Degeneracy Loci in CR Geometry

.1 Introduction

The geometry of complex subspaces contained within real subspaces of a complex vector space is described in detail in Section 2. This leads to an analysis of real subbundles of a complex vector bundle in Section 3, using a generalized Gauss map. These geometric constructions lend themselves to a cohomological formalism of Thom and [Porteous], and consequences of the theory are examined in Section 4. A general formula unifies relations described by [Wells], [Lai], [W_{86}], [HL_{93}], and [Domrin] among the pontrjagin classes of a real oriented m-subbundle T, the chern classes of the ambient complex n-bundle F, and chern numbers of complex bundles over the CR-singular sets in a smooth base space M. The initial study of the grassmannian as a universal case shows that the codimensions appearing in these cohomological formulas are indeed the numbers "expected" in geometrically generic configurations.

The theory of degeneracy loci is classical in algebraic geometry ([Fulton_{84}]); a differential-topological approach is considered in a recent article of of Harvey and Lawson [HL_{95}]. In particular, [HL_{95}] also arrives at a more general version of Theorem 4.5, and phrases it in terms of singular differential forms and approximating families. Here, Theorem 4.5 is generalized in different directions, requiring that degeneracy loci be, in a certain sense, submanifolds arising from transverse intersections. The examples make explicit how some formulas regarding complex tangencies follow from [HL_{95}]. Some new formulas (Theorems 4.12, 4.18) follow from theorems of [HT] and [Pragacz] on chern numbers of kernel bundles, and of [Fulton_{91}] on degeneracy loci for flags of bundles.

The theory is applied to smooth maps between complex bundles, and the loci where such maps are complex-linear on subspaces. Corollary 5.2 generalizes a lemma of [EW] and the invariants for pairs of complex structures of [HL_{93}] and [HL_{95}].

.2 Planes in Planes in Planes

Consider a complex vector space F^n as an oriented real vector space $F_{\mathbb{R}}$ of real dimension $2n$, with a complex structure J, $J^2 = -1$ and a compatible positive definite metric g. The compatibility condition is the equation $g(X,Y) = g(JX,JY)$, and if (J,g) does not already have this property, then g can be replaced by a positive multiple of $g(X,Y)+g(JX,JY)$. For $0 \le m \le 2n$, denote by $SG(m,F)$ the manifold of all oriented real-linear m-subspaces in $F_{\mathbb{R}}$. $SG(m,F)$ has real dimension $m(2n-m)$.

Definition 2.1 $V \in SG(m,F)$ has (exactly) j complex directions if $\dim_{\mathbb{R}} V \cap JV \ge 2j$ $(=2j)$. Define $j_0 = \max\{0, m-n\}$ and $k = \llbracket m/2 \rrbracket$.

Lemma 2.2

$$j \in \{j_0, ..., k\} \iff \exists V \in SG(m,F) \text{ with exactly } j \text{ complex directions.}$$

PROOF: Note that $H := V \cap JV$ is the largest J-invariant subspace of $F_{\mathbb{R}}$ contained in V and $J|_H$ is a complex structure. The inequality $j_0 \le j \le k$ follows from simple linear algebra. ∎

Definition 2.3 If V has $j = k = m/2$ complex directions, then V is J-invariant, and is called a "complex j-subspace" of F if its orientation is compatible with its induced complex structure, or an "anticomplex j-subspace" otherwise.

Definition 2.4 In the case where V has exactly zero complex directions, V is said to be totally real. V with exactly j_0 complex directions will be called CR-regular.

For $j_0 \le j \le k$, define $D_j = \{V \in SG(m,F) \text{ with } j \text{ complex directions}\}$, and $C_j = \{V \in SG(m,F) \text{ with exactly } j \text{ complex directions}\}$; then

- $C_k = D_k$, and for $j_0 \le j \le k-1$, $C_j = D_j \setminus D_{j+1}$.
- $SG(m,F) = D_{j_0} \supseteq \ldots \supseteq D_j \supseteq \ldots \supseteq D_k$
- $SG(m,F)$ is the disjoint union $C_{j_0} \cup \ldots \cup C_j \cup \ldots \cup C_k$.

It is convenient to define $C_{k+1} = \emptyset$ and $D_j = SG(m,F)$ for $j < j_0$.

D_j is a subvariety of $SG(m,F)$, with singular locus C_{j+1}, and can be described as follows. Given a complex j-plane (an element of $\mathbb{C}G(j,n)$), one could consider the set of real m-planes containing it; this set forms a $SG(m-2j, n-2j)$. This process gives a bundle of grassmannians over $\mathbb{C}G(j,n)$, but if an m-plane has $j+1$ complex directions then it appears in this grassmann bundle more than once. D_j is the set obtained by making the appropriate identifications on this manifold. The smooth locus of D_j is C_j, which is a fibered manifold with base $\mathbb{C}G(j,n)$ and fiber a dense open subset of $SG(m-2j, n-2j)$. In particular, C_{j_0} is a dense open subset of $SG(m,F)$. If m is even, $C_k = \mathbb{C}G(k,n) \times S^0$, and if m is odd, then C_k has base $\mathbb{C}G(k,n)$ and fiber $SG(1, 2n-m+1) = S^{2n-m}$.

Lemma 2.5 A brief calculation gives the codimension of C_j in $SG(m,F)$,

$$\mathrm{codim}_{\mathbb{R}} C_j = 2j(n-m+j), \tag{1}$$

and that this number is ≥ 0 and strictly increasing with j. ∎

Example 2.6 $m = n = 2 \Rightarrow 0 \leq j \leq 1$. [Chern-Spanier] show that $SG(2, \mathbb{R}^4) = S^2 \times S^2 \subseteq \Lambda^2\mathbb{R}^4 = \mathbb{R}^6$, given by $x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2 = 1$. If the $\mathbb{R}^4$ has complex structure operator

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

then $C_1 = \mathbb{C}P^1 \times S^0$ is the parallel pair of 2-spheres given by $x_3 = \pm 1$. (cf also [Banchoff-Farris], [IO], and [Chen-Morvan])

Example 2.7 The metric on $F_{\mathbb{R}}$ gives the familiar diffeomorphism from $SG(m, F)$ to $SG(2n - m, F)$, where $V \mapsto V^\perp$. Consider $V \in C_j$:

$$\begin{aligned} \dim V \cap JV = 2j \quad &\Rightarrow \quad \dim V^\perp \cap JV = \dim JV^\perp \cap V = m - 2j \\ &\Rightarrow \quad \dim V^\perp \cap JV^\perp = (2n - m) - (m - 2j) = 2(n - m + j) \\ &\Rightarrow \quad V^\perp \in C'_{j'} \subseteq SG(m', F), \end{aligned}$$

where, denoting $j' = n - m + j$, $m' = 2n - m$, etc., $C'_{j'} = \{V \in SG(m', F)$ with exactly j' complex directions$\}$. The inequality $j'_0 \leq j' \leq k'$ follows from $j_0 \leq j \leq k$, and the codimension of $C'_{j'}$ in $SG(m', F)$ is $2j'(n - m' + j') = 2j(n - m + j)$. So, the diffeomorphism $\perp$ preserves the decomposition into C_j's, possibly re-indexing them.

