Analytic Normal Form for CR Singular Surfaces in $\mathbb{C}^3$

Adam Coffman
Department of Mathematical Sciences
Indiana University - Purdue University Fort Wayne
Fort Wayne, IN, USA 46805-1499
CoffmanA@ipfw.edu

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Abstract

A real analytic surface inside complex 3-space with an isolated, non-degenerate complex tangent is shown to be biholomorphically equivalent to a fixed real algebraic variety. The analyticity of the normalizing transformation is proved using a rapid convergence argument. Real surfaces in higher dimensions are also shown to have an algebraic normal form. [1] [2]

1 Introduction

This paper contributes to the program of studying the "local equivalence problem for real submanifolds of $\mathbb{C}^n$," as described in the survey paper, [BER]. The main result (Theorem 5.7) establishes a normal form for real analytic 2-manifolds in $\mathbb{C}^3$, near a suitably non-degenerate complex tangent, under biholomorphic coordinate changes. The non-degeneracy condition is an easily checked property of the quadratic terms of the equations that define the submanifold. The normal form is a fixed real algebraic variety, so there are no biholomorphic invariants for real surfaces near a non-degenerate complex tangent in $\mathbb{C}^3$. We will also point out some analogies between this normal form and some of Whitney's equations from the singularity theory of smooth maps from surfaces to $\mathbb{R}^3$.

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[2]Key words: normal form, CR singularity, complex tangent, rapid convergence

To outline the plan for the proof of the main result, let M be a real analytic surface in $\mathbb{C}^3$, which has an "isolated CR singularity" at the point $\vec{x} \in M$, that is, the tangent plane at $\vec{x}$ is a complex line, but the tangent plane at all other points in a neighborhood of $\vec{x}$ is totally real. By a complex affine coordinate change, it can be assumed that the singularity is at the origin, and that the tangent plane is one of the three complex coordinate axes. The local geometry of M, in some polydisc Δ centered at the singularity, will be considered first by specifying a quadratic non-degeneracy condition (the inequality (1) in Section 2), and in the case where M satisfies this condition, setting up a system of functional equations whose solution, if it exists, is a "normalizing transformation," meaning a coordinate change so that in the new coordinate system, the defining equations for M are in the real algebraic normal form (equation (4) in Section 2). Since an exact solution to these non-linear equations (equations (8) and (9) in Section 3) is hard to find in just one step, an approximation $\vec{p}$ to this solution will be found by solving a related system of linear equations ((10) and (11)). The solution technique for the linear equations is a straightforward comparison of coefficients, and $\vec{p}$ and its derivatives will turn out to be bounded in a neighborhood comparable in size to Δ. The size of the domain of $\vec{p}$ is a crucial ingredient of the analysis, so that when we get a sequence of such approximate solutions, each of them will be analytic on a certain fixed polydisc. The rapid convergence technique employed in Section 5 closely follows the construction of [M], where a convergent normalizing transformation was proved to exist for a surface in $\mathbb{C}^2$ with a special kind of CR singularity.

In the final Section, a cubic normal form is derived for surfaces in $\mathbb{C}^n$, $n > 3$. Again, being real analytic and suitably non-degenerate will be enough to guarantee the existence of a coordinate system on an open set, in which the surface is real algebraic.

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2 The non-degeneracy condition, and a quadratic normal form

Let (z_1, z_2, z_3) be a coordinate system for $\mathbb{C}^3$, and assume M is a real analytic surface in $\mathbb{C}^3$ with a complex tangent at $\vec{0} \in M$. By a complex linear transformation, the tangent plane $T_{\vec{0}}M$ can be assumed to be the z_1-axis. (z_1 will hereafter be abbreviated as z.) Then there is some polydisc Δ centered at $\vec{0}$ so that the defining equations of M in Δ are in the form of a graph over a disc in the z-axis:

$$\begin{aligned} z_2 &= h_2(z, \bar{z}), \\ z_3 &= h_3(z, \bar{z}), \end{aligned}$$

where h_2 and h_3 are complex-valued real analytic functions defined in a neighborhood of $z = 0$, and vanishing to second order at $z = 0$. These functions can be expressed as the restriction to $\{\zeta = \bar{z}\}$ of the two-variable series:

$$\begin{aligned} h_2(z, \zeta) &= \alpha_2 z^2 + \beta_2 z\zeta + \gamma_2 \zeta^2 + \sum_{a+b\geq 3} h_2^{a,b} z^a \zeta^b, \\ h_3(z, \zeta) &= \alpha_3 z^2 + \beta_3 z\zeta + \gamma_3 \zeta^2 + \sum_{a+b\geq 3} h_3^{a,b} z^a \zeta^b, \end{aligned}$$

each of which converges on the set $\{(z, \zeta) : |z| < R, |\zeta| < R\}$ to a complex analytic function. The terms $\alpha_2 z^2$ and $\alpha_3 z^2$ can be eliminated by a holomorphic quadratic coordinate change (which may change the above radius R). From this point, M will be assumed to satisfy the non-degeneracy condition:

$$\det \begin{pmatrix} \beta_2 & \gamma_2 \\ \beta_3 & \gamma_3 \end{pmatrix} \neq 0, \tag{1}$$

so that there is a complex linear transformation of z_2 and z_3 bringing M to the following quadratic normal form:

$$z_2 = \bar{z}^2 + e_2(z, \bar{z}), \tag{2}$$

$$z_3 = z\bar{z} + e_3(z, \bar{z}), \tag{3}$$

with e_2 and e_3 complex-valued real analytic functions vanishing to third order at 0.

Let $\widetilde{M}$ denote the real algebraic surface where e_2 and e_3 are identically zero:

$$\widetilde{M} = \{(z, z_2, z_3) \in \mathbb{C}^3 : z_2 = \bar{z}^2, \ z_3 = z\bar{z}\}. \tag{4}$$

The main result (Theorem 5.7) is that if M satisfies the non-degeneracy condition, and is described by equations of the form (2), (3), then there exist holomorphic functions $p_1(z, z_2, z_3)$, $p_2(z, z_2, z_3)$, $p_3(z, z_2, z_3)$ defined on some polydisc centered at $\vec{0}$ so that in the following new local coordinate system for $\mathbb{C}^3$:

$$\tilde{z} = z + p_1 \tag{5}$$

$$\tilde{z}_2 = z_2 + p_2 \tag{6}$$

$$\tilde{z}_3 = z_3 + p_3, \tag{7}$$

M is defined, in a neighborhood of $\vec{0}$, by $\tilde{z}_2 = \bar{\tilde{z}}^2$, $\tilde{z}_3 = z\bar{\tilde{z}}$; that is, M and $\widetilde{M}$ are locally analytically equivalent.

It is worth mentioning that $\widetilde{M}$ is totally real except at the origin, and that it is contained in the singular complex hypersurface $z^2z_2 - z_3^2 = 0$. The equations for $\widetilde{M}$ apparently resemble Whitney's parametric equations for a surface with a cross-cap singularity in $\mathbb{R}^3$ ([W]),

$$(u, v) \mapsto (x, y, z) = (u, v^2, uv),$$

and the equation for the complex hypersurface resembles the implicit equation $x^2y - z^2 = 0$ for the Whitney umbrella variety in $\mathbb{R}^3$.

Before starting the proof of the main result, we conclude this Section by comparing the situation of surfaces in $\mathbb{C}^3$ with the situation in higher dimensions.

In [C_1], the local equivalence problem was considered for m-submanifolds of $\mathbb{C}^n$, with $\frac{2}{3}(n+1) \leq m < n$ (note that the case $m = 2$, $n = 3$ is outside this range). The approach of [C_1] was, as in this paper, to first establish non-degeneracy conditions (analogous to (1)) for the quadratic part of the defining equations, and then to show there is a polynomial normal form, such that near a non-degenerate complex tangent of a real analytic m-submanifold, there exists a formal coordinate change so that the new defining equations are in the polynomial normal form. For example, in the $m = 4$, $n = 5$ case, the normal form variety in the 5-space with coordinates $(z_1, z_2, z_3, z_4, z_5)$, $z_a = x_a + iy_a$, is a graph over the real subspace with coordinates z_1, x_2, x_3:

$$\{y_2 = 0,\ y_3 = 0,\ z_4 = (\bar{z}_1 + x_2 + ix_3)^2,\ z_5 = z_1(\bar{z}_1 + x_2 + ix_3)\}.$$

The obvious difference between these equations and those defining the surface $\widetilde{M}$ is that the normal form for the surface can be expressed using only monomials. This is significant, because the analysis of 4-manifolds in [C_1] was complicated by the presence of multinomial coefficients appearing in powers of the quantity $(\bar{z}_1 + x_2 + ix_3)$.

In fact, the approximate normalizing transformations for the local equivalence problem in [C_1] were not suitable for the "rapid convergence" technique. While a formal power series for the normalizing transformation could be constructed as a formal composition of a sequence of holomorphic approximations, each of these approximations was defined on a polydisc significantly smaller than the previous one. So, the intersection of the domains of the transformations was just the single point at the origin. The question of convergence or divergence of the formal coordinate change for higher dimensions remains open. In the case of surfaces considered in this article, the rate of decrease of the size of the domain of the approximations is so small that we are able to apply the rapid convergence technique, to show that the approximations do indeed converge to an analytic coordinate change on an open set.

Some of the calculations of the next Section will be similar to those in [C_1], but simpler and with fewer variables. The goal is, given real analytic functions e_2 and e_3 (as in equations (2), (3)), to construct complex analytic functions p_1, p_2, p_3 (as in (5), (6), (7)). This goal will not be met until Section 5, but along the way we will state each major step as a "Theorem." The first Theorem decomposes e_2 and e_3 into even and odd parts, for the purposes of a comparison of coefficients, to arrive at an approximate normalizing transformation as the solution of a linearized functional equation. The even/odd decomposition also has some similarities with the argument of [W] proving the stability of the cross-cap singularity.

