn centered at $C_1 = -\frac{d}{c} = T^{-1}(\infty)$.

To show that the $c \neq 0$ case leads to a contradiction, note that T can be written as a composition of a translation $T_1(Z) = Z - C_1$, a map $T_2(Z) = \frac{1}{Z}$, and a similarity transformation, T_3:

$$\frac{aZ+b}{cZ+d} = (T_3 \circ T_2 \circ T_1)(Z) = \frac{1}{Z - (-\frac{d}{c})} \cdot \frac{bc - ad}{c^2} + \frac{a}{c}.$$

$T_1(\mathcal{E})$ and $T_3^{-1}(\mathcal{E}')$ are both ellipses centered at the origin, and related by the transformation T_2. However, this contradicts Theorem 1, so the case $c \neq 0$ is ruled out. ∎

Theorem 4 can also be applied to curves which are images of ellipses under Möbius transformations, such as the elliptic lemniscates or limaçons.

Corollary 9. *Curves of the form $T(\mathcal{E})$ and $T'(\mathcal{E}')$ in $\mathbb{C} \cup \{\infty\}$ are related by some Möbius transformation if and only if $\mathcal{E}$ and $\mathcal{E}'$ have the same eccentricity.* ∎

The node of an ellipse

Our final approach to Theorem 4 is to find yet another way to describe the ellipse, this time as the "real part" of a complex curve. We should point out now that there is more than one way to add "points at infinity" or "complex points" to the real plane of Euclidean geometry. One method, which is used by many both old and new books on plane curves (for example, [10]), is to put the real plane inside the "complex projective plane." This coordinate system introduces a "line at infinity," which contains some so-called "circular points at infinity," and authors using these coordinates call our real biquadratic curves "bicircular quartics." Another approach, which is described in this section, is to use the product of two Riemann spheres, where ordered pairs of (extended) complex numbers form a coordinate system which contains, as a "real diagonal" subset, an image of the inversive plane. So, our construction is somewhat different from the complex equations for ellipses described by [12], or from projective coordinate systems involving a "line at infinity." See [23] for a comparison of the two coordinate systems.

As in the previous section, it is useful to put the ellipse in standard position, (3), but it is also convenient to choose just one ellipse of each eccentricity, by scaling an ellipse in standard position so that its foci are at $\pm 1 + 0i$. This family of ellipses depends on only one real parameter, instead of two (the radii A and B).

A nice way to write a parametric equation for such an ellipse is

$$P_k : \mathbb{R} \to \mathbb{C} : u \mapsto \frac{2(k^2+1)u + i(k^2-1)(1-u^2)}{2k(1+u^2)}, \tag{9}$$

for $k > 1$. This is just Equation (6), with coefficients chosen so the Euclidean invariants of the ellipse are rational functions of k: the major radius is $A = \frac{k^2+1}{2k} > 1$, the minor radius is $B = \frac{k^2-1}{2k} > 0$, the eccentricity is $e = \frac{2k}{k^2+1}$, $0 < e < 1$, and Equation (7) in complex coordinates becomes

$$\mathcal{E} = \{Z : Z^2 + \overline{Z}^2 - \left(\frac{k^4+1}{k^2}\right) Z\overline{Z} + \frac{(k^4-1)^2}{4k^4} = 0\}. \tag{10}$$

What might seem to be a straightforward approach to Theorem 4 or Corollary 8 would be to use the composite $T \circ P_k$ to parametrize $T(\mathcal{E})$, and then see if it satisfies the implicit equation of some ellipse — this was the technique of Theorem 1 and Lemma 5. (For more about curves in the inversive plane parametrized by rational functions, see [7], [19], and [5].) However, for an arbitrary linear fractional transformation T, plugging $(T \circ P_k)(u)$ into an implicit equation, say (10), gives such a complicated expression that this method is impractical.