The following table describes some configurations allowed by Lemma 2.2 and (1), arranged by the codimension of C_j in $SG(m, F)$.

j	m	n	codim	j	m	n	codim	j	m	n	codim
j_0	$0 \leq m \leq 2n$	≥ 0	0	1	$n-4$	≥ 6	10	1	$n-7$	≥ 9	16
1	n	≥ 2	2	5	$n+4$	≥ 6	10	2	$n-2$	≥ 6	16
1	$n-1$	≥ 3	4	1	$n-5$	≥ 7	12	4	$n+2$	≥ 6	16
2	$n+1$	≥ 3	4	2	$n-1$	≥ 5	12	8	$n+7$	≥ 9	16
1	$n-2$	≥ 4	6	3	$n+1$	≥ 5	12	1	$n-8$	≥ 10	18
3	$n+2$	≥ 4	6	6	$n+5$	≥ 7	12	3	n	≥ 6	18
1	$n-3$	≥ 5	8	1	$n-6$	≥ 8	14	9	$n+8$	≥ 10	18
2	n	≥ 4	8	7	$n+6$	≥ 8	14				
4	$n+3$	≥ 5	8						...		

Another description of D_j detects J-invariant subspaces as subspaces of a certain kernel. Fix $V \in SG(m, F)$, containing $H = V \cap JV$, and denote the inclusion $\iota : V \hookrightarrow F_{\mathbb{R}}$. Complexifying (applying $\otimes_{\mathbb{R}}\mathbb{C}$ to) H, V, and $F_{\mathbb{R}}$ extends ι to $\iota_{\mathbb{C}}$, and J to $J_{\mathbb{C}} : F_{\mathbb{R}} \otimes \mathbb{C} \to F_{\mathbb{R}} \otimes \mathbb{C}$. $F_{\mathbb{R}} \otimes \mathbb{C}$ splits into the $+i$ and $-i$ eigenspaces of $J_{\mathbb{C}}$, as $F \oplus \overline{F}$. The "**p**rojection onto the **h**olomorphic part" $\varphi : F \oplus \overline{F} \to F$ is given by the operator $(\frac{1}{2} - \frac{i}{2}J_{\mathbb{C}})$. Since $H \otimes \mathbb{C} \hookrightarrow V \otimes \mathbb{C}$ is $J_{\mathbb{C}}$-invariant, it similarly is isomorphic to $H_{\mathbb{C}} \oplus \overline{H}_{\mathbb{C}}$, where $H_{\mathbb{C}} := \iota_{\mathbb{C}}(V \otimes \mathbb{C}) \cap F$, and $\overline{H}_{\mathbb{C}} := \iota_{\mathbb{C}}(V \otimes \mathbb{C}) \cap \overline{F}$. The composite $\varphi\iota_{\mathbb{C}} : V \otimes \mathbb{C} \to F$ has kernel $\overline{H}_{\mathbb{C}}$.

Lemma 2.8 $V \in D_j \iff$ some complex j-subspace of $V \otimes \mathbb{C}$ is mapped to $\{\vec{0} \in F\}$ by $\varphi\iota_{\mathbb{C}}$.

PROOF: A subspace, regardless of its orientation, is mapped to zero if and only if it is contained in the kernel of the map, $\overline{H}_{\mathbb{C}}$. $\overline{H}_{\mathbb{C}}$ has the property $\dim_{\mathbb{R}} V \cap JV = 2\dim_{\mathbb{C}} \overline{H}_{\mathbb{C}} \geq 2j$. ∎

.3 Degeneracy Loci and Grassmann Bundles

Now let V be the real oriented tautological m-bundle over $SG(m, F)$, and abbreviate as F the trivial complex n-bundle $SG(m, F) \times F$. Taking all complex j-subspaces of fibers of the complexified bundle $V \otimes \mathbb{C}$, form the complex grassmann bundle $\pi_j : \mathbb{C}G(j, V \otimes \mathbb{C}) \to SG(m, F)$. Let U^j be the tautological j-bundle over $\mathbb{C}G(j, V \otimes \mathbb{C})$, and form the pullbacks $\mathbb{V} := \pi_j^*(V \otimes \mathbb{C})$ and $\mathbb{F} := \pi_j^* F$.

The inclusion of U^j into $\mathbb{V}$, followed by the map $\varphi\iota_{\mathbb{C}}$ on fibers is a bundle map $U^j \to \mathbb{F}$, and defines a section s_j of $\mathrm{Hom}(U^j, \mathbb{F})$. By Lemma 2.7, if this section vanishes at x then $\pi_j(x) \in D_j$, and if $x \in D_j$, then $s_j = 0$ for some element of $\pi_j^{-1}(x)$. Denote the zero set $\mathrm{Div}(s_j) := \{x | s_j(x) = 0\}$; then $\pi_j(\mathrm{Div}(s_j))$ is called the degeneracy locus of the bundle map $\varphi\iota_{\mathbb{C}} : V \otimes \mathbb{C} \to F$.

$\mathrm{Div}(s_j) \subseteq \pi_j^{-1} D_j$ is a real algebraic subvariety of $\mathbb{C}G(j, V \otimes \mathbb{C})$. For $x \in C_l$, $\pi_j^{-1}(x) \cap \mathrm{Div}(s_j) = \mathbb{C}G(j, l)$. If $Z_j := \mathrm{Div}(s_j) \setminus \pi_j^{-1} D_{j+1}$, then $\pi_j|_{Z_j}$ is a diffeomorphism onto C_j, and so

$$\mathrm{codim}_{\mathbb{R}} \mathrm{Div}(s_j) = 2j(n - m + j) + 2j(m - j) = 2jn,$$

as expected for a section of this bundle. The projection of $\mathrm{Div}(s_j)$ onto D_j is a partial desingularization which undoes some of the identifications made in the earlier description of D_j.

The above paragraphs and the preceding section generalize to the situation where F is a smooth complex vector bundle over a smooth manifold M. Let $F_{\mathbb{R}}$ be the underlying real oriented $2n$-bundle, with some Riemannian metric, and $\mu : SG(m, F) \to M$ the grassmann bundle formed by all the real oriented m-subspaces of fibers $(F_{\mathbb{R}})_x$. If the tautological m-bundle over $SG(m, F)$ is $\iota : V \hookrightarrow \mu^* F_{\mathbb{R}}$, then any real oriented m-subbundle T of $F_{\mathbb{R}}$ is of the form $\gamma_T^* \iota : T = \gamma_T^* V \hookrightarrow F_{\mathbb{R}}$, for some smooth section $\gamma_T : M \to SG(m, F)$.

Example 3.1 If an m-dimensional manifold is immersed $g : M \to \mathbb{C}^n$, then the immersion induces a map of real tangent bundles, $TM \to T\mathbb{C}^n$, and an inclusion of TM in the trivial bundle $g^* T\mathbb{C}^n = M \times \mathbb{R}^{2n}$. Then $\gamma_T : M \to M \times SG(m, 2n)$ is the graph of the oriented Gauss map of the immersion.

The subsets D_j and C_j of $SG(m, F)$ are defined in the same way, e. g. $D_j = \{x \in SG(m, F) | \dim_{\mathbb{R}} V_x \cap \mu^* J_x V_x \geq 2j\}$, and the codimension formula (1) still applies.

Definition 3.2 The CR-singular set of $(j - j_0)^{\text{th}}$ order, N_j, of a subbundle T in F, is the set $\mu(\gamma_T(M) \cap D_j) \subseteq M$. Points of $\mu(\gamma_T(M) \cap C_{j_0})$ are called CR-regular.