It should also be remarked that in the dimension range $\frac{2}{3}(n+1) \leq m < n$, CR singularities of m-submanifolds of $\mathbb{C}^n$ are "topologically stable," roughly meaning that if a submanifold in sufficiently general position has a point where the tangent space contains a complex line, then small smooth perturbations of the manifold will also have this property. (The notion of general position can be defined in terms of intersection properties of the gauss map from the manifold to the grassmannian of real subspaces in $\mathbb{C}^n$ — see [C_1], [G].) Surfaces in $\mathbb{C}^3$ do not enjoy this topological stability property: a CR singular surface can always be perturbed to a nearby surface which is totally real at every point.

3 A functional equation

The following notation will be convenient:

Notation 3.1. For $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$, with all $r_k > 0$, define a polydisc (with center $\vec{0}$) in $\mathbb{C}^n$ by

$$\mathbb{D}_{\mathbf{r}} = \{(z_1, \ldots, z_n) : |z_k| < r_k\}.$$

As special cases, let

$$D_r = \mathbb{D}_{(r,r)}$$

and

$$\Delta_r = \mathbb{D}_{(r,r^2,r^2)}.$$

Often in this paper, we will discuss both multivariable power series with center $\vec{0}$ and complex coefficients, which converge on a polydisc $\mathbb{D}_{\mathbf{r}}$, and complex valued functions which are complex analytic on a polydisc $\mathbb{D}_{\mathbf{r}}$. It is well-known that these two notions are equivalent, so we will not always explicitly mention that. It is also well-known that a function which is real analytic on a "real polydisc," $\mathbb{D}_{\mathbf{r}} \cap \mathbb{R}^n$, may have a series expansion with center $\vec{0}$ that does not converge on all of $\mathbb{D}_{\mathbf{r}} \cap \mathbb{R}^n$, so when working with real analytic functions, we will be careful about the domain of convergence, mostly by assuming that the real polydisc is small enough so that the function is the restriction of a complex analytic function on a complex polydisc.

Notation 3.2. For a complex-valued function $e(z, \zeta)$ of two complex variables, which is defined on some set containing the polydisc D_r, define the norm

$$|e|_r = \sup_{(z,\zeta)\in D_r} |e(z, \zeta)|.$$

For a pair $\vec{e} = (e_2, e_3)$, define

$$|\vec{e}|_r = \max\{|e_2|_r, |e_3|_r\}.$$

For a complex-valued function $p(z, z_2, z_3)$ of three complex variables, which is defined on some set containing the polydisc Δ_r, define the norm

$$\|p\|_r = \sup_{(z,z_2,z_3)\in\Delta_r} |p(z, z_2, z_3)|.$$

Notation 3.3. For a series in two variables $e(z, \zeta) = \sum e^{a,b} z^a \zeta^b$, define its "degree" as the lowest integer $a + b$ so that $e^{a,b} \neq 0$. (Also, the "degree" of a pair of series is the lower of the two degrees.) For a series $p(z, z_2, z_3) = \sum p^{abc} z^a z_2^b z_3^c$, define its "weight" as the lowest integer $a + 2b + 2c$ so that $p^{abc} \neq 0$.

With this notation, and a real analytic manifold M defined by equations (2), (3), we can assume that there is some $r > 0$ so that $|(e_2(z, \zeta), e_3(z, \zeta))|_r$ is finite. Given $\vec{e} = (e_2, e_3)$ with degree ≥ 3, the goal is to find some $\tilde{r}$, $0 < \tilde{r} \leq r$, and some holomorphic functions $\vec{p} = (p_1, p_2, p_3)$ on $\Delta_{\tilde{r}}$, so that the transformation

$$\Psi(z, z_2, z_3) = (\tilde{z}, \tilde{z}_2, \tilde{z}_3)$$

defined by (5), (6), (7) is a biholomorphism with domain $\Delta_{\tilde{r}}$ taking M to $\widetilde{M}$. That is, if $(z, z_2, z_3) \in M \cap \Delta_{\tilde{r}}$, then $\tilde{z}_2 = \bar{\tilde{z}}^2$ and $\tilde{z}_3 = \tilde{z}\bar{\tilde{z}}$. The weight of p_1 will be ≥ 2 and the weight of p_2 and p_3 will be ≥ 3.

For now, suppose Ψ is any (not necessarily invertible) map of the form (5), (6), (7). The following equations must hold for any $(z, z_2, z_3) \in M$ close enough to $\vec{0}$ so that both LHS and RHS are defined:

$$
\begin{aligned}
\tilde{z}_2 - \bar{\tilde{z}}^2 &= z_2 + p_2(z, z_2, z_3) - \overline{(z + p_1(z, z_2, z_3))}^2 \\
&= e_2(z, \bar{z}) + p_2(z, \bar{z}^2 + e_2, z\bar{z} + e_3) \qquad (8) \\
&\quad -(2\bar{z}\overline{p_1(z, \bar{z}^2 + e_2, z\bar{z} + e_3)} + \overline{p_1(z, \bar{z}^2 + e_2, z\bar{z} + e_3)}^2), \\
\tilde{z}_3 - \tilde{z}\bar{\tilde{z}} &= z_3 + p_3(z, z_2, z_3) - (z + p_1(z, z_2, z_3))\overline{(z + p_1(z, z_2, z_3))} \\
&= e_3(z, \bar{z}) + p_3(z, \bar{z}^2 + e_2, z\bar{z} + e_3) \qquad (9) \\
&\quad -(z\overline{p_1(z, \bar{z}^2 + e_2, z\bar{z} + e_3)} + \bar{z}p_1(z, \bar{z}^2 + e_2, z\bar{z} + e_3)) \\
&\quad -|p_1(z, \bar{z}^2 + e_2, z\bar{z} + e_3)|^2.
\end{aligned}
$$

If there are holomorphic functions $\vec{p}$ on some $\Delta_{\tilde{r}}$ so that Ψ is a biholomorphism, and the quantities (8), (9) are identically 0 (as formal series in z, $\bar{z}$), then Ψ is a normalizing transformation. As a first step in solving for $\vec{p}$ in terms of $\vec{e}$, consider the following two simpler equations:

$$
\begin{aligned}
0 &= e_2(z, \bar{z}) + p_2(z, \bar{z}^2, z\bar{z}) - 2\bar{z}\overline{p_1(z, \bar{z}^2, z\bar{z})}, \qquad (10) \\
0 &= e_3(z, \bar{z}) + p_3(z, \bar{z}^2, z\bar{z}) - z\overline{p_1(z, \bar{z}^2, z\bar{z})} - \bar{z}p_1(z, \bar{z}^2, z\bar{z}). \qquad (11)
\end{aligned}
$$

To see how the new equations are related to the original system, suppose $\vec{e}$ has degree $d \geq 3$, and that $\vec{p} = (p_1, p_2, p_3)$ is a solution of (10), (11) so that p_1 has weight $\geq d-1$, and p_2 and p_3 have weight $\geq d$. Using this solution $\vec{p}$ to evaluate the RHS of (8) and (9) evidently results in expressions of degree $\geq 2d-2$. So, Ψ defined by $\vec{p}$ may not bring M to normal form, but it will approximately double the degree of the higher-order terms.

Theorem 3.4. *Given $r > 0$ and a pair of complex analytic functions $\vec{e} = (e_2(z, \zeta), e_3(z, \zeta))$ defined on D_r with $|\vec{e}|_r < \infty$ and degree $d \geq 3$, there exists a triple of power series $\vec{p} = (p_1, p_2, p_3)$ which is complex analytic on Δ_r, and which exactly solves equations* (10), (11). *This solution satisfies the estimates*

$$\|p_1\|_r \leq \frac{3}{2r}|\vec{e}|_r, \ \|p_2\|_r \leq 5|\vec{e}|_r, \ \|p_3\|_r \leq 5|\vec{e}|_r.$$

Proof. First, notice that if $\vec{p}(z, z_2, z_3)$ is a formal series solution of (10), (11), it certainly does not follow that $\vec{p}$ is convergent. For example, $p_2(z, z_2, z_3)$ and $p_2 + (z^2 z_2 - z_3^2) \cdot \mathrm{Q}$ are formally the same when restricted to $\{z_2 = \bar{z}^2, z_3 = z\bar{z}\}$, for any (possibly divergent) series $\mathrm{Q}(z, z_2, z_3)$. So, if one formal solution

$\vec{p}$ exists, there exist infinitely many divergent solutions. The following choice of normalization will not only determine $\vec{p}$ uniquely, it will also result in the claimed convergence and bounds.

$$\begin{aligned} p_1(z, z_2, z_3) &= p_1^H(z) + p_1^A(z_2) \\ p_2(z, z_2, z_3) &= p_2^E(z, z_2) + z_3 p_2^O(z, z_2) \\ p_3(z, z_2, z_3) &= p_3^E(z, z_2) + z_3 p_3^O(z, z_2) \end{aligned}$$

Also, p_1^H must be an even function of z: $p_1^H(z) = p_1^H(-z)$, and the weights of p_1, p_2, p_3 must be $\geq d-1$, d, d, respectively.

This choice of normalization can be motivated by inspecting the set of formal transformations which leave $\widetilde{M}$ invariant; the normalizing transformations for M should in some sense be complementary to this set. This topic will be deferred to Section 6; for now, notice only that any formal series $P(z, z_2, z_3)$ can, by the Weierstrass Division Theorem, be uniquely expanded in the form

$$P = (z^2 z_2 - z_3^2) \cdot \mathrm{Q} + z_3 P^O(z, z_2) + P^E(z, z_2).$$

The last two "remainder" terms are as in the normalization of p_2 and p_3. Also, any monomial $e^{a,b} z^a \bar{z}^b$ fits into one of the three cases:

- $e^{a,2k} z^a (\bar{z}^2)^k = P^E(z, \bar{z}^2)$, for b even, or
- $(z\bar{z}) e^{a,2k+1} z^{a-1} (\bar{z}^2)^k = z\bar{z} P^O(z, \bar{z}^2)$, for b odd and $a > 0$, or
- $e^{0,2k+1} \bar{z}^{2k+1}$, with $a = 0$, b odd.