Instead, the maps taking one ellipse to another can be characterized using the isolated singular point, which was observed in Equation (5). This "node" was useful in detecting which Möbius transformations take one hyperbola to another, but for an ellipse, the singularity of the implicit equation is "at infinity," and not in the image of the real parametric map, or even its closure. Both of these properties suggest that it could be useful to view Theorem 4 in the setting of complex projective geometry.

So, we very briefly review the homogeneous coordinate system. Define the projective space $\mathbb{C}P^n$ as the set of one-dimensional subspaces of $\mathbb{C}^{n+1}$. Denote by $[z_0 : \ldots : z_n]$ the line spanned by the non-zero vector $(z_0, \ldots, z_n)$, so for a non-zero scalar $\lambda \in \mathbb{C}$, $[\lambda \cdot z_0 : \ldots : \lambda \cdot z_n] = [z_0 : \ldots : z_n]$.

The map $\mathbb{C}^n \to \mathbb{C}P^n$ defined by $(z_1, \ldots, z_n) \mapsto [1 : z_1 : \ldots : z_n]$ is one-to-one, and its image is an example of an "affine neighborhood." The complement of this affine neighborhood is the set $\{[0 : z_1 : \ldots : z_n]\}$, a complex projective space of one lower dimension. The complex number line we have been working with in the previous sections, which contained $\mathcal{E}$, is the affine neighborhood $\{[1 : Z]\}$ in $\mathbb{C}P^1$, the Riemann sphere. The point ∞ is the element $[0 : 1]$.

A map $F : \mathbb{C}^{m+1} \to \mathbb{C}^{n+1}$ induces a "well-defined" map $F : \mathbb{C}P^m \to \mathbb{C}P^n$ if it takes lines to lines, i.e., for every non-zero vector $\vec{z} \in \mathbb{C}^{m+1}$, F has the following two properties: first, $F(\vec{z}) \neq \vec{0}$, and second, for any non-zero scalars λ_1, $\lambda_2 \in \mathbb{C}$, there exist non-zero scalars μ_1, $\mu_2 \in \mathbb{C}$ so that

$$\mu_1 \cdot F(\lambda_1 \cdot \vec{z}) = \mu_2 \cdot F(\lambda_2 \cdot \vec{z}).$$

As an example of a well-defined map from $\mathbb{C}P^1$ to $\mathbb{C}P^1$, a linear fractional transformation of the form (1) acts as a linear map on the homogeneous coordinates:

$$T : \mathbb{C}P^1 \to \mathbb{C}P^1 : \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix} \mapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix} \cdot \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix}. \qquad (11)$$

There is no need to treat the point at ∞ separately in this coordinate system; for example, $T([0 : 1]) = [c : a]$, which equals $[1 : a/c]$ if $c \neq 0$, consistent with the earlier observation $T(\infty) = a/c$.

Another well-defined map is a complex homogeneous version of Equation (9), depending on the real parameter $k > 1$:

$$P_k : [z_0 : z_1] \mapsto [2k(z_0^2 + z_1^2) : 2(k^2 + 1)z_0 z_1 + i(k^2 - 1)(z_0^2 - z_1^2)],$$

which, as in the previous example, takes $\mathbb{C}P^1$ to $\mathbb{C}P^1$, but P_k is not one-to-one. Points of the form $[1 : u]$, for $u \in \mathbb{R}$, have an image in the affine neighborhood $\{[Z_0 : Z_1] : Z_0 \neq 0\}$, so this map P_k restricts to the one-to-one, real parametric curve (9). The homogeneous coordinates again conveniently handle the point at infinity: $P_k([0 : 1]) = [2k : -i(k^2 - 1)] = [1 : -iB]$, as in (6).

At this point, we need some terminology from algebraic geometry, and although some of the rigorous definitions are technical, terms like "irreducible" and "dimension" correspond to intuitive concepts, and we refer the reader to [6] for an accessible account of the details. For the rest of this section, the term "dimension" refers to the complex dimension of a complex projective algebraic variety, as defined in [6] §9.3.