If T is the tangent bundle of M and F is the pullback bundle by an immersion $M \hookrightarrow A$ of TA, the tangent bundle of the almost complex manifold A, then both points in N_{j_0+1} and the subspaces H of fibers of T over such points are called "complex tangents" of the immersion.

Definition 3.3 T is generically included in F if $\gamma_T(M)$ and C_j intersect transversely in $SG(m, F)$ for each j.

Lemma 3.4 If T is generically included in F, $N_j \setminus N_{j+1}$ is a smooth submanifold of M, of codimension $2j(n - m + j)$. $\blacksquare$

Definition 3.5 T is a CR-subbundle of F of CR codimension $m - 2j$ if $\gamma_T(M) \subseteq C_j$. If $j = 0$ then T is called totally real, and if $j = k = m/2$, T is a complex (or anticomplex, on each connected component) j-subbundle of F. A generic CR-subbundle has CR-codimension $m - 2j_0$; such subbundles are totally real if $m \leq n$, or have totally real orthogonal complement if $m \geq n$. If T is the tangent bundle as above, similar adjectives apply to the manifold M.

Example 3.6 If $0 < m < 2n$ then a generically included subbundle is not complex or anticomplex. Otherwise, if $m = 0$ or $2n$, every subbundle is trivially both generically included and complex (or anticomplex, on connected components). If $m = 2n - 1$, then $j_0 = k = n - 1$, and every subbundle T is a generically included CR-subbundle of CR codimension 1. Every real line subbundle ($m = 1$) is totally real. If $\dim_{\mathbb{R}} M < 2(n - m + 1)$, then a generically included T is totally real. In the case $T = TM^m$, the tangent bundle of a real m-manifold, this inequality is similar to bounds in embedding theorems of [Whitney$_{44}$] and [Haefliger]: if $m < \frac{2}{3}(n + 1)$, then any m-manifold immersed in $\mathbb{C}^n$ can be perturbed so that there are no complex tangents.

To describe D_j as a degeneracy locus inside the grassmann bundle $SG(m, F)$, again form the grassmann bundle π_j of complex j-subspaces of the bundle $V \otimes \mathbb{C}$. The tautological complex j-bundle U^j over the total space $\mathbb{C}G(j, V \otimes \mathbb{C})$ is included in the pullback bundle $\mathbb{V} := \pi_j^*(V \otimes \mathbb{C})$. This inclusion, composed with the map given on fibers by $\varphi\iota_{\mathbb{C}}$, is a bundle map $U^j \to \mathbb{F} := \pi_j^* \mu^* F$, defining a section s_j of the bundle $\mathrm{Hom}(U^j, \mathbb{F})$. $\mathrm{Div}(s_j)$ projects by π_j to D_j.

$$\begin{tikzcd}
\mathrm{Hom}(U^j, \mathbb{F}) \arrow[d, shift left] & V \arrow[d] & F \arrow[d] \\
\mathbb{C}G(j, V \otimes \mathbb{C}) \arrow[u, shift left, "s_j"] \arrow[r, "\pi_j"] & SG(m, F) \arrow[r, shift left, "\mu"] & M \arrow[l, shift left, "\gamma_T"]
\end{tikzcd}$$

Over the set $N_j \setminus N_{j+1}$ lies an obvious complex j-bundle, defined at a point x by $T_x \cap J_x T_x$, with the complex structure induced by J_x. In terms of the above construction, it is a pullback bundle, as follows. $\gamma_T|_{N_j \setminus N_{j+1}}$ is a diffeomorphism onto $\gamma_T(M) \cap C_j$, and $\pi_j|_{Z_j}^{-1}$ is a diffeomorphism $C_j \to Z_j$. Finally, include $Z_j \hookrightarrow \mathbb{C}G(j, V \otimes \mathbb{C})$, and denote the (injective) composition $\kappa_j : N_j \setminus N_{j+1} \to \mathbb{C}G(j, V \otimes \mathbb{C})$. The fiber of U^j above an element y of Z_j is the j-plane mapped to $\vec{0}$ in $\mathbb{F}_y$, which is, by Lemma 2.7, of the form $(\overline{H}_{\mathbb{C}})_y$, where $(H_{\mathbb{C}})_y$ is isomorphic to the $\pi_j^* \mu^* J_y$-invariant real $2j$-plane contained in $\pi_j^* V_y$. So, define the complex

j-bundle H^j over $N_j \setminus N_{j+1}$ by $\kappa_j^* \overline{U^j}$. If m is even, the connected components of H^k are complex or anticomplex k-subbundles of components of $F|_{N_k}$.

REMARK: At this point, assume M is a smooth compact oriented manifold without boundary, of real dimension d. The orientations on T, F, and TM, the real tangent bundle of M, induce orientations on almost everthing else. At a point x of the grassmann bundle $SG(m, F)$, the tangent space is $T_{\mu(x)} M \oplus V_x^* \otimes V_x^\perp$, where V is the oriented tautological bundle. At a point y of $\mathbb{C}G(j, V \otimes \mathbb{C})$, the tangent space is $T_{\pi_j(y)} SG(m, F) \oplus U_y^{j*} \otimes (\mathbb{V}_y / U_y^j)$. The tangent space to $\mathrm{Div}(s_j)$ is oriented so that $\mathrm{Hom}(U^j, \mathbb{F})|_{\mathrm{Div}(s_j)} \oplus T\mathrm{Div}(s_j) = T\mathbb{C}G(j, V \otimes \mathbb{C})|_{\mathrm{Div}(s_j)}$. This orientation induces orientations for the submanifolds C_j and $N_j \setminus N_{j+1}$. ∎

REMARK: In the generic case, the normal bundle of the submanifold $N_j \setminus N_{j+1}$ is isomorphic to $\mathrm{Hom}(\ker \varphi\iota_{\mathbb{C}}, \mathrm{coker}\ \varphi\iota_{\mathbb{C}}) \cong \bar{H}^j \otimes_{\mathbb{C}} H_\perp^{n-m+j}$, where $H_\perp$ is the complex tangent space $T^\perp \cap JT^\perp$ of the bundle normal to T in F. ([GG], p. 145) This description of the normal bundle is another geometric feature of the codimension formula (1). A complex structure on $\nu(N_1 \setminus N_2)$ was observed in the $m = n$ case in [W_{85}]. ∎

REMARK: If X is connected without boundary, but not orientable, then it admits a connected, orientable two-fold cover $X^o \to X$. CR singularities of a bundle T over X induce CR singularities when T and F are pulled back to X^o. ∎

It will frequently be the case that CR-singularities of high order are to be avoided; this is denoted by "$N_{j+1} = \emptyset$." This condition may follow from the hypothesis "T generic," (for large j), and otherwise is not inconsistent with genericity. If T is generically included in F and $N_{j+1} = \emptyset$, then $\gamma_T(M)$ and C_l intersect transversely in $SG(m, F)$ for each $l \leq j$, and $\gamma_T(M) \cap D_{j+1} = \emptyset$.

Example 3.7 If $d < 2(j+1)(n-m+j+1)$, then $N_{j+1} = \emptyset$ for T generically included in F. If $N_{j+1} = \emptyset$ and T is generic, then N_j is a closed submanifold of M. Example 3.6 is a special case of this phenomenon.