In the last case, the monomial is not equal to any expression of the form $P(z, \bar{z}^2, z\bar{z})$. So, the main idea of this proof will be to construct a function p_1 which simultaneously solves the $e^{0,b}\bar{z}^b$, b odd, components of equations (10), (11).

Combining equation (10) with the normalization conditions gives:

$$\sum e_2^{a,b} z^a \bar{z}^b \quad = \quad 2\bar{z}\overline{(p_1^H(z) + p_1^A(\bar{z}^2))} - p_2(z, \bar{z}^2, z\bar{z}).$$

Since $2\bar{z}\overline{p_1^H}$ is the only term on the RHS that could have monomials of the form $z^0 \bar{z}^b$, with b odd, p_1^H is determined uniquely and is even in z:

$$\begin{aligned} 2\bar{z}\overline{p_1^H(z)} &= \sum_{b=2k+1} e_2^{0,b} \bar{z}^b \\ 2z p_1^H(z) &= \overline{\sum_{b=2k+1} e_2^{0,b} \bar{z}^b} \\ p_1^H(z) &= \frac{1}{4z} \overline{(e_2(0, \bar{z}) - e_2(0, -\bar{z}))}. \end{aligned}$$

Of course, for $|z| < r$, the expression $e_2(0, \bar{z})$ refers to the complex analytic function $e_2(z, \zeta)$, evaluated at $(0, \bar{z}) \in D_r$; it doesn't mean $z = \bar{z} = 0$. Also, the division by z in the above expression for p_1^H should be taken formally, by subtraction of exponents in the formal power series, so that after the division, the series still has only positive exponents and converges to a complex analytic function on a domain containing $z = 0$. These conventions will be observed in some constructions that follow.

It follows from the Schwarz Lemma ([A]) that for $|z| < r$,

$$|\overline{e_2(0, \bar{z})}| \leq \frac{|z|}{r} \sup_{|z|<r} |\overline{e_2(0, \bar{z})}| \leq \frac{|z|}{r} |e_2|_r,$$

and,

$$\|p_1^H(z)\|_r = \sup_{|z|<r} \left| \frac{\overline{(e_2(0, \bar{z}) - e_2(0, -\bar{z}))}}{4z} \right| \leq \frac{1}{2r} |e_2|_r.$$

A similar substitution into equation (11):

$$\sum e_3^{a,b} z^a \bar{z}^b \quad = \quad \bar{z}(p_1^H(z) + p_1^A(\bar{z}^2)) + z\overline{p_1(z, \bar{z}^2)} - p_3(z, \bar{z}^2, z\bar{z}),$$

determines $p_1^A(z_2)$ uniquely:

$$\begin{aligned} \bar{z} p_1^A(\bar{z}^2) &= \sum_{b=2k+1} e_3^{0,b} \bar{z}^b \\ \zeta p_1^A(\zeta^2) &= \frac{1}{2}(e_3(0, \zeta) - e_3(0, -\zeta)) \\ p_1^A(z_2) &= \sum_k e_3^{0,2k+1} z_2^k. \end{aligned}$$

Again, the Schwarz Lemma gives a bound for the values:

$$\begin{aligned} \|p_1^A(z_2)\|_r &= \sup_{|\zeta|<r} |p_1^A(\zeta^2)| = \sup_{|\zeta|<r} \left| \frac{e_3(0, \zeta) - e_3(0, -\zeta)}{2\zeta} \right| \\ &\leq \sup_{|\zeta|<r} \frac{2\frac{|\zeta|}{r} |e_3|_r}{|2\zeta|} = \frac{1}{r} |e_3|_r. \end{aligned}$$

It is now possible to solve equation (10) for p_2:

$$\begin{aligned} p_2^E(z, \bar{z}^2) + z\bar{z} p_2^O(z, \bar{z}^2) &= -e_2(z, \bar{z}) + 2\bar{z}\overline{(p_1^H(z) + p_1^A(\bar{z}^2))} \\ &= -e_2(z, \bar{z}) + \sum_{b=2k+1} e_2^{0,b} \bar{z}^b + 2\frac{\bar{z}}{z} \overline{\sum_{b=2k+1} e_3^{0,b} \bar{z}^b}. \end{aligned}$$

Replacing $\bar{z}$ with ζ and separating both RHS and LHS into the even and odd parts with respect to ζ determines p_2^E and p_2^O:

$$\begin{aligned}
p_2^E(z,\zeta^2) &= -\frac{1}{2}(e_2(z,\zeta)+e_2(z,-\zeta)),\\
p_2^E(z,z_2) &= -\sum_{b=2k} e_2^{a,b} z^a z_2^k,\\
\|p_2^E\|_r &\le |e_2|_r.\\
z\zeta p_2^O(z,\zeta^2) &= -\frac{1}{2}(e_2(z,\zeta)-e_2(z,-\zeta))+\frac{1}{2}(e_2(0,\zeta)-e_2(0,-\zeta))\\
&\quad +2\frac{\zeta}{z}\overline{\frac{1}{2}(e_3(0,\bar{z})-e_3(0,-\bar{z}))},\\
p_2^O(z,\zeta^2) &= \frac{e_2(z,-\zeta)-e_2(z,\zeta)+e_2(0,\zeta)-e_2(0,-\zeta)}{2z\zeta}\\
&\quad +\frac{\overline{e_3(0,\bar{z})-e_3(0,-\bar{z})}}{z^2},\\
p_2^O(z,z_2) &= \sum_{a\ge 1, b=2k+1} e_2^{a,b} z^{a-1} z_2^k + \sum_{b=2k+1} \overline{e_3^{0,b}} z^{2k-1},\\
\|p_2^O\|_r &\le \sup_{|z|<r}\sup_{|\zeta|<r}\left|\frac{e_2(z,-\zeta)-e_2(z,\zeta)+e_2(0,\zeta)-e_2(0,-\zeta)}{2z\zeta}\right|\\
&\quad +\sup_{|z|<r}\left|\frac{\overline{e_3(0,\bar{z})-e_3(0,-\bar{z})}}{z^2}\right|\\
&\le \frac{2}{r^2}|e_2|_r+\frac{2}{r^2}|e_3|_r,
\end{aligned}$$

the last inequality following from several applications of the Schwarz Lemma. The calculation and bounds for p_3 follow from a similar treatment of equation (11).

$$\begin{aligned}
p_3^E(z,\bar{z}^2)+z\bar{z}p_3^O(z,\bar{z}^2) &= -e_3(z,\bar{z})+\bar{z}p_1(z,\bar{z}^2)+z\overline{p_1(z,\bar{z}^2)}\\
&= -e_3(z,\bar{z})+\frac{\bar{z}}{2z}\overline{\sum_{b=2k+1} e_2^{0,b}\bar{z}^b}+\sum_{b=2k+1} e_3^{0,b}\bar{z}^b\\
&\quad +\frac{z}{2\bar{z}}\sum_{b=2k+1} e_2^{0,b}\bar{z}^b+\overline{\sum_{b=2k+1} e_3^{0,b}\bar{z}^b}.
\end{aligned}$$

$$
\begin{aligned}
p_3^E(z,\zeta^2) &= -\frac{1}{2}(e_3(z,\zeta)+e_3(z,-\zeta))+\frac{z}{4\zeta}(e_2(0,\zeta)-e_2(0,-\zeta)) \\
&\quad +\frac{1}{2}\overline{(e_3(0,\bar{z})-e_3(0,-\bar{z}))}, \\
\|p_3^E\|_r &\leq 2|e_3|_r+\frac{1}{2}|e_2|_r. \\
z\zeta p_3^O(z,\zeta^2) &= -\frac{1}{2}(e_3(z,\zeta)-e_3(z,-\zeta))+\frac{1}{2}(e_3(0,\zeta)-e_3(0,-\zeta)) \\
&\quad +\frac{\zeta}{4z}\overline{(e_2(0,\bar{z})-e_2(0,-\bar{z}))}, \\
p_3^O(z,\zeta^2) &= \frac{e_3(z,-\zeta)-e_3(z,\zeta)+e_3(0,\zeta)-e_3(0,-\zeta)}{2z\zeta} \\
&\quad +\frac{\overline{e_2(0,\bar{z})-e_2(0,-\bar{z})}}{4z^2}, \\
\|p_3^O\|_r &\leq \frac{2}{r^2}|e_3|_r+\frac{1}{2r^2}|e_2|_r.
\end{aligned}
$$

∎

It is not yet claimed that $\vec{p}$ defines a biholomorphism Ψ; this will be shown later (Theorem 4.4), under certain conditions on $\vec{e}$ and r. In Section 7, the above procedure for finding $\vec{p}$ will be demonstrated for a specific choice of $\vec{e}$.

Corollary 3.5. *There is a constant $c_1 > 0$ such that for any $\vec{p}$ and $\vec{e}$ as in Theorem 3.4, and any radii ρ, r with $\frac{1}{2} < \rho < r \leq 1$, the following estimates hold:*

$$\max_{j=1,2,3}\{\|p_j\|_r\} \leq c_1|\vec{e}|_r,$$

$$\max_{j=1,2,3}\left\{\sum_{i=1}^{3}\left\|\frac{dp_i}{dz_j}\right\|_\rho\right\} \leq \frac{c_1|\vec{e}|_r}{r-\rho}.$$

Proof. The bound $\|p_1\|_r < 3|\vec{e}|_r$ follows from the estimate from the previous Theorem and $r > \frac{1}{2}$. The bounds for the derivatives of p_i follow from an estimate which is a consequence of Cauchy's estimate ([A]):

If $0 < R_2 < R_1$ and $f(z)$ is holomorphic and bounded by M for $|z| < R_1$, then $\frac{df}{dz}$ is bounded by $\frac{M}{R_1-R_2}$ for $|z| < R_2$.