The following maps are needed to describe a "complexification" of the ellipse: the involution of $\mathbb{C}P^1$ induced by complex conjugation,

$$C([Z_0 : Z_1]) = ([\bar{Z}_0 : \bar{Z}_1]),$$

the "totally real diagonal" embedding,

$$\begin{aligned} \Delta : \mathbb{C}P^1 &\rightarrow \mathbb{C}P^1 \times \mathbb{C}P^1 \\ [Z_0 : Z_1] &\mapsto ([Z_0 : Z_1], C([Z_0 : Z_1])), \end{aligned}$$

and the Segre embedding,

$$\begin{aligned} s : \mathbb{C}P^1 \times \mathbb{C}P^1 &\rightarrow \mathbb{C}P^3 \\ ([Z_0 : Z_1], [W_0 : W_1]) &\mapsto [Z_0W_0 : Z_0W_1 : Z_1W_0 : Z_1W_1]. \end{aligned}$$

The fixed point set of C is the real projective line, $\mathbb{R}P^1 \subseteq \mathbb{C}P^1$. The image of Δ is exactly the real submanifold

$$\Delta(\mathbb{C}P^1) = \{(Z, W) : W = C(Z)\}.$$

The image of s is exactly the set

$$s(\mathbb{C}P^1 \times \mathbb{C}P^1) = \{[\zeta_0 : \zeta_1 : \zeta_2 : \zeta_3] : \zeta_0\zeta_3 - \zeta_1\zeta_2 = 0\},$$

and this is a smooth hypersurface ([6] §8.6) in $\mathbb{C}P^3$, and therefore an irreducible variety of dimension 2 ([6] §9.4).

The following parametric map, G_k, "complexifies" $\mathcal{E} \subseteq \mathbb{C}P^1$, in the sense that its image and its target space, $G_k(\mathbb{C}P^1) \subseteq \mathbb{C}P^1 \times \mathbb{C}P^1$, are both complex varieties, which contain, respectively, as real subvarieties, the images of the real ellipse and the inversive plane, $\Delta(\mathcal{E}) \subseteq \Delta(\mathbb{C}P^1)$. The use of complex curves described by pairs of homogeneous coordinates to study real curves in the inversive plane is an old idea ([11], [23]), but we give a perhaps more modern presentation, following [31].

$$\begin{aligned}
G_k : \mathbb{C}P^1 &\to \mathbb{C}P^1 \times \mathbb{C}P^1 \\
[z_0 : z_1] &\mapsto (P_k([z_0 : z_1]), (C \circ P_k \circ C)([z_0 : z_1]) \\
&= ([2k(z_0^2 + z_1^2) : 2(k^2+1)z_0 z_1 + i(k^2-1)(z_0^2 - z_1^2)], \\
&\quad\ [2k(z_0^2 + z_1^2) : 2(k^2+1)z_0 z_1 - i(k^2-1)(z_0^2 - z_1^2)]).
\end{aligned}$$

This map is well-defined because P_k is well-defined. Note that if $[z_0 : z_1] \in \mathbb{R}P^1$, then $G_k([z_0 : z_1])$ is in the image of Δ, and $\Delta(\mathcal{E}) = G_k(\mathbb{R}P^1)$. So, the image of G_k contains infinitely many points, but it is not an onto map: for example, $([1 : 0], [0 : 1])$ is not in the image.