.4 Determinantal Formulas

Definition 4.1 For two complex vector bundles R, S over a manifold M, let $c(R - S)$ denote the quotient of total chern classes $cR(cS)^{-1}$ in the cohomology ring $H^*(M; \mathbb{C})$, with $c_i(R - S)$ the degree $2i$ part.

Example 4.2 Abbreviate $F = \mu^* F$ over $SG(m, F)$. Then

$$\begin{aligned}
& c(F - V \otimes \mathbb{C}) \\
:= \quad & cF(c(V \otimes \mathbb{C}))^{-1} && (2) \\
\equiv \quad & (\sum_{a=0}^{n} c_a F)(\sum_{b=0}^{k} (-1)^b p_b V)^{-1} && (3) \\
= \quad & (\sum_{a=0}^{n} c_a F)(\sum_{i=0}^{\infty} (\sum_{b=1}^{k} (-1)^{b+1} p_b V)^i) && (4) \\
= \quad & 1 + c_1 F + (c_2 F + p_1 V) + (c_3 F + c_1 F p_1 V) + && (5) \\
& (c_4 F + c_2 F p_1 V - p_2 V + p_1^2 V) + \\
& (c_5 F + c_3 F p_1 V - c_1 F p_2 V + c_1 F p_1^2 V) + \\
& (c_6 F + c_4 F p_1 V - c_2 F p_2 V + c_2 F p_1^2 V + p_3 V - 2 p_1 V p_2 V + p_1^3 V) + \ldots
\end{aligned}$$

(3) is the definition of pontrjagin classes. (4) is a formal power series expansion. The terms of (5) are grouped by total (even) degree.

.4.1 Formulas for isolated complex tangents

A determinantal formula developed over a long history by Giambelli, Thom, and [Porteous] about degeneracy loci of bundle maps applies to the geometry of complex tangents as developed previously. Both of the following propositions are from [Porteous]:

Proposition 4.3 The equation of degree $2jn$ cohomology classes

$$[\mathrm{Div}(s_j)] = c_{jn} \mathrm{Hom}(U^j, \mathbb{F}) \tag{6}$$

pushes forward by π_j to the equation

$$\begin{aligned}
\pi_{j*}[\mathrm{Div}(s_j)] \quad &= \quad \Delta^{(j)}_{n-m+j}(c(\mu^* F - V \otimes \mathbb{C})) && (7) \\
&:= \quad \det(c_{n-m+j+a-b}(\mu^* F - V \otimes \mathbb{C}))_{1 \le a, b \le j}\,, && (8)
\end{aligned}$$

of degree $2j(n - m + j)$ cohomology classes. ∎

Lemma 4.4 $\pi_{j*}[\mathrm{Div}(s_j)]|_{SG(m,F) \setminus D_{j+1}} = [D_j]|_{SG(m,F) \setminus D_{j+1}}$. ∎

Theorem 4.5 If $d = \dim_{\mathbb{R}} M = 2j(n-m+j)$, and T is generically included in F, then

$$\sum_{x \in N_j} \mathrm{ind}(x) = \int_M \Delta^{(j)}_{n-m+j}(c(F - T \otimes \mathbb{C})), \tag{9}$$

where $\mathrm{ind}(x)$ is the oriented intersection number of $\gamma_T(M)$ and C_j at $\gamma_T(x)$.

PROOF: This equation is the integral of the product of both sides of (7) with $[\gamma_T(M)]$. The chern and pontrjagin classes on the RHS pull back to M by functoriality. The hypothesis T generic assures that $\gamma_T(M)$ is transverse to C_j, the smooth part of the support of $\pi_{j*}[\mathrm{Div}(s_j)]$, and the codimension formula (1) implies the intersection is only in isolated points of C_j. ∎

Example 4.6 ([W_{86}]) Consider the situation where T is a real 2-subbundle of a complex 2-bundle F over a smooth compact oriented surface M. (This will be described by $d = m = n = 2$.) If T is generically included in F, the set N_1 of points x where the fiber T_x is complex or anticomplex is a codimension 2 submanifold, i.e., finite. Formula (9) then reads:

$$\sum_{x \in N_1} \mathrm{ind}(x) = \int_M c_1 F. \tag{10}$$

In fact, this holds for $d = 2,\ m = n \geq 2$. The local geometry of this index when $T = TM$ is also considered by [IO].

Example 4.7 In the case $d = m = 4$, $n = 5$, [Domrin] calculated the index sum

$$\sum_{x \in N_1} \mathrm{ind}(x) = \int_M c_2 F + p_1 T.$$

For example, the complex projective plane can be embedded in $\mathbb{C}^3 \oplus \mathbb{R}^3$ by the map $P : [z_1 : z_2 : z_3] \mapsto \frac{1}{N}(z_2\bar{z}_3, \bar{z}_1 z_3, z_1\bar{z}_2, z_1\bar{z}_1, z_2\bar{z}_2, z_3\bar{z}_3)$, with $N(z_1, z_2, z_3) = |z_1|^2 + |z_2|^2 + |z_3|^2$. If the target space $\mathbb{R}^9$ is included in $\mathbb{C}^5$, the image $\mathbb{C}P^2$, or generic smooth perturbations of it, has isolated points at which the tangent space contains a complex line. The index sum is $p_1 \mathbb{C}P^2 = 3$.

Example 4.8 $m = n \geq 4,\ j = 2,\ d = 8$. Abbreviate by c_i the $2i$-degree part of $c(F - T \otimes \mathbb{C})$, corresponding to the i^{th} term of (5).

$$\sum_{x \in N_2} \mathrm{ind}(x) = \int_M \begin{vmatrix} c_2 & c_1 \\ c_3 & c_2 \end{vmatrix} \tag{11}$$

$$= \int_M (c_2 F + p_1 T)^2 - c_1 F(c_3 F + c_1 F p_1 T). \tag{12}$$

In particular, if F is trivial, as in the case where M is immersed in $\mathbb{C}^8$, $T = TM$, and F is the pullback by the immersion of $T\mathbb{C}^8$, then ([Lai])

$$\sum_{x \in N_2} \mathrm{ind}(x) = \int_M p_1^2 T. \tag{13}$$

Example 4.9 $m - n + 1 = j = 2l > 0,\ d = 2j$.

$$\sum_{x \in N_j} \mathrm{ind}(x) = \int_M \Delta_1^{(j)}(c(F - T \otimes \mathbb{C})), \tag{14}$$

This can be a complicated polynomial. One of several useful determinantal identities in [Fulton_{84}] applies,

$$\Delta_a^{(b)}(c(R - S)) = (-1)^{ab} \Delta_b^{(a)}(c(S - R)),$$

so the characteristic class can be rewritten $\Delta^{(1)}_j(T \otimes \mathbb{C} - F)$, which simplifies to $(-1)^l p_l T$ if F is trivial. Equation (14) becomes this corollary of Theorem 4.5: Let M be a $4l$-dimensional manifold, and let T be a generic $(n+2l-1)$-subbundle of the trivial complex n-bundle. Then (note $j_0 = 2l-1$, so N_{2l} is the first CR-singular set)

$$\sum_{x \in N_{2l}} \mathrm{ind}(x) = (-1)^l \int_M p_l T, \tag{15}$$

and, letting $n = 2l+1$, and $T = TM$ yields a formula ([HL$_{93}$]) about $2l$-complex and anticomplex tangencies of generic immersions $M^{4l} \to \mathbb{C}^{2l+1}$:

$$\sum_{x \in N_{2l}} \mathrm{ind}(x) = (-1)^l \int_M p_l M. \tag{16}$$

Example 4.10 $m = 2,\ j = 1,\ d = 2(n-1) \geq 0.$

$$\begin{aligned} \sum_{x \in N_1} \mathrm{ind}(x) &= \int_M \Delta^{(1)}_{n-1}(c(F - T \otimes \mathbb{C})) = \int_M c_{n-1}(F - T \otimes \mathbb{C}) && (17) \\ &= \int_M ((cF)(1 - p_1 T)^{-1})_{2n-2} = \int_M \sum_{b=0}^{\infty} c_{n-1-2b} F (p_1 T)^b. && (18) \end{aligned}$$