This fact can be applied immediately to the bound $\|p_j\|_r \leq 5|\vec{e}|_r$, with $R_1 - R_2 = r - \rho$ for the z derivatives, and $R_1 - R_2 = r^2 - \rho^2 > r - \rho$ for the z_2, z_3 derivatives. By inspecting the derivatives of the normalized functions, one could find the constant c_1, or improve upon the bounds claimed by the Theorem. Some estimates are recorded below, for use in the proof of Theorem 4.2. Note that not all the terms have $r-\rho$ in the denominator, but $\frac{1}{r-\rho} > 2$, and it will be simpler in some of the later Theorems to treat all the quantities in the same way instead of using the sharpest estimates available.

$$\begin{aligned}
\left\|\frac{dp_1}{dz}\right\|_\rho &\le \frac{\frac{1}{2r}|e_2|_r}{r-\rho} \\
\left\|\frac{dp_1}{dz_2}\right\|_\rho &\le \frac{\frac{1}{r}|e_3|_r}{r^2-\rho^2} \\
\frac{dp_1}{dz_3} &= 0, \\
\left\|\frac{dp_2}{dz}\right\|_\rho &\le \left\|\frac{dp_2^E}{dz}\right\|_\rho + \left\|z_3\frac{dp_2^O}{dz}\right\|_\rho \le \frac{|e_2|_r}{r-\rho} + \frac{\rho^2}{r^2}\frac{2(|e_2|_r+|e_3|_r)}{r-\rho} \\
\left\|\frac{dp_2}{dz_2}\right\|_\rho &\le \left\|\frac{dp_2^E}{dz_2}\right\|_\rho + \left\|z_3\frac{dp_2^O}{dz_2}\right\|_\rho \le \frac{|e_2|_r}{r^2-\rho^2} + \frac{\rho^2}{r^2}\frac{2(|e_2|_r+|e_3|_r)}{r^2-\rho^2} \\
\left\|\frac{dp_2}{dz_3}\right\|_r &= \|p_2^O\|_r \le \frac{2}{r^2}(|e_2|_r+|e_3|_r) \\
\left\|\frac{dp_3}{dz}\right\|_\rho &\le \left\|\frac{dp_3^E}{dz}\right\|_\rho + \left\|z_3\frac{dp_3^O}{dz}\right\|_\rho \le \left(1+\frac{\rho^2}{r^2}\right)\frac{2|e_3|_r+\frac{1}{2}|e_2|_r}{r-\rho} \\
\left\|\frac{dp_3}{dz_2}\right\|_\rho &\le \left\|\frac{dp_3^E}{dz_2}\right\|_\rho + \left\|z_3\frac{dp_3^O}{dz_2}\right\|_\rho \le \left(1+\frac{\rho^2}{r^2}\right)\frac{2|e_3|_r+\frac{1}{2}|e_2|_r}{r^2-\rho^2} \\
\left\|\frac{dp_3}{dz_3}\right\|_r &= \|p_3^O\|_r \le \frac{2}{r^2}|e_3|_r + \frac{1}{2r^2}|e_2|_r.
\end{aligned}$$

∎

The lower bound $r > \frac{1}{2}$ was important for the previous Corollary, but it is not a significant *a priori* restriction on the manifold M. By a real rescaling $(z, z_2, z_3) \mapsto (sz, s^2 z_2, s^2 z_3)$, $s > 0$, the equations (2), (3) can be assumed to define M for $|z| < 1$, and for any $\eta > 0$, there is a rescaling making $|\vec{e}|_1 < \eta$. Such scalings form a subgroup of the stabilizer of the normal form, considered in Section 6.

4 The new defining equations and some estimates

The following Lemma on the ℓ_1 norm in $\mathbb{C}^n$ will be used several times:

Lemma 4.1. *Let $f(\vec{z}) = (f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))$ be a holomorphic map $f : \mathbb{D}_{\mathbf{r}} \to \mathbb{C}^n$, with*

$$\max_{j=1,\ldots,n} \left\{ \sum_{i=1}^{n} \sup_{\vec{z} \in \mathbb{D}_{\mathbf{r}}} \left| \frac{df_i}{dz_j}(\vec{z}) \right| \right\} \leq K.$$

Then, for $\vec{z}$, $\vec{z}\,' \in \mathbb{D}_{\mathbf{r}}$,

$$\sum_{i=1}^{n} |f_i(\vec{z}\,') - f_i(\vec{z})| \leq K \sum_{i=1}^{n} |z_i' - z_i|.$$

Proof. The single variable case, with $n = 1$ and $f : \mathbb{D}_{(r_1)} \to \mathbb{C}$, follows from integrating $\frac{df}{dz_1}$ along a path γ which is contained in the disc and connects z to z':

$$|f(z') - f(z)| = \left| \int_\gamma \frac{df}{dz_1} dz_1 \right| \leq K \cdot (\text{length } \gamma).$$

Since this holds in particular when γ is the shortest path, with length $|z' - z|$, the claimed inequality holds in this case.

The $n > 1$ case follows from the $n = 1$ case:

$$\begin{aligned}
\sum_{i=1}^{n} |f_i(\vec{z}\,') - f_i(\vec{z})| &= \sum_{i=1}^{n} |f_i(z_1', \ldots, z_n') - f_i(z_1', \ldots, z_{n-1}', z_n) \\
&\quad + f_i(z_1', \ldots, z_{n-1}', z_n) - f_i(z_1', \ldots, z_{n-2}', z_{n-1}, z_n) \\
&\quad + \ldots + f_i(z_1', z_2, \ldots, z_n) - f_i(z_1, \ldots, z_n)| \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sup_{\vec{z} \in \mathbb{D}_{\mathbf{r}}} \left| \frac{df_i}{dz_j}(\vec{z}) \right| \right) |z_j' - z_j| \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \sup_{\vec{z} \in \mathbb{D}_{\mathbf{r}}} \left| \frac{df_i}{dz_j}(\vec{z}) \right| \right) |z_j' - z_j| \\
&\leq \sum_{j=1}^{n} \left(\max_{l=1,\ldots,n} \left\{ \sum_{i=1}^{n} \sup_{\vec{z} \in \mathbb{D}_{\mathbf{r}}} \left| \frac{df_i}{dz_l}(\vec{z}) \right| \right\} \right) |z_j' - z_j| \\
&\leq K \sum_{j=1}^{n} |z_j' - z_j|.
\end{aligned}$$

∎

Theorem 3.4 showed that an approximate solution of (8), (9) exists; the estimates from Corollary 3.5 will measure how close the approximation is to an actual solution. Substituting the normalized solution $\vec{p}$ from the Theorem into e_2, e_3 in the RHS of (8), (9) gives two quantities depending on z, $\bar{z}$:

$$\begin{aligned}
q_2(z,\bar{z}) &= e_2(z,\bar{z}) + p_2(z,\bar{z}^2+e_2, z\bar{z}+e_3) \\
&\quad -(2\bar{z}\overline{p_1(z,\bar{z}^2+e_2)} + \overline{p_1(z,\bar{z}^2+e_2)}^2) \\
&= p_2^E(z,\bar{z}^2+e_2) - p_2^E(z,\bar{z}^2) \\
&\quad +(z\bar{z}+e_3)p_2^O(z,\bar{z}^2+e_2) - z\bar{z}p_2^O(z,\bar{z}^2) \\
&\quad -2\bar{z}\overline{(p_1(z,\bar{z}^2+e_2)-p_1(z,\bar{z}^2))} - \overline{p_1(z,\bar{z}^2+e_2)}^2,
\end{aligned} \tag{12}$$

$$\begin{aligned}
q_3(z,\bar{z}) &= e_3(z,\bar{z}) + p_3(z,\bar{z}^2+e_2, z\bar{z}+e_3) \\
&\quad -z\overline{p_1(z,\bar{z}^2+e_2)} - \bar{z}p_1(z,\bar{z}^2+e_2) - |p_1(z,\bar{z}^2+e_2)|^2 \\
&= p_3^E(z,\bar{z}^2+e_2) - p_3^E(z,\bar{z}^2) \\
&\quad +(z\bar{z}+e_3)p_3^O(z,\bar{z}^2+e_2) - z\bar{z}p_3^O(z,\bar{z}^2) \\
&\quad -z\overline{(p_1(z,\bar{z}^2+e_2)-p_1(z,\bar{z}^2))} - \bar{z}(p_1(z,\bar{z}^2+e_2)-p_1(z,\bar{z}^2)) \\
&\quad -|p_1(z,\bar{z}^2+e_2)|^2.
\end{aligned} \tag{13}$$

If $\vec{p}(z,z_2,z_3)$ is complex analytic on Δ_r, and $\sigma^2+|\vec{e}|_\sigma < r^2$, then $\vec{q} = (q_2(z,\bar{z}), q_3(z,\bar{z}))$ is a real analytic function for $|z|<\sigma$. If $\vec{q}(z,\bar{z})$ happens to be identically zero, the manifold M has been brought to normal form by the functions $\vec{p}$; if not, the degree of $\vec{q}$ is at least $2d-2$ as mentioned previously, and its values can be bounded in terms of $\vec{e}$. Define $\vec{q}(z,\zeta) = (q_2(z,\zeta), q_3(z,\zeta))$ by (12), (13), with ζ formally substituted for $\bar{z}$.