The interesting property of G_k is that it is one-to-one, except for a double point. The map $P_k : \mathbb{C}P^1 \to \mathbb{C}P^1$ is generically two-to-one: for any $z = [z_0 : z_1]$, the point

$$z' = [-i(1-k^2)z_0 + (1+k^2)z_1 : (1+k^2)z_0 + i(1-k^2)z_1]$$

has the same image, $P_k(z) = P_k(z')$. Similarly, the map $C \circ P_k \circ C$ is generically two-to-one, where

$$z'' = [i(1-k^2)z_0 + (1+k^2)z_1 : (1+k^2)z_0 - i(1-k^2)z_1]$$

satisfies $(C \circ P_k \circ C)(z) = (C \circ P_k \circ C)(z'')$. If $G_k(z) = G_k(w)$, then either $w = z$, or $w = z' = z''$, and the only solutions of $z' = z''$ are $[1 : i]$ and $[1 : -i]$, so, independent of k,

$$G_k([1 : i]) = G_k([1 : -i]) = ([0 : 1], [0 : 1]).$$

This "double point" of the complex curve G_k corresponds to the real ellipse's isolated point at infinity, which inverted to 0 in Equation (5).

The implicit equation of the curve $G_k(\mathbb{C}P^1)$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$ is given in the bi-homogeneous coordinates $([Z_0 : Z_1], [W_0 : W_1])$ by

$$Z_1^2 W_0^2 + W_1^2 Z_0^2 - \left(\frac{k^4+1}{k^2}\right) Z_1 W_1 Z_0 W_0 + \frac{(k^4-1)^2}{4k^4} Z_0^2 W_0^2 = 0. \qquad (12)$$

This is just Equation (10), with $\bar{Z}$ replaced by W_1, and made bi-homogeneous by introducing Z_0 and W_0. Equation (12) is also one of the "normal forms" of [11], representing a "Species IIa" biquadratic curve.

Even if the $[Z_0 : Z_1]$ and $[W_0 : W_1]$ variables separately undergo complex linear transformations (as in (11)), the above expression remains quadratic in each pair (explaining the term "biquadratic"). We recover a real curve, as in Definition 3, when $[Z_0 : Z_1]$ and $[W_0 : W_1]$ are replaced by $[1 : Z]$ and $[1 : \bar{Z}]$ — this is the intersection of the complex biquadratic with the product of affine neighborhoods and the real diagonal set $\{W = C(Z)\}$.

The composition $s \circ G_k : \mathbb{C}P^1 \to \mathbb{C}P^3$ is well-defined, so the image of this parametrized space curve is an irreducible variety ([6] §8.5 — comments on the irreducibility appear after the last proof). It has infinitely many points (it includes the image of the ellipse, $(s \circ \Delta)(\mathcal{E})$) but since G_k is not onto, $(s \circ G_k)(\mathbb{C}P^1)$ is not all of the Segre variety $s(\mathbb{C}P^1 \times \mathbb{C}P^1)$, so it has dimension 1 ([6] §9.4). Since s is one-to-one, the map $s \circ G_k$ has exactly one double point, at $[0 : 0 : 0 : 1]$ in $\mathbb{C}P^3$. The existence and uniqueness of the singular point of the image could be double-checked by a long but straightforward computation with its implicit equations.

Fourth Proof of Theorem 4. We are actually only going to prove Corollary 8, for linear fractional transformations T, but it was already shown that this is enough. Now that the geometry of the complexified ellipse has been considered, the action of the direct Möbius transformations can also be complexified.

Given a linear (fractional) transformation as in (11), define

$$\begin{aligned} T_c : \mathbb{C}P^1 \times \mathbb{C}P^1 &\to \mathbb{C}P^1 \times \mathbb{C}P^1 \\ (Z, W) &\mapsto (T(Z), (C \circ T \circ C)(W)). \end{aligned}$$

Note that T_c is well-defined, one-to-one, and algebraic (involving only Z and W, and not $\bar{Z}$ or $\bar{W}$), and $T_c \circ \Delta = \Delta \circ T$. The composition $s \circ T_c \circ G_k$ has exactly one double point, and, as previously, its image is an irreducible, one-dimensional subvariety of $\mathbb{C}P^3$. If T is a symmetry of $\mathcal{E}$ as in the hypothesis of Corollary 8, and $Z \in \mathcal{E}$, then $T(Z) \in \mathcal{E}$, and $(T_c \circ \Delta)(Z) = (\Delta \circ T)(Z) \in \Delta(\mathcal{E})$. So, the image of $s \circ T_c \circ G_k$ contains $(s \circ \Delta)(\mathcal{E})$, an infinite set which is also a subset of the image of $s \circ G_k$. Since the intersection of these two images is a variety of positive dimension, the images must coincide ([6] §9.4). In particular, the images of G_k and $T_c \circ G_k$ must have the double point at the same place,