Note that Example 4.6 is the special case $n = 2$. If $T = \nu M$, the normal 2-bundle of a generic immersion of a $2(n-1)$-manifold M in an almost complex n-manifold, and F is the restriction to M of the ambient tangent bundle, then $p_1 \nu M = \tilde{\Omega}^2$, where $\tilde{\Omega}$ is the euler class of the normal bundle. In this scenario, (18) is a theorem of [Lai]:

$$\sum_{x \in N_1} \mathrm{ind}(x) = \int_M \sum_{b=0}^{\infty} \tilde{\Omega}^{2b} c_{n-1-2b} F. \tag{19}$$

If n is even and F is trivial, then (RHS 18) $= 0$. If $n = 2l+1$, and F is trivial, such as when M is generically immersed in $\mathbb{C}^n$, then (19) becomes

$$\sum_{x \in N_1} \mathrm{ind}(x) = \int_M \tilde{\Omega}^{2l}. \tag{20}$$

For a given immersion of M, equations (16) and (20) give the same information. The gauss maps of the tangent and normal bundle are related by $\gamma_{TM} = \perp \circ \gamma_{\nu M}$; by Example 2.6, the CR-singular sets are the same, and so are the intersection indices. Since $pTMp\nu M = p\mathbb{C}^{2l+1} = 1$, $pTM = (1 + p_1 \nu M)^{-1} \Rightarrow p_l TM = (-1)^l (p_1 \nu M)^l$.

More generally, If T^m is generic in F^n, then so is $T^\perp$; the set N_j of CR-singularities of T is the same set as the N_{n-m+j} corresponding to $T^\perp$ in F, and either sum of indices is given by Theorem 4.5. The equality of the chern numbers is also a consequence of determinantal identities:

$$\int_M \Delta_j^{(n-m+j)}(c(F-T^\perp\otimes\mathbb{C})) = \int_M \Delta_j^{(n-m+j)}\left(\frac{cF}{c((F/T)\otimes\mathbb{C})}\right)$$

$$= \int_M \Delta_j^{(n-m+j)}\left(\frac{cF}{cFc\bar{F}c(T\otimes\mathbb{C})^{-1}}\right) = \int_M \Delta_j^{(n-m+j)}(c(T\otimes\mathbb{C}-\bar{F}))$$

$$= (-1)^{j(n-m+j)}\int_M \Delta_j^{(n-m+j)}(c(T\otimes\mathbb{C}-F)) = \int_M \Delta_{n-m+j}^{(j)}(c(F-T\otimes\mathbb{C})).$$

Example 4.11 In the case of an oriented four-manifold M generically immersed in an almost complex 3-manifold A, the characteristic classes of $T = TM$, $\nu = T^\perp$, and $F = TA|_M$ are related by $p_1T + p_1\nu = c_1^2F - 2c_2F$. The sum of the indices of complex tangents for T is $\int_M c_1^2F - c_2F - p_1T$, and for ν is $\int_M c_2F + p_1\nu$. These numbers are equal, and zero if M is a CR submanifold of A. For any such M^4, the pontrjagin number $\int_M p_1T$ is three times the signature ([Hirzebruch]), and if $A = \mathbb{C}^3$, then $\int_M p_1\nu$ is three times the algebraic number of triple points of the immersion ([Herbert]). If M is isotopic to a complex submanifold Y, the chern classes of Y pull back to M; adjunction formulas for almost complex hypersurfaces can be used ([GH], p. 601). For example, if Y is a smooth, degree d complex hypersurface in $A = \mathbb{C}P^3$, and H is the hyperplane class in the cohomology of $\mathbb{C}P^3$, $c_2(T\mathbb{C}P^3|_Y) = 6H^2$, and $c_1T^\perp Y = dH$. Since $\int_Y H^2 = d$, the chern number $\int_Y c_2(T\mathbb{C}P^3|_Y) + (c_1T^\perp Y)^2$ is $6d + d^3$, which equals $\int_M c_2F + p_1\nu$ if M is isotopic to Y. This number is always positive; there are no CR submanifolds of $\mathbb{C}P^3$ isotopic to smooth complex hypersurfaces. In particular, a generically immersed submanifold isotopic to a complex hyperplane $\mathbb{C}P^2$ has seven complex tangents, counted with multiplicity.

.4.2 Formulas for non-isolated complex tangents

A procedure for finding the chern numbers of a kernel bundle was given by [HT]. Later, [Pragacz] gave a closed formula in terms of the segre class $s(F - T\otimes\mathbb{C}) = 1 + s_1 + s_2 + \ldots$, where the first few terms of the segre class are as follows:

$$\begin{aligned} s &= (1 - c_1 + c_2 - c_3 + c_4 - c_5 + \ldots)^{-1} \\ &= 1 + c_1 + (c_1^2 - c_2) \\ &\quad +(c_1^3 - 2c_1c_2 + c_3) \\ &\quad +(c_1^4 - 3c_1^2c_2 + 2c_1c_3 + c_2^2 - c_4) + \ldots \\ &= 1 + c_1F + (c_1^2F - c_2F - p_1T) \\ &\quad +(c_1^3F - 2c_1Fc_2F + c_3F - c_1Fp_1T) \\ &\quad +(c_1^4F - 3c_1^2Fc_2F + 2c_1Fc_3F + c_2^2F - c_4F - c_1^2Fp_1T + c_2Fp_1T + p_2T) + \ldots . \end{aligned}$$

The segre classes will appear in determinants labeled with a sequence of integers $\mathbf{J} = (j_1, j_2, \ldots, j_r)$, with

$$\Delta_{\mathbf{J}}(s(R-S)) = \det(s_{j_p-p+q})_{1\le p,q\le r}.$$

Theorem 4.12 If T is generic in $F \to M$, $N_{j+1} = \emptyset$, and the product of chern classes $\prod c_i(K)^{\beta_i}$ is expressed as a sum with nonnegative integer coefficients $\prod c_i(K)^{\beta_i} = \sum m_{\mathbf{J}} \Delta_{\mathbf{J}}(s(K))$, where $\sum i\beta_i = \frac{1}{2}\dim_{\mathbb{R}}(N_j) = d$, then

$$\int_{N_j} \prod_i c_i(H)^{\beta_i} = \int_M \sum_{\mathbf{J}} m_{\mathbf{J}} \Delta_{j^{n-m+j},\tilde{\mathbf{J}}}(s(F - T \otimes \mathbb{C})),$$

where $\tilde{\mathbf{J}}$ denotes the conjugate partition $(i_1, i_2, \ldots)$, $i_a = \operatorname{card}\{h : j_h \geq a\}$.