Theorem 4.2. *There are some constants $c_2>0$ and $\delta_1>0$ such that if $\frac{1}{2}<\sigma<r\le 1$, and $\vec{e}$ is as in Theorem 3.4, with $|\vec{e}|_r \le \delta_1(r-\sigma)$, then*

$$|\vec{q}|_\sigma \le \frac{c_2|\vec{e}|_r^2}{r-\sigma}.$$

Proof. Note that if $\delta_1\le 1$, the formal series for $\vec{q}$ is in fact convergent on D_σ, since then $|\zeta^2+e_2(z,\zeta)| < \sigma^2+\delta_1(r-\sigma) < \sigma^2+(r-\sigma)(r+\sigma) = r^2$, and $\vec{p}$ is convergent on Δ_r by Theorem 3.4. To find a bound on the norm of $\vec{q}$, the following corollary of the $n=1$ case of Lemma 4.1 will apply:

If $0<R_2<R_1$ and $f(z)$ is holomorphic for $|z|<R_1$, and $g: X\to\mathbb{C}$ is any function bounded by $M\le R_1-R_2$, then for $(z,x)\in\{|z|<R_2\}\times X$,

$$|f(z+g(x))-f(z)| \le M \sup_{|z|<R_1}\left|\frac{df}{dz}\right|.$$

Using this estimate and the bounds for derivatives from Corollary 3.5, with $R_1=\rho=\frac{1}{2}(\sigma+r)$, $R_2=\sigma$, $g=e_2$, and $M=|e_2|_\sigma\le\delta_1(r-\sigma)\le\rho-\sigma$

for $\delta_1 \leq \frac{1}{2}$ gives the following inequalities:

$$
\begin{aligned}
|q_2|_\sigma &\leq |e_2|_\sigma \left\| \frac{dp_2^E}{dz_2} \right\|_\rho + \sigma^2 |e_2|_\sigma \left\| \frac{dp_2^O}{dz_2} \right\|_\rho \\
&\quad + |e_3|_\sigma \|p_2^O\|_\rho + 2\sigma |e_2|_\sigma \left\| \frac{dp_1}{dz_2} \right\|_\rho + \|p_1\|_\rho^2 \\
&\leq |e_2|_\sigma \frac{|e_2|_r}{r^2 - \rho^2} + \frac{\sigma^2}{r^2} |e_2|_\sigma \frac{2(|e_2|_r + |e_3|_r)}{r^2 - \rho^2} \\
&\quad + \frac{2}{r^2} |e_3|_\sigma (|e_2|_r + |e_3|_r) + 2\frac{\sigma}{r} |e_2|_\sigma \frac{|e_3|_r}{r^2 - \rho^2} + \left(\frac{1}{2r} |e_2|_r + \frac{1}{r} |e_3|_r \right)^2, \\
|q_3|_\sigma &\leq |e_2|_\sigma \left\| \frac{dp_3^E}{dz_2} \right\|_\rho + \sigma^2 |e_2|_\sigma \left\| \frac{dp_3^O}{dz_2} \right\|_\rho \\
&\quad + |e_3|_\sigma \|p_3^O\|_\rho + 2\sigma |e_2|_\sigma \left\| \frac{dp_1}{dz_2} \right\|_\rho + \|p_1\|_\rho^2 \\
&\leq |e_2|_\sigma \frac{2|e_3|_r + \frac{1}{2}|e_2|_r}{r^2 - \rho^2} + \frac{\sigma^2}{r^2} |e_2|_\sigma \frac{2|e_3|_r + \frac{1}{2}|e_2|_r}{r^2 - \rho^2} \\
&\quad + |e_3|_\sigma \frac{4|e_3|_r + |e_2|_r}{2r^2} + 2\frac{\sigma}{r} |e_2|_\sigma \frac{|e_3|_r}{r^2 - \rho^2} + \left(\frac{1}{2r} |e_2|_r + \frac{1}{r} |e_3|_r \right)^2.
\end{aligned}
$$

∎

The purpose of constructing the functions $\vec{p}$ was to define a new coordinate system. To see how close the defining equations in the new coordinates are to the normal form equations, some more estimates will be needed. In particular, the new coordinate system will only be defined on some polydisc Δ, and the new equations $\tilde{z}_2 = \bar{\tilde{z}}^2 + \tilde{e}_2(\tilde{z}, \bar{\tilde{z}})$, $\tilde{z}_3 = \tilde{z}\bar{\tilde{z}} + \tilde{e}_3(\tilde{z}, \bar{\tilde{z}})$ will only be defined for $\tilde{z}$ in some disc of radius $\tilde{r}$.

The following Lemma, the "standard iteration procedure" for inverse functions, ([M], [SM] §§26, 33), will be used twice, in the construction of both the new coordinate system and the new defining equations. It is stated in general, with a sketch of a proof.

Lemma 4.3. *Suppose* $0 < R_2^i < R_1^i$ *for* $i = 1, \ldots, n$, *so that*

$$\mathbb{D}^2 = \mathbb{D}_{(R_2^1, \ldots, R_2^n)} \subseteq \mathbb{D}^1 = \mathbb{D}_{(R_1^1, \ldots, R_1^n)}.$$

Let $f(\vec{z}) = (f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))$ *be holomorphic on* $\mathbb{D}^1$, *with*

$$\max_{j=1,\ldots,n} \left\{ \sum_{i=1}^n \sup_{\vec{z} \in \mathbb{D}^1} \left| \frac{df_i}{dz_j}(\vec{z}) \right| \right\} \leq K < 1,$$

and

$$\sum_{i=1}^{n} \sup_{\vec{z}\in\mathbb{D}^2} |f_i(\vec{z})| \leq (1-K) \min_{i=1,\ldots,n} \{R_1^i - R_2^i\}.$$

Then, given $\vec{w} \in \mathbb{D}^2$, there exists a unique solution $\vec{z} \in \mathbb{D}^1$ of the equation

$$\vec{w} = \vec{z} + f(\vec{z}),$$

and this solution satisfies

$$\sum_{i=1}^{n} |z_i - w_i| \leq \frac{1}{1-K} \sum_{i=1}^{n} |f_i(\vec{w})|.$$

Proof. The idea is to construct a sequence $\vec{z}^{\,(k)} = (z_1^{(k)}, \ldots, z_n^{(k)})$, $k = 0, 1, 2, \ldots$, converging to $\vec{z}$ so that each $\vec{z}^{\,(k)} \in \mathbb{D}^1$. Start with an initial approximate solution $\vec{z}^{\,(0)} = \vec{w}$. Inductively, assume $\vec{z}^{\,(m)} \in \mathbb{D}^1$ for $0 \leq m \leq k$, and define

$$\vec{z}^{\,(k+1)} = \vec{w} - f(\vec{z}^{\,(k)}) \tag{14}$$

Then, $\vec{z}^{\,(k+1)} \in \mathbb{D}^1$, and for $k \geq 1$, Lemma 4.1 applies:

$$\begin{aligned} \sum_{i=1}^{n} |z_i^{(k+1)} - z_i^{(k)}| &= \sum_{i=1}^{n} |f_i(\vec{z}^{\,(k)}) - f_i(\vec{z}^{\,(k-1)})| \\ &\leq K \sum_{j=1}^{n} |z_j^{(k)} - z_j^{(k-1)}| \\ &\leq K^k \sum_{j=1}^{n} |z_j^{(1)} - z_j^{(0)}| = K^k \sum_{j=1}^{n} |f_j(\vec{w})|. \end{aligned}$$

Since $K < 1$, it follows that $\vec{z}^{\,(k)}$ forms a Cauchy sequence, and that

$$\begin{aligned} \sum_{i=1}^{n} |z_i^{(k)} - w_i| &\leq \sum_{i=1}^{n} (|z_i^{(k)} - z_i^{(k-1)}| + |z_i^{(k-1)} - z_i^{(k-2)}| + \cdots + |z_i^{(1)} - w_i|) \\ &\leq (K^{k-1} + \cdots + K + 1) \sum_{i=1}^{n} |f_i(\vec{w})| \\ &= \frac{1-K^k}{1-K} \sum_{i=1}^{n} |f_i(\vec{w})| \leq \min_{i=1,\ldots,n} \{R_1^i - R_2^i\}, \end{aligned}$$

so $\vec{z} = \lim_{k\to\infty} \vec{z}^{\,(k)}$ exists, with $|z_i - w_i| \leq R_1^i - R_2^i$, and so $\vec{z} \in \mathbb{D}^1$, and $\vec{z}$ satisfies the claimed estimate. To see that $\vec{z}$ is a solution of the original equation, take the $k \to \infty$ limit of equation (14).

Finally, to show that if $\vec{z}$ is unique, suppose $\vec{z}\,'$ is another solution; then by Lemma 4.1 again,

$$\sum_{i=1}^{n} |z_i - z_i'| = \sum_{i=1}^{n} |f_i(\vec{z}) - f_i(\vec{z}\,')| \leq K \sum_{j=1}^{n} |z_j - z_j'|,$$

contradicting $K < 1$. In fact, $\vec{z}$ is a holomorphic function of $\vec{w}$ on the polydisc $\mathbb{D}^2$. ∎

Theorem 4.4. *There is some constant $\delta_2 > 0$ so that for any radii $\frac{1}{2} < \sigma < r \leq 1$, and $\vec{e}$, $\vec{p}$ as in Theorem 3.4, with $|\vec{e}|_r \leq \delta_2(r-\sigma)$ and $\rho = \frac{1}{2}(r+\sigma)$, the transformation*

$$\Psi : (z, z_2, z_3) \mapsto (\tilde{z}, \tilde{z}_2, \tilde{z}_3) = (z + p_1, z_2 + p_2, z_3 + p_3)$$

has a holomorphic inverse $\psi(\tilde{z}, \tilde{z}_2, \tilde{z}_3) = (z, z_2, z_3)$ such that if $(\tilde{z}, \tilde{z}_2, \tilde{z}_3) \in \Delta_\sigma$, then $\psi(\tilde{z}, \tilde{z}_2, \tilde{z}_3) \in \Delta_\rho$.