$$([0:1],[0:1]) = T_c([0:1],[0:1]) = (T([0:1]),(C \circ T \circ C)([0:1])),$$

which implies T fixes the point $[0:1]$.

The conclusion is that $c = 0$, so T is a similarity. ∎

We can also conclude from this construction that, since T_c fixes $\Delta(\mathbb{C}P^1)$, any Möbius transformation of an ellipse, $T(\mathcal{E})$, has a real biquadratic implicit equation with an acnode at $T(\infty)$.

The crucial step in the above proof was the irreducibility of the curves, which allowed us to conclude that if the two complex curves intersect on the real ellipse, then they coincide everywhere. So, we needed the theorem from [6], which states that a well-defined polynomial map from one complex projective space to another has an image equal to an irreducible variety. This is a great way to establish the irreducibility of a locus, since checking a parametric map is "well-defined" is often easier than attempting to find, and then factor, an implicit equation. We already saw, in Equation (5), that this theorem does not hold for real parametric maps, where the image might be contained in, but not equal to, a real variety.

The last interesting point is to see how this same argument fails when applied to circles, which have many symmetries that are not similarities. If we extend the parametric equation of a circle (8) to complex homogeneous coordinates, to get

$$P : [z_0 : z_1] \mapsto [z_0^2 + z_1^2 : 2rz_0z_1 + U(z_0^2 + z_1^2) + i(r(z_0^2 - z_1^2) + V(z_0^2 + z_1^2))],$$

it is not well-defined: every point on the line spanned by $(1, -i)$ is mapped to $(0, 0)$. In fact, the components have a common factor:

$$P : [z_0 : z_1] \mapsto [(z_0 + iz_1)(z_0 - iz_1) : ((U + V + ir)z_0 + (r + i(U + V))z_1)(z_0 - iz_1)],$$

and if we cancel off the factor $z_0 - iz_1$, then the map $P : \mathbb{C}P^1 \to \mathbb{C}P^1$ extends to a linear fractional transformation. So, the circle does not have a node, even in this complex coordinate system.

References

[1] E. Beutel, *Algebraische Kurven II: Theorie und Kurven Dritter und Vierter Ordnung*, G. J. Göschen'sche Verlagshandlung, Leipzig, 1911.

[2] J. Booth, *Researches on the geometrical properties of elliptic integrals*, Philosophical Transactions of the Royal Society of London, **142** (1852), 311–416, and **144** (1854), 53–69.

[3] J. Booth, *A Treatise on Some New Geometrical Methods*, Longmans, Green, Reader, and Dyer, London, Vol. I, 1873, and Vol. II, 1877.

[4] G. Cairns and R. Sharpe, *On the inversive differential geometry of plane curves*, Enseign. Math. (2) **36** (1990), 175–196.

[5] A. Coffman, *Real congruence of complex matrix pencils and complex projections of real Veronese varieties*, Linear Algebra and its Applications (to appear).

[6] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992.

[7] R. Deaux, *Introduction to the Geometry of Complex Numbers*, transl. from the revised French ed. by H. Eves, Ungar Pub. Co., New York, 1957.

[8] R. Ferréol, *Encyclopédie des Formes Mathématiques Remarquables*, `perso.club-internet.fr/rferreol/encyclopedie/courbes2d/courbes2d.shtml`

[9] K. Fladt, *Analytische Geometrie Spezieller Ebener Kurven*, Akademische Verlagsgesellschaft, Frankfurt, 1962.