Proof: Recall that M is assumed smooth, compact, and oriented, and by Example 3.7, N_j is a closed, oriented submanifold. The formula follows from the equality of cohomology classes ([Pragacz] Lemma 5.1 and Prop. 5.3)

$$\Delta_{\mathbf{J}}(s(-K))[N_j] = \Delta_{j^{n-m+j},\mathbf{J}}(s(F - T \otimes \mathbb{C})),$$

which by a determinantal identity implies

$$\Delta_{\mathbf{J}}(s(K))[N_j] = (-1)^d \Delta_{j^{n-m+j},\tilde{\mathbf{J}}}(s(F - T \otimes \mathbb{C})),$$

where K is the kernel bundle over N_j of the map $T \otimes \mathbb{C} \to F$. The sign $(-1)^d$ is cancelled when writing formulas for the chern number of $H = \bar{K}$. ∎

The segre classes are sometimes more natural than the chern classes in enumerative constructions such as these. The determinants in this formula could immediately be rewritten in terms of chern classes using the determinantal identities $\Delta_{\mathbf{I}}(s(E - F)) = (-1)^{|I|}\Delta_{\tilde{\mathbf{I}}}(s(F - E)) = \Delta_{\tilde{\mathbf{I}}}(c(E - F))$.

Example 4.13 For T generic, $d = 12$, $j = 2$, $m = n \geq 4$, the characteristic numbers are expressed as:

$$\int_{N_2} c_2 H^2 = \int_M \begin{vmatrix} s_2 & s_3 & s_4 \\ s_1 & s_2 & s_3 \\ 1 & s_1 & s_2 \end{vmatrix} = \int_M c_3^2 - c_2 c_4,$$

$$\int_{N_2} c_1^2 H^2 = \int_M \begin{vmatrix} s_2 & s_3 & s_4 \\ s_1 & s_2 & s_3 \\ 1 & s_1 & s_2 \end{vmatrix} + \begin{vmatrix} s_2 & s_3 & s_4 & s_5 \\ s_1 & s_2 & s_3 & s_4 \\ 0 & 1 & s_1 & s_2 \\ 0 & 0 & 1 & s_1 \end{vmatrix} = \int_M c_3^2 - c_1 c_5.$$

These are two possibly different obstructions to $N_2 = \emptyset$.

Example 4.14 $d = 2j(n + q - m + j)$, $q \geq 0$: suppose T is a generic subbundle of F such that $N_{j+1} = \emptyset$. To find a power of the top chern class of H^j, the determinantal formula can be written in terms of chern classes,

$$\int_{N_j} c_j^q H^j = \int_M \Delta^{(j)}_{n+q-m+j}(c(F - T \otimes \mathbb{C})), \tag{21}$$

and the special case $q = 0$ is Theorem 4.5.

Example 4.15 $j=1,\ m=n=4,\ q=1,\ d=4,\ T$ generic ([W_{86}])

$$\int_{N_1} c_1 H^1 = \int_M \Delta_2^{(1)}(c(F - T\otimes\mathbb{C})) = \int_M c_2 F + p_1 T.$$

More generally, for T generic, $j=1,\ m=n\geq 2,\ d=2(q+1)$, and $N_2=\emptyset$ (which is implied by generic for $d=4$ or 6),

$$\int_{N_1} c_1^q H^1 = \int_M \Delta_{q+1}^{(1)}(c(F - T\otimes\mathbb{C})) = \int_M c_{q+1}(F - T\otimes\mathbb{C}).$$

For example, [$\mathrm{Whitney}_{44}$] showed the embedding of $\mathbb{C}P^2$ from Example 4.7 can be composed with a projection resulting in an embedding into $\mathbb{R}^7$. Generic perturbations of this map give a 4-manifold in $\mathbb{C}^4$ such that N_1 is a smooth surface and $\int_{N_1} c_1 H^1 = p_1\mathbb{C}P^2 = 3$.

Example 4.16 Suppose T is a totally real n-subbundle of the trivial bundle $F=\mathbb{C}^n$. Then all its pontrjagin classes except p_0 are zero. This follows directly from the definition of pontrjagin classes since F and $T\otimes\mathbb{C}$ are isomorphic, but is also a consequence of the formula. $\gamma_T(M)$ does not intersect the support of $\pi_{1*}([\mathrm{Div}(s_1)]c_1^q U^{1*})$, so

$$0 = \Delta_{q+1}^{(1)}(c(F - T\otimes\mathbb{C})) = (c(T\otimes\mathbb{C})^{-1})_{q+1}$$

for $q\geq 0$, so $pT = (1+0+0+\dots)^{-1} = 1$. This was noticed by Kobayashi and [Wells] in the $T=TM$ case. The same conclusion holds if T is a totally real $n-1$-subbundle of $\mathbb{C}^n$; in this case $0=\Delta_{q+2}^{(1)}(c(F-T\otimes\mathbb{C}))$, and $c_1=0$ by hypothesis. This also gives a result of [Lai]: if T is a CR-regular $n+1$-subbundle of $\mathbb{C}^n$, then $T^\perp$ is a totally real $n-1$-subbundle, and $p\mathbb{C}^n = pTpT^\perp = pT = 1$.

Example 4.17 Suppose T is a generic CR-subbundle of F, with $m>n$, so $F_\mathbb{R}$ decomposes as a direct sum of smooth subbundles $T^\perp\oplus JT^\perp\oplus H^{m-n}$. The chern numbers of H can be calculated in terms of cF and pT using the sum formula

$$cF = c(T^\perp\otimes\mathbb{C})cH,$$

and the kernel bundle formula gives the same result when applied to $N_{m-n}=M$:

$$\begin{aligned}\prod c_i^{\beta_i} H &= \sum m_{\mathbf{J}}\Delta_{\tilde{\mathbf{J}}}(s(F - T\otimes\mathbb{C}))\\ &= \sum m_{\mathbf{J}}\Delta_{\mathbf{J}}(c(F - T\otimes\mathbb{C}))\\ &= \sum m_{\mathbf{J}}\Delta_{\mathbf{J}}(s(F - T^\perp\otimes\mathbb{C}))\\ &= \prod c_i^{\beta_i}(F - T^\perp\otimes\mathbb{C}).\end{aligned}$$

In the case where T is the normal bundle of a totally real submanifold, the topology of H is considered in [Forstnerič].

.4.3 Flags of subbundles

As another generalization of Theorem 4.5, a "flag of real subbundles" inside the complex bundle F is considered. To fix notation, $t \geq 1$ subspaces (or subbundles) are indexed:

$$T_1^{m_1} \subseteq T_2^{m_2} \subseteq \ldots \subseteq T_t^{m_t} \subseteq F^n,$$

with $0 \leq m_1 \leq m_2 \leq \ldots \leq m_r \leq 2n$. The interesting loci of complex tangents are indexed by sequences of $j_i = \dim_{\mathbb{C}} T_i \cap JT_i$:

$$0 < j_1 < j_2 \quad < \quad \ldots < j_t \tag{22}$$

$$m_1 - j_1 < m_2 - j_2 \quad < \quad \ldots < m_t - j_t < n. \tag{23}$$

Flags with (at least) j_i complex directions in T_i, $1 \leq i \leq t$, form a locus $D_{(j_1 \ldots j_t)}$ in the (partial) flag manifold $SFl(m_1, \ldots, m_n, F)$ of oriented real subspaces of $F_{\mathbb{R}}^{2n}$.