Proof. By Corollary 3.5,

$$\max_{j=1,2,3} \left\{ \sum_{i=1}^{3} \left\| \frac{dp_i}{dz_j} \right\|_\rho \right\} \leq \frac{c_1 |\vec{e}|_r}{r-\rho} \leq \frac{c_1 \delta_2 (r-\sigma)}{r-\rho} = 2\delta_2 c_1 \leq \frac{1}{2} = K,$$

if $\delta_2 \leq \frac{1}{4c_1}$. Also by Corollary 3.5,

$$\|p_1\|_\sigma + \|p_2\|_\sigma + \|p_3\|_\sigma \leq 3c_1 |\vec{e}|_r \leq 3c_1 \delta_2 (r-\sigma) \leq (1-K)(\rho-\sigma),$$

if $\delta_2 \leq \frac{\rho-\sigma}{6c_1(r-\sigma)} = \frac{1}{12c_1}$. The hypotheses of Lemma 4.3 are satisfied, so given $(\tilde{z}, \tilde{z}_2, \tilde{z}_3) \in \Delta_\sigma$, there exists a unique $(z, z_2, z_3) \in \Delta_\rho$ such that $(\tilde{z}, \tilde{z}_2, \tilde{z}_3) = (z + p_1, z_2 + p_2, z_3 + p_3)$. This defines ψ so that $\Psi \circ \psi$ is the identity map on Δ_σ. ∎

Theorem 4.5. *There is some constant $\delta_3 > 0$ so that for any radii $\frac{1}{2} < r' < r \leq 1$, and $\vec{e}$, $\vec{p}$ as in Theorem 3.4, with $|\vec{e}|_r \leq \delta_3(r-r')$, and $\sigma = r' + \frac{1}{3}(r-r')$, the transformation*

$$(z, \zeta) \mapsto (\tilde{z}, \tilde{\zeta}) = (z + p_1(z, \zeta^2 + e_2(z, \zeta)), \zeta + \overline{p_1(\bar{\zeta}, \bar{z}^2 + e_2(\bar{\zeta}, \bar{z}))}) \qquad (15)$$

has a holomorphic inverse $\phi(\tilde{z}, \tilde{\zeta}) = (z, \zeta)$ such that if $(\tilde{z}, \tilde{\zeta}) \in D_{r'}$, then $\phi(\tilde{z}, \tilde{\zeta}) \in D_\sigma$.

Proof. Let $\rho = r' + \frac{2}{3}(r - r')$, so $\sigma - r' = \rho - \sigma = r - \rho = \frac{1}{3}(r - r')$. If $(z, \zeta) \in D_\sigma$, and $\delta_3 \leq \frac{2}{3}$, then $|\vec{e}|_\sigma \leq r^2 - \sigma^2$, and the map (15) is well-defined and holomorphic on D_σ. Omitting the details, Cauchy's estimate can be used in calculations similar to those in Corollary 3.5, so that there is some constant $c_3 > 0$ (not depending on $\vec{e}$), with

$$\begin{aligned} & \max\left\{\left|\frac{d}{dz}p_1(z, \zeta^2 + e_2(z, \zeta))\right|_\sigma + \left|\frac{d}{dz}\overline{p_1(\bar{\zeta}, \bar{z}^2 + e_2(\bar{\zeta}, \bar{z}))}\right|_\sigma,\right. \\ & \left.\left|\frac{d}{d\zeta}p_1(z, \zeta^2 + e_2(z, \zeta))\right|_\sigma + \left|\frac{d}{d\zeta}\overline{p_1(\bar{\zeta}, \bar{z}^2 + e_2(\bar{\zeta}, \bar{z}))}\right|_\sigma\right\} \\ \leq \; & \frac{c_3|\vec{e}|_r}{r - \rho} \leq \frac{c_3\delta_3(r - r')}{r - \rho} \leq \frac{1}{2}, \end{aligned}$$

if $\delta_3 \leq \frac{r-\rho}{2c_3(r-r')} = \frac{1}{6c_3}$. It follows from Corollary 3.5 that

$$\begin{aligned} & |p_1(z, \zeta^2 + e_2(z, \zeta))|_{r'} + |\overline{p_1(\bar{\zeta}, \bar{z}^2 + e_2(\bar{\zeta}, \bar{z}))}|_{r'} \\ \leq \; & 2\|p_1(z, z_2)\|_r \leq 2c_1|\vec{e}|_r \leq 2c_1\delta_3(r - r') \leq \frac{1}{2}(\sigma - r'), \end{aligned}$$

if $\delta_3 \leq \frac{\sigma - r'}{4c_1(r-r')} = \frac{1}{12c_1}$. So, by Lemma 4.3, given $(\tilde{z}, \tilde{\zeta}) \in D_{r'}$, there exists a unique $(z, \zeta) \in D_\sigma$ such that

$$(\tilde{z}, \tilde{\zeta}) = (z + p_1(z, \zeta^2 + e_2(z, \zeta)), \zeta + \overline{p_1(\bar{\zeta}, \bar{z}^2 + e_2(\bar{\zeta}, \bar{z}))}).$$

∎

By inspection of the form of (15), if $(z, \zeta) \in D_\sigma$, and $(z, \zeta) \mapsto (\tilde{z}, \tilde{\zeta})$, then $(\bar{\zeta}, \bar{z}) \mapsto (\bar{\tilde{\zeta}}, \bar{\tilde{z}})$. If, further, $(\tilde{z}, \tilde{\zeta}) = (\bar{\tilde{\zeta}}, \bar{\tilde{z}}) \in D_{r'}$, then $(z, \zeta) = (\bar{\zeta}, \bar{z})$ by uniqueness of the inverse. In particular, if $|\tilde{z}| < r'$, then $\phi(\tilde{z}, \bar{\tilde{z}})$ is of the form $(z, \bar{z})$ for some z with $|z| < \sigma$. Also, if there exists some other z' such that $|z'| < \sigma$ and $\tilde{z} = z' + p_1(z', (\bar{z}')^2 + e_2(z', \bar{z}'))$, then

$$\begin{aligned} \bar{\tilde{z}} &= \overline{z' + p_1(z', (\bar{z}')^2 + e_2(z', \bar{z}'))} \\ &= \bar{z}' + \overline{p_1(\overline{\bar{z}'}, (\bar{z}')^2 + e_2(\overline{\bar{z}'}, \bar{z}'))}, \end{aligned}$$

so $\phi(\tilde{z}, \bar{\tilde{z}}) = (z', \bar{z}') = (z, \bar{z})$ by uniqueness.

Theorem 4.6. *There exist constants $c_4 > 0$ and $\delta_4 > 0$ such that for any $\frac{1}{2} < r' < r \leq 1$ (with σ, ρ as in the previous Theorem), and any $\vec{e}$ as in Theorem 3.4 with $|\vec{e}|_r \leq \delta_4(r - r')$, there exist a biholomorphic map $\Psi : (z, z_2, z_3) \mapsto (\tilde{z}, \tilde{z}_2, \tilde{z}_3)$, with the domain of $\psi = \Psi^{-1}$ containing Δ_σ,* *and holomorphic functions $\tilde{e}_2$, $\tilde{e}_3$, with domain containing $D_{r'}$, such that the defining equations for M are*

$$\begin{aligned}\tilde{z}_2 &= \bar{\tilde{z}}^2 + \tilde{e}_2(\tilde{z}, \bar{\tilde{z}}),\\ \tilde{z}_3 &= \tilde{z}\bar{\tilde{z}} + \tilde{e}_3(\tilde{z}, \bar{\tilde{z}}),\end{aligned}$$

for $|\tilde{z}| < r'$. Further, the degree of $\tilde{e} = (\tilde{e}_2, \tilde{e}_3)$ is at least $2d-2$, and

$$|\tilde{e}|_{r'} \leq \frac{c_4|\vec{e}|_r^2}{r - r'}.$$

Proof. Initially, choose $\delta_4 \leq \min\{\frac{2}{3}\delta_1, \frac{2}{3}\delta_2, \delta_3\}$, so that Theorems 4.2, 4.4, 4.5 apply, and define Ψ, ψ, $\vec{q}$, and ϕ in terms of the given $\vec{e}$ and the functions $\vec{p}$ constructed in Theorem 3.4. Define $\tilde{e}(\tilde{z}, \tilde{\zeta}) = (\vec{q} \circ \phi)(\tilde{z}, \tilde{\zeta})$, so that $\tilde{e}$ is a pair of compositions of holomorphic functions, with domain containing $D_{r'}$, and

$$|\tilde{e}|_{r'} \leq |\vec{q}|_\sigma \leq \frac{c_2|\vec{e}|_r^2}{r - \sigma} = \frac{c_2|\vec{e}|_r^2}{\frac{2}{3}(r - r')}.$$

Since $\phi(\tilde{z}, \bar{\tilde{z}})$ has no constant terms, and $\vec{q}$ has degree $\geq 2d-2$ by construction, $\tilde{e}$ also has degree at least $2d-2$.

Suppose $|\tilde{z}| < r'$, and define $\tilde{z}_2 = \bar{\tilde{z}}^2 + \tilde{e}_2(\tilde{z}, \bar{\tilde{z}})$, and $\tilde{z}_3 = \tilde{z}\bar{\tilde{z}} + \tilde{e}_3(\tilde{z}, \bar{\tilde{z}})$. The claim of the Theorem is that $\psi(\tilde{z}, \tilde{z}_2, \tilde{z}_3) \in M$.

If $\delta_4^2 \leq \frac{2}{9c_2}$, then $|\tilde{e}|_{r'} \leq \frac{c_2(\delta_4(r-r'))^2}{r-\sigma} \leq \frac{1}{3}(r - r') \leq \sigma^2 - (r')^2$, so $(\tilde{z}, \tilde{z}_2, \tilde{z}_3) \in \Delta_\sigma$.