[10] H. Hilton, *Plane Algebraic Curves*, Oxford, 1920.

[11] E. Kasner, *The invariant theory of the inversion group: geometry upon a quadric surface*, Trans. Amer. Math. Soc. (4) **1** (1900), 430-498.

[12] K. Kendig, *Stalking the wild ellipse*, Amer. Math. Monthly (9) **102** (1995), 782–787.

[13] J. D. Lawrence, *A Catalog of Special Plane Curves*, Dover, 1972.

[14] X. Lee, *A Visual Dictionary of Special Plane Curves*, `xahlee.org/SpecialPlaneCurves_dir/specialPlaneCurves.html`

[15] E. Lockwood, *A Book of Curves*, Cambridge, 1961.

[16] G. Loria, *Spezielle Algebraische und Transzendente Ebene Kurven, Theorie und Geschichte*, Vol. I, transl. F. Schütte, Teubner, Leipzig, 1910.

[17] W. Massey, *Algebraic Topology: an Introduction*, GTM **56**, Springer, New York, 1967.

[18] R. Mathews, *Concyclic points on an equilateral hyperbola and on its inverses*, Amer. Math. Monthly (9) **29** (1922), 347–348, and (4) **30** (1923), 198.

[19] F. Morley and F. V. Morley, *Inversive Geometry*, Chelsea, New York, 1954.

[20] F. Morley and B. Patterson, *On algebraic inversive invariants*, American J. of Math. (2) **52** (1930), 413–424.

[21] J. O'Connor, E. Robertson, and B. Soares, Famous Curves Index, MacTutor History of Mathematics Archive, `www-history.mcs.st-and.ac.uk/history/Curves/Curves.html`

[22] B. Patterson, *The differential invariants of inversive geometry*, American J. of Math. (4) **50** (1928), 553-568.

[23] B. Patterson, *The inversive plane*, Amer. Math. Monthly (9) **48** (1941), 589–599.

[24] B. Patterson, *Jacobian circles of the biquadratic*, Amer. Math. Monthly (5) **49** (1942), 304–309.

[25] Proclus, *A Commentary on the First Book of Euclid's Elements*, transl. G. Morrow, Princeton, 1970.

[26] E. Rice, *On the foci of plane algebraic curves with applications to symmetric cubic curves*, Amer. Math. Monthly (10) **43** (1936), 618-630.

[27] S. Richardson, *Some Hele-Shaw flows with time-dependent free boundaries*, J. Fluid Mech. **102** (1981), 263–278.

[28] H. Schmidt, *Die Inversion und ihre Anwendungen*, Verlag von R. Oldenbourg, München, 1950.

[29] H. Schupp and H. Dabrock, *Höhere Kurven, Situative, Mathematische, Historische und Didaktische Aspekte*, Lehrbücher und Monographien zur Didaktik der Mathematik **28**, Bibliographisches Institut, Mannheim, 1995.

[30] D. Struik, *Lectures on Classical Differential Geometry*, 2nd ed., Dover, New York, 1961.

[31] S. Webster, *Double valued reflection in the complex plane*, Enseign. Math. (2) **42** (1996), 25–48.

[32] H. Wieleitner, *Theorie der Ebenen Algebraischen Kurven Höherer Ordnung*, G. J. Göschen'sche Verlagshandlung, Leipzig, 1905.

[33] H. Wieleitner, *Spezielle Ebene Kurven*, G. J. Göschen'sche Verlagshandlung, Leipzig, 1908.

[34] J. Wilker, *When is the inverse of an ellipse convex?*, Utilitas Math. **17** (1980), 45–50.

[35] J. Wilker, *Möbius equivalence and Euclidean symmetry*, Amer. Math. Monthly (4) **91** (1984), 225–247.

[36] R. Yates, *Curves and Their Properties*, NCTM Classics in Mathematics Education, 1974.

[37] C. Zwikker, *The Advanced Geometry of Plane Curves and their Applications*, Dover, New York, 1963.