It is not too restrictive to consider strictly increasing indices j_i and $m_i - j_i$—consider $T_1^{m_1} \subseteq T_2^{m_2} \subseteq F^n$ with $m_2 \leq n$. If T_1 has exactly j_1 complex directions, any T_2 will have at least that many, and generically will have an equal number, $j_2 = j_1$. So, the locus in SFl where $0 < j_1 \leq j_2$ is the inverse image of $D_{j_1} \subseteq SG(m_1, F)$ under the projection forgetting T_2, and enumerative questions about the locus $D_{(j_1 j_2)}$ reduce to the single subbundle case if $j_2 = j_1$. Similarly, if $n \leq m_1$, the number of complex directions in the normal space, $n - m_i + j_i$, generically does not increase for $i = 1, 2$, so the configurations which do not reduce to the grassmannian case are those with $m_i - j_1 < m_2 - j_2$.

The formula for the real codimension of the locus $D_{(j_1 \ldots j_t)}$ in SFl is

$$2j_1(n - m_1 + j_1) + 2(j_2 - j_1)(n - m_2 + j_2) + \ldots + 2(j_t - j_{t-1})(n - m_t + j_t).$$

Note that this reduces to formula (1) when $t = 1$, and that each term is non-zero by (22), (23). The proof of the special case $t = 2$ illustrates the general idea.

Example 4.18 $SFl(m_1, m_2, F^n)$ is a fiber space with base $SG(m_1, F)$, the oriented m_1-planes in $F_{\mathbb{R}}^{2n}$. The m_2-planes containing a fixed $T_1^{m_1}$ form the fiber over T_1, a grassmannian of quotients of $m_2 - m_1$ dimensions inside F/T_1 The total dimension of SFl is $m_1(2n - m_1) + (m_2 - m_1)((2n - m_1) - (m_2 - m_1))$. Those T_1 that contain any complex j_1-dimensional subspace form a codimension $2j_1(n - m_1 + j_1)$ locus in the base space $SG(m_1, F)$, and so a dimension $2m_1 n - m_1^2 - 2j_1 n + 2j_1 m_1 - 2j_1^2$ base variety. Those $H_2^{j_2}$ that contain the complex subspace $H_1^{j_1}$ form a fiber with real dimension $2(j_2 - j_1)(n - j_1 - (j_2 - j_1))$. Generically, H_2 and T_1 intersect only in H_1 and the real dimension of their sum ($H_2 + T_1$ as real vector subspaces) is $m_1 + 2j_2 - 2j_1$. The space of $T_2^{m_2}$ that contain both T_1 and H_2 forms a fiber of real dimension $(m_2 - (m_1 + 2j_2 - 2j_1))(2n - (m_1 + 2j_2 - 2j_1) - (m_2 - (m_1 + 2j_2 - 2j_1)))$. The total parameter count for the space $D_{(j_1 j_2)}$ is $2j_1(n - m_1 + j_1) + 2(j_2 - j_1)(n - m_2 + j_2)$ less than the dimension of SFl. This description of $D_{(j_1 j_2)}$ omits some identifications that must be made when some of the subspaces intersect non-transversely, causing singularities in the locus but not decreasing the dimension.

A flag of subbundles over M is "generic" if its image under the Gauss map to the flag bundle $SFl(m_1, \ldots, m_t, F) \to M$ meets the smooth parts of the loci $D_{\mathbf{j}}$ transversely for each $\mathbf{j}$ satisfying (22), (23). The locus in M mapped to $D_{\mathbf{j}}$ will be called $N_{\mathbf{j}}$. Define the partition

$$\mu = ((n - m_1 + j_1)^{j_1}, (n - m_2 + j_2)^{j_2 - j_1}, \ldots, (n - m_t + j_t)^{j_t - j_{t-1}}),$$

and, for $1 \leq a \leq j_t$, let $\rho(a) = \min\{s : 1 \leq s \leq t,\ a \leq j_s\}$.

Theorem 4.19 If $T_1 \subseteq \ldots \subseteq T_t \subseteq F$ is a generic flag of subbundles, then $N_{\mathbf{j}}$ has codimension $2|\mu| = 2j_1(n - m_1 + j_1) + \ldots + 2(j_t - j_{t-1})(n - m_t + j_t)$, and the current defined by $N_{\mathbf{j}}$ has cohomology class

$$[N_{\mathbf{j}}] = \det(c_{\mu_a - a + b}(F - T_{\rho(a)} \otimes \mathbb{C}))_{1 \leq a, b \leq j_t}.$$

∎

The formula follows from [Fulton$_{91}$]; the real degree of the cohomology class is $2|\mu|$, and the dimension count in the universal flag bundle is necessary to show that generically $N_{\mathbf{j}}$ is a subset with a smooth locus of codimension $2|\mu|$.

Example 4.20 The simplest example with $t = 2$ is $j_1 = 1$, $j_2 = 2$, $m_1 = 2$, $m_2 = 4$, $n = 3$. In this case, F^3 is a complex 3-plane bundle, containing T_2^4 as a real subbundle, which in turn contains T_1^2. T_1^2 is generically totally real in F^3, but is a complex line along a locus of real codimension 4. T_2^4 generically contains a complex line, but is a complex 2-plane along a locus of real codimension 4. T_1^2 and T_2^4 are simultaneously complex along the locus $N_{(12)}$, which generically has codimension $2 \cdot 1 \cdot (3 - 2 + 1) + 2 \cdot (2 - 1) \cdot (3 - 4 + 2) = 6$.

The formula uses the partition $\mu = (2, 1)$ and the function $\rho(1) = 1$, $\rho(2) = 2$:

$$\begin{aligned} [N_{(12)}] &= \begin{vmatrix} c_2(F - T_1 \otimes \mathbb{C}) & c_3(F - T_1 \otimes \mathbb{C}) \\ 1 & c_1(F - T_2 \otimes \mathbb{C}) \end{vmatrix} \\ &= c_1 F(c_2 F + p_1 T_1) - (c_3 F + c_1 F p_1 T_1) = c_1 F c_2 F - c_3 F. \end{aligned}$$

Consider $M = \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ as a real 6-manifold, with complex tangent bundle F^3. If $p_1 : M \to \mathbb{C}P^1$ is projection onto the first factor, and $p_2 : M \to \mathbb{C}P^1 \times \mathbb{C}P^1$ is projection onto the first two factors, then $p_1^* T\mathbb{C}P^1 \subseteq p_2^* T(\mathbb{C}P^1 \times \mathbb{C}P^1) \subseteq F^3$ is a flag of complex subbundles tangent to the product foliations of M. A real perturbation of the subbundles (or the foliations) can give a generic flag of real subbundles $T_1^2 \subseteq T_2^4 \subseteq F^3$, with $N_{(12)}$ expected to be isolated. Denoting by a, b, c the generators of the cohomology of each of the three factors $\mathbb{C}P^1$ pulled back to M, the total chern class of F is $(1+2a)(1+2b)(1+2c)$. The class $c_1 F c_2 F - c_3 F$ is $16abc$, so the index sum of the isolated complex tangents of $T_1 \subseteq T_2 \subseteq F$ is 16.