By Theorem 4.5, there exists a unique z such that $|z| < \sigma$ and $\tilde{z} = z + p_1(z, \bar{z}^2 + e_2(z, \bar{z}))$. (This z is the first component of $(z, \bar{z}) = \phi(\tilde{z}, \bar{\tilde{z}})$.) Then, define $z_2 = \bar{z}^2 + e_2(z, \bar{z})$, $z_3 = z\bar{z} + e_3(z, \bar{z})$; since $|z| < \sigma < r$, $(z, z_2, z_3) \in M$, and if $\delta_4 \leq \frac{1}{3} < \frac{\rho^2-\sigma^2}{3(\rho-\sigma)} = \frac{\rho^2-\sigma^2}{r-r'}$, then $|z_2| < \sigma^2 + \delta_4(r-r') < \rho^2$, and similarly $|z_3| < \rho^2$, so $(z, z_2, z_3) \in \Delta_\rho$.

$$\begin{aligned}\Psi(z, z_2, z_3) &= (z + p_1(z, \bar{z}^2 + e_2(z, \bar{z})),\\ &\quad\ \bar{z}^2 + e_2(z, \bar{z}) + p_2(z, \bar{z}^2 + e_2(z, \bar{z}), z\bar{z} + e_3(z, \bar{z})),\\ &\quad\ z\bar{z} + e_3(z, \bar{z}) + p_3(z, \bar{z}^2 + e_2(z, \bar{z}), z\bar{z} + e_3(z, \bar{z})))\\ &= (\tilde{z}, \overline{(\tilde{z} - p_1)}^2 + e_2 + p_2, (\tilde{z} - p_1)\overline{(\tilde{z} - p_1)} + e_3 + p_3)\\ &= (\tilde{z}, \bar{\tilde{z}}^2 + q_2(z, \bar{z}), \tilde{z}\bar{\tilde{z}} + q_3(z, \bar{z}))\\ &= (\tilde{z}, \bar{\tilde{z}}^2 + q_2(\phi(\tilde{z}, \bar{\tilde{z}})), \tilde{z}\bar{\tilde{z}} + q_3(\phi(\tilde{z}, \bar{\tilde{z}})))\\ &= (\tilde{z}, \tilde{z}_2, \tilde{z}_3),\end{aligned}$$

by construction of $\vec{p}$, $\vec{q}$, and $\tilde{z}_2$, $\tilde{z}_3$. By the uniqueness of Theorem 4.4, $\psi(\tilde{z}, \tilde{z}_2, \tilde{z}_3) = (z, z_2, z_3) \in M$. ∎

5 Composition of approximate solutions

The previous Theorem's quadratic estimate on the size of $\tilde{e}$ in terms of $\vec{e}$ allows for the rapid convergence of a sequence of approximations. A few technical lemmas will be needed to control the behavior of composite mappings. The main result, Theorem 5.7, uses these lemmas and the estimates of the previous Section to prove convergence of a sequence of transformations, following the ideas of [M] and [SM] §§26, 32.

Notation 5.1. For a 3×3 matrix of complex-valued functions $F = (F_{ij}(\vec{z}))$ on Δ_r, define

$$|||F|||_r = \max_{j=1,2,3}\left\{\sum_{i=1}^{3}\sup_{\vec{z}\in\Delta_r}|F_{ij}(\vec{z})|\right\}.$$

This "maximum column sum" norm appeared already, in Corollary 3.5 and Lemmas 4.1, 4.3, in the case where $F = \mathrm{D}f = \mathrm{D}_{\vec{z}}f$, the Jacobian matrix of some map $f:\Delta_r\to\mathbb{C}^3$ at $\vec{z}\in\Delta_r$.

Lemma 5.2. *If $|||A|||_r < 1$, then $I + A$ is invertible (where I is the 3×3 identity matrix), and*

$$|||(I+A)^{-1}|||_r \le \frac{1}{1-|||A|||_r}.$$

Proof. It is enough (cf [HJ]) to check that $|||I|||_r = 1$ and that this norm is "submultiplicative," $|||F\cdot G|||_r \le |||F|||_r\cdot|||G|||_r$. The following calculation is similar to the steps of Lemma 4.1, and generalizes to $\mathbb{D}_{\mathbf{r}}\subseteq\mathbb{C}^n$.

$$\begin{aligned}
|||F\cdot G|||_r &= \max_{j=1,2,3}\left\{\sum_{i=1}^{3}\sup_{\vec{z}\in\Delta_r}\left|\sum_{k=1}^{3}F_{ik}(\vec{z})\cdot G_{kj}(\vec{z})\right|\right\}\\
&\le \max_{j=1,2,3}\left\{\sum_{i=1}^{3}\sup_{\vec{z}\in\Delta_r}\sum_{k=1}^{3}|F_{ik}(\vec{z})|\cdot|G_{kj}(\vec{z})|\right\}\\
&\le \max_{j=1,2,3}\left\{\sum_{i=1}^{3}\sum_{k=1}^{3}\left(\sup_{\vec{z}\in\Delta_r}|F_{ik}(\vec{z})|\right)\cdot\left(\sup_{\vec{z}\in\Delta_r}|G_{kj}(\vec{z})|\right)\right\}\\
&= \max_{j=1,2,3}\left\{\sum_{k=1}^{3}\left(\sum_{i=1}^{3}\sup_{\vec{z}\in\Delta_r}|F_{ik}(\vec{z})|\right)\cdot\left(\sup_{\vec{z}\in\Delta_r}|G_{kj}(\vec{z})|\right)\right\}\\
&\le \max_{j=1,2,3}\left\{\sum_{k=1}^{3}\left(\max_{l=1,2,3}\left\{\sum_{i=1}^{3}\sup_{\vec{z}\in\Delta_r}|F_{il}(\vec{z})|\right\}\right)\cdot\left(\sup_{\vec{z}\in\Delta_r}|G_{kj}(\vec{z})|\right)\right\}\\
&= |||F|||_r\cdot|||G|||_r
\end{aligned}$$

$\blacksquare$

Also, the following elementary fact from the calculus of one real variable will be used.

Lemma 5.3. *If a_i is a sequence such that $0 \le a_i < 1$ and $\sum_{i=0}^{\infty} a_i$ is a convergent series, then the sequence of partial products*

$$\prod_{i=0}^{N} \frac{1}{1-a_i}$$

is bounded above by some positive limit. $\blacksquare$

The following notation will be convenient.

Notation 5.4. For $\nu = 0, 1, 2, \ldots$, define a sequence $\{1, \frac{3}{4}, \frac{4}{6}, \frac{5}{8}, \ldots\}$ by the formula

$$r_\nu = \frac{1}{2}\left(1 + \frac{1}{\nu+1}\right).$$

Note that $\frac{1}{2} < r_\nu \le 1$, and the sequence is decreasing, with

$$\begin{aligned} r_\nu - r_{\nu+1} &= \frac{1}{2(\nu+1)(\nu+2)}, \\ \frac{r_{\nu+1} - r_{\nu+2}}{r_\nu - r_{\nu+1}} &= \frac{\nu+1}{\nu+3} \ge \frac{1}{3}. \end{aligned}$$

Notation 5.5. Define $\sigma_\nu = r_{\nu+1} + \frac{1}{3}(r_\nu - r_{\nu+1})$, $\rho_\nu = r_{\nu+1} + \frac{2}{3}(r_\nu - r_{\nu+1})$, as in Theorem 4.5.

Recall that given $\eta > 0$, there is some scaling transformation so that $M \cap \Delta_1$ is defined by equations (2), (3), with $\vec{e}$ holomorphic on D_1, degree $d \ge 3$, and $|\vec{e}|_1 \le \eta$.

Notation 5.6. Denote $\vec{e}_0 = \vec{e}$ (so $|\vec{e}_0|_{r_0} = |\vec{e}|_1 \le \eta$), and inductively define the formal series $\vec{e}_{\nu+1}(z,\zeta)$ in terms of $\vec{e}_\nu(z,\zeta)$, by the $\vec{e} \mapsto \tilde{e}$ procedure of Theorem 4.6, with $r = r_\nu$, $r' = r_{\nu+1}$. Each $\vec{e}_\nu$ defines, as in the previous Theorems, functions $\vec{p}_\nu$, $\vec{q}_\nu$, Ψ_ν, ψ_ν, ϕ_ν, and the degree of $\vec{e}_\nu$ is denoted d_ν.

Also recall that the degree $d_{\nu+1}$ of $\vec{e}_{\nu+1}$ is at least $2d_\nu - 2$; it can be checked that this, together with $d_0 = d \ge 3$, implies $d_\nu \ge 2^\nu + 2$.

The plan is to show that the estimate in the hypothesis of Theorem 4.6 holds for all ν, to get a sequence of transformations $\psi_\nu : \Delta_{\sigma_\nu} \to \Delta_{\rho_\nu}$, so that the composition $\psi_0 \circ \ldots \circ \psi_{\nu-1} \circ \psi_\nu : \Delta_{\sigma_\nu} \to \Delta_{\rho_0}$ is well-defined, $\vec{e}_\nu$ is holomorphic on D_{r_ν}, and

$$\lim_{\nu\to\infty} |\vec{e}_\nu|_{r_\nu} = 0.$$

Theorem 5.7. *There exists $\eta > 0$ so that if $\vec{e}_0$ and M are as described above, then there exists a transformation $\Psi : \Delta_1 \to \mathbb{C}^3$, which has a holomorphic inverse $\psi : \Delta_{\frac{1}{2}} \to \Delta_1$, and such that if $(\tilde{z}, \tilde{z}_2, \tilde{z}_3) \in \widetilde{M} \cap \Delta_{\frac{1}{2}}$, then $\psi(\tilde{z}, \tilde{z}_2, \tilde{z}_3) \in M$.*

Proof. Let $\delta_5 = \min\{\delta_4, \frac{1}{3c_4}\}$. It will be shown that

$$|\vec{e}_\nu|_{r_\nu} \le \delta_5(r_\nu - r_{\nu+1}) \implies |\vec{e}_{\nu+1}|_{r_{\nu+1}} \le \delta_5(r_{\nu+1} - r_{\nu+2}).$$