.5 Coincidence as CR Singularities of a Graph

A smooth real-linear map α between complex vector bundles $T^r = (T_{\mathbb{R}}, J^T)$ and $F^n = (F_{\mathbb{R}}, J^F)$ of real dimension $2r = m$ and $2n$ over M may be complex-linear on some vectors $\vec{v}_x \in T_x$, meaning $\alpha_x(J_x^T \vec{v}_x) = J_x^F \alpha_x(\vec{v}_x)$. In the fiber T_x, the set of vectors where α_x is complex-linear is a complex subspace K_x, and the subset of points $x \in M$ where $\dim_{\mathbb{C}} K_x$ is at least j defines the "coincidence locus" Q_j.

The map $\alpha : T_{\mathbb{R}} \to F_{\mathbb{R}}$ defines its graph $\underline{\alpha}$ as a real-linear inclusion of the image $\underline{\alpha}T$ in $T_{\mathbb{R}} \oplus F_{\mathbb{R}}$ of the map $\vec{v} \mapsto (\vec{v}, \alpha(\vec{v}))$. $T_{\mathbb{R}} \oplus F_{\mathbb{R}}$ has the direct sum complex structure, $J_x^{\oplus}(\vec{v}, \vec{w}) = (J_x^T \vec{v}, J_x^F \vec{w})$. Denote $T \oplus F = (T_{\mathbb{R}} \oplus F_{\mathbb{R}}, J^{\oplus})$.

Lemma 5.1 $K_x \cong \underline{\alpha}T_x \cap J_x^{\oplus}\underline{\alpha}T_x$.

PROOF: The claim is that $\underline{\alpha}_x$ is a $\mathbb{C}$-linear isomorphism when restricted to K_x, and that its image is the maximal $J_x^{\oplus}$-complex subspace of $\underline{\alpha}T_x$.

$$\begin{aligned} \vec{v} \in K \quad &\iff \quad (J^T\vec{v}, J^F\alpha\vec{v}) = (J^T\vec{v}, \alpha J^T \vec{v}) \iff J^{\oplus}(\vec{v}, \alpha\vec{v}) \in \underline{\alpha}T \\ &\iff \quad (\vec{v}, \alpha\vec{v}) \in \underline{\alpha}T \cap J^{\oplus}\underline{\alpha}T. \end{aligned}$$

∎

So, a map α such that $\underline{\alpha}T$ is a generically included subbundle of real rank $m = 2r$ in the complex bundle $T \oplus F$ of complex rank $n + r$ has coincidence loci $Q_j \setminus Q_{j+1}$ of real codimension $2j((n + r) - 2r + j)$ in M, corresponding to the complex tangent loci of $\underline{\alpha}T$. The following corollary of Theorem 4.5 describes the the coincidence locus in terms of the chern classes of (T, J^T) and (F, J^F):

Corollary 5.2

$$\begin{aligned} [Q_j] = [N_j] \quad &= \quad \Delta^{(j)}_{(n+r)-2r+j}(c(T \oplus F - T \otimes \mathbb{C})) \\ &= \quad \Delta^{(j)}_{n-r+j}\left(\frac{cTcF}{cTc\bar{T}}\right) = \Delta^{(j)}_{n-r+j}(c(F - \bar{T})). \end{aligned}$$

∎

The relationship between complex coincidence and the CR structure of the graph seems to be well-known, but not formulated as explicitly as this in the literature. (cf [Freeman] and §4.2, [Chirka])

Example 5.3 If T is the conjugate bundle of F, i.e., $T_{\mathbb{R}} = F_{\mathbb{R}}$ and $J^T = -J^F$, then the identity map is complex-linear on no vector— $Q_j = \emptyset$ for $j > 0$. The graph of the identity map is a totally real inclusion and $c(F - \bar{T}) = 1 + 0 + 0 + \ldots$.

Example 5.4 Two complex structures J^T and J^F on the same real vector bundle are relatively generic if the graph of the identity map is a generic inclusion in the sum. Invariants for such a pair were calculated in [HL_{93}] and [HL_{95}]:

$$[Q_j] = \Delta^{(j)}_j(c(F - \bar{T})).$$

Example 5.5 The graph of a smooth map $f : M \to A$ (not necessarily an immersion) defines an embedding $\underline{f} : M \to M \times A$. The coincidence locus Q_j of df is the same as the locus N_j of complex tangents of the image of $d\underline{f} = \underline{df}$ in $T(M \times A)$.

Example 5.6 ([EW]) If f is a smooth map between connected, compact, oriented Riemannian surfaces, $f : (M, g_M) \to (A, g_A)$, the degree of f is an

Figure 1: A smooth map $\mathbb{C}P^1 \to \mathbb{C}$, conformal at two points

integer. Giving M and A complex structures compatible with the metrics, and using Webster's formula (Example 4.6) for the image of the graph of f in $(M \times A, J^{\oplus})$,

$$\sum_{x \in N_1} \mathrm{ind}(x) = \int_M c_1(TM \oplus f^*TA) = \chi M + (\deg\, f)\chi A.$$

Reversing the orientation and complex structure on A changes the index sum to $\chi M - (\deg\, f)\chi A$, and similarly, reversing the orientation and complex structure on M gives index sum $-\chi M + (\deg\, f)\chi A$. The sign of the index differs from the [EW] formulas, which use a bundle map $T^{1,0}M \to T^{0,1}A$ instead of the projection onto the $(1,0)$ part as in Lemma 2.7.

Example 5.7 As an example of a smooth map between Riemann surfaces, the unit sphere in $\mathbb{R}^3$ can be projected radially onto the x,y-plane from the point $(0,0,2)$, giving a degree zero map with one (index ± 2) point each of direct and indirect conformality. Varying the point of projection along the line $\{(r,0,2)\}$ projects the sphere onto a region whose boundary is an ellipse. Not too surprisingly, the map is conformal at exactly those points projected onto the foci of the ellipse. In the figure, the lines connecting the foci to the vertex of the cone meet the sphere at four points, two each of direct and indirect conformality. Two of the four points are the poles $(0,0,\pm 1)$. The minor semiaxis of the ellipse, parallel to the y-axis, has constant length $2/\sqrt{3}$. The major semiaxis has length $\frac{2}{3}\sqrt{r^2+3}$, and the foci are at $r/3$ and $-r$. It is a well-known theorem that, in this case, the ellipses defined by the intersections of the planes $z = 1$ and $z = -1$ with the cone each have a focus at a pole of the sphere, and that they are similar (and so conformal) to any parallel ellipse on the cone.

Example 5.8 If M and A are complex manifolds, with real dimensions 4 and 6, and complex structures (TM, J^T), (TA, J^A), a smooth map $f : M \to A$ generically will satisfy $df \circ J^T = J^A \circ df$ only on complex lines tangent to a discrete set Q_1. The graph of f is an embedding of a real 4-manifold in $M \times A$, a complex 5-manifold, and if f is real-analytic, so is the image $f(M)$. By Example 4.7, the index sum of Q_1 is

$$\begin{aligned} \sum_{x \in Q_1} \mathrm{ind}(x) &= \int_M p_1 M + c_2(TM \oplus f^*TA) \\ &= \int_M c_1^2 M - c_2 M + c_1 M f^* c_1 A + f^* c_2 A. \end{aligned}$$

This example shows that the geometry of a generic map f from a complex surface to a complex 3-manifold reduces to the study of isolated complex tangents in the graph of f.

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