By Theorem 4.6, $|\vec{e}_\nu|_{r_\nu} \le \delta_4(r_\nu - r_{\nu+1})$ and $|\vec{e}_\nu|_{r_\nu} \le \frac{1}{3c_4}(r_\nu - r_{\nu+1})$ imply

$$|\vec{e}_{\nu+1}|_{r_{\nu+1}} \le \frac{c_4 |\vec{e}_\nu|^2_{r_\nu}}{r_\nu - r_{\nu+1}} \le \frac{1}{3}|\vec{e}_\nu|_{r_\nu};$$

this already suggests a geometric decrease in the sequence of norms. Then, using the properties of the sequence r_ν,

$$\frac{1}{3}|\vec{e}_\nu|_{r_\nu} \le \frac{1}{3}\delta_5(r_\nu - r_{\nu+1}) \le \delta_5(r_{\nu+1} - r_{\nu+2}),$$

which proves the claimed implication. Using this as an inductive step, and starting the induction with $|\vec{e}_0|_{r_0} \le \delta_5(r_0 - r_1) = \frac{1}{4}\delta_5 = \eta$, the hypothesis of Theorem 4.6 is satisfied for all ν. One conclusion from Theorem 4.6 is that $\vec{e}_\nu$ is holomorphic on D_ν, with degree $d_\nu \ge 2^\nu + 2$, and $|\vec{e}_\nu|_{r_\nu} \le 3^{-\nu}\eta$. The other conclusion is that $\Psi_\nu \circ \ldots \circ \Psi_0$ is holomorphic, with inverse $\psi_0 \circ \ldots \circ \psi_\nu : \Delta_{\sigma_\nu} \to \Delta_{\rho_0}$ such that if $|\tilde{z}| < r_{\nu+1}$, and $\tilde{z}_2 = \bar{\tilde{z}}^2 + (\vec{e}_{\nu+1})_2(\tilde{z}, \bar{\tilde{z}})$, and $\tilde{z}_3 = \tilde{z}\bar{\tilde{z}} + (\vec{e}_{\nu+1})_3(\tilde{z}, \bar{\tilde{z}})$, then $(\psi_0 \circ \ldots \circ \psi_\nu)(\tilde{z}, \tilde{z}_2, \tilde{z}_3) \in M$. For $(z, z_2, z_3) \in \Delta_{\frac{1}{2}}$, the sequence (depending on ν) $(\psi_0 \circ \ldots \circ \psi_{\nu-1} \circ \psi_\nu)(z, z_2, z_3)$ is contained in Δ_1. The following argument, beginning with several applications of Lemma 4.1, shows this sequence is a Cauchy sequence, and converges to some value

$\psi(z, z_2, z_3)$.

$$\begin{aligned}
& \sum_{i=1}^{3} |(\psi_0 \circ \ldots \circ \psi_{\nu+1})_i(z, z_2, z_3) - (\psi_0 \circ \ldots \circ \psi_\nu)_i(z, z_2, z_3)| \\
= & \sum_{i=1}^{3} |(\psi_0)_i((\psi_1 \circ \ldots \circ \psi_{\nu+1})(z, z_2, z_3)) - (\psi_0)_i((\psi_1 \circ \ldots \circ \psi_\nu)(z, z_2, z_3))| \\
\leq & \; |||\mathrm{D}\psi_0|||_{\rho_1} \cdot \sum_{j=1}^{3} |(\psi_1 \circ \ldots \circ \psi_{\nu+1})_j(z, z_2, z_3) - (\psi_1 \circ \ldots \circ \psi_\nu)_j(z, z_2, z_3)| \\
\leq & \left(\prod_{m=0}^{\nu} |||\mathrm{D}\psi_m|||_{\rho_{m+1}} \right) \cdot \sum_{j=1}^{3} |(\psi_{\nu+1})_j(z, z_2, z_3) - z_j|. \qquad (16)
\end{aligned}$$

By the estimate from Lemma 4.3, with $f = \vec{p}_{\nu+1}$ and $K = \frac{1}{2}$ from the proof of Theorem 4.4, and then using the bound for $\vec{p}$ from Corollary 3.5,

$$\begin{aligned}
\sum_{j=1}^{3} |(\psi_{\nu+1})_j(z, z_2, z_3) - z_j| & \leq \frac{1}{1 - \frac{1}{2}} \sum_{j=1}^{3} |(\vec{p}_{\nu+1})_j(z, z_2, z_3)| \\
& \leq 2 \sum_{j=1}^{3} \|(\vec{p}_{\nu+1})_j\|_{\frac{1}{2}} \\
& \leq 6c_1 |\vec{e}_{\nu+1}|_{r_{\nu+1}} \leq \frac{6c_1\eta}{3^{\nu+1}}.
\end{aligned}$$

It follows from $\mathrm{D}_{\vec{z}}\psi_m = (I + \mathrm{D}_{\psi_m(\vec{z})}\vec{p}_m)^{-1}$ and Lemma 5.2 that:

$$\begin{aligned}
|||\mathrm{D}\psi_m|||_{\rho_{m+1}} & = |||(I + \mathrm{D}_{\psi_m(\vec{z})}\vec{p}_m)^{-1}|||_{\rho_{m+1}} \\
& \leq |||(I + \mathrm{D}\vec{p}_m)^{-1}|||_{\rho_m} \\
& \leq \frac{1}{1 - |||\mathrm{D}\vec{p}_m|||_{\rho_m}}.
\end{aligned}$$

Then, by Lemma 5.3, the product from (16) is bounded above by some constant c_5, since by Corollary 3.5,

$$\sum_{m=0}^{\infty} |||\mathrm{D}\vec{p}_m|||_{\rho_m} \leq \sum_{m=0}^{\infty} \frac{c_1 |\vec{e}_m|_{r_m}}{r_m - \rho_m} \leq \sum_{m=0}^{\infty} \frac{2(m+1)(m+2)c_1\eta}{3^{m-1}},$$

a convergent infinite series.

The inequality

$$\sum_{i=1}^{3} |(\psi_0 \circ \ldots \circ \psi_{\nu+1})_i(z, z_2, z_3) - (\psi_0 \circ \ldots \circ \psi_\nu)_i(z, z_2, z_3)| \leq \frac{2c_1c_5\eta}{3^\nu}$$

is enough to show that the sequence of composite functions converges pointwise and uniformly to a function ψ on $\Delta_{\frac{1}{2}}$. ∎

6 Stabilizer of the normal form

As an appendix to the main result, the set of formal transformations of $\mathbb{C}^3$ which fix the origin and preserve the defining equations of $\widetilde{M}$ is constructed. This is intended to motivate the choice of normalization in Section 3.

For $\vec{z} = (z, z_2, z_3)$, define $(\tilde{z}, \tilde{z}_2, \tilde{z}_3) = \Psi(\vec{z}) = A\vec{z} + (p_1, p_2, p_3)$, as in equations (5), (6), (7), but with a 3×3 constant matrix A of complex coefficients on the linear terms. Since the linear part of Ψ should fix the z-axis, the tangent plane of $\widetilde{M}$, the matrix A is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

The a_{12}, a_{13} entries are coefficients on terms of the form $\tilde{z} = z + a_{12}z_2 + a_{13}z_3$, and such a "weight 2" transformation does not contribute quadratic terms to the new defining equations, and might be more conveniently considered as part of the $p_1(z, z_2, z_3)$ function. The only matrix entries which affect the quadratic terms of the defining equations are a_{11} and the $a_{22}, a_{23}, a_{32}, a_{33}$ block, and since Ψ should be (formally) invertible, $\det A \neq 0$, so a_{11} can be any nonzero $s \in \mathbb{C}$, and $a_{22}a_{33} - a_{23}a_{32} \neq 0$. If Ψ stabilizes the defining equations, $z_2 = \bar{z}^2$, $z_3 = z\bar{z}$, then $\tilde{z}_2 = \bar{\tilde{z}}^2$ and $\tilde{z}_3 = \tilde{z}\bar{\tilde{z}}$, so

$$\begin{aligned} a_{22}z_2 + a_{23}z_3 + p_2 &= \overline{(sz + p_1)}^2 \\ &= a_{22}\bar{z}^2 + a_{23}z\bar{z} + p_2, \\ a_{32}z_2 + a_{33}z_3 + p_3 &= (sz + p_1)\overline{(sz + p_1)} \\ &= a_{32}\bar{z}^2 + a_{33}z\bar{z} + p_3. \end{aligned}$$

Comparing the quadratic coefficients gives $a_{22} = \bar{s}^2$, $a_{23} = a_{32} = 0$, and $a_{33} = s\bar{s}$, so Ψ is of the form

$$(z, z_2, z_3) \mapsto (sz, \bar{s}^2z_2, |s|^2z_3) + (p_1, p_2, p_3),$$

with p_1 weight ≥ 2, and p_2, p_3 weight ≥ 3. Constructing $\vec{q}$ as in Section 4 gives:

$$\begin{aligned}
\tilde{z}_2 - \bar{\tilde{z}}^2 &= p_2(z, \bar{z}^2, z\bar{z}) - 2\bar{s}\bar{z}\overline{p_1(z, \bar{z}^2, z\bar{z})} - (\overline{p_1(z, \bar{z}^2, z\bar{z})})^2, \\
\tilde{z}_3 - \tilde{z}\bar{\tilde{z}} &= p_3(z, \bar{z}^2, z\bar{z}) - sz\overline{p_1(z, \bar{z}^2, z\bar{z})} - \bar{s}\bar{z}p_1(z, \bar{z}^2, z\bar{z}) - |p_1(z, \bar{z}^2, z\bar{z})|^2.
\end{aligned}$$

Setting RHS equal to 0 determines $p_2(z, \bar{z}^2, z\bar{z})$ and $p_3(z, \bar{z}^2, z\bar{z})$ in terms of p_1, however, p_1 cannot be arbitrary, and $p_2(z, z_2, z_3)$, $p_3(z, z_2, z_3)$ are not uniquely defined by their restriction to $\widetilde{M}$.

Let $\mathcal{Q}$ be the subalgebra of $\mathbb{C}[[z, \bar{z}]]$ of formal series of monomials of the form $z^a(\bar{z}^2)^k(z\bar{z})^c$; any monomial can be written in this form except $\bar{z}^b$ with b odd.

Let $\mathcal{J}$ denote the ideal generated by $(z^2z_2 - z_3^2)$ in $\mathbb{C}[[z, z_2, z_3]]$. As mentioned in the proof of Theorem 3.4, any formal series p can be written as $(z^2z_2 - z_3^2) \cdot \mathrm{Q} + z_3p^O(z, z_2) + p^E(z, z_2)$. A series in $\mathbb{C}[[z, \bar{z}]]$ is in $\mathcal{Q}$ if and only if it is the restriction of some $z_3p^O(z, z_2) + p^E(z, z_2)$ to $z_2 = \bar{z}^2$, $\bar{z}_3 = z\bar{z}$.

In order for p_1, p_2, p_3 to satisfy:

$$\begin{aligned}
p_2(z, \bar{z}^2, z\bar{z}) &= 2\bar{s}\bar{z}\overline{p_1(z, \bar{z}^2, z\bar{z})} + (\overline{p_1(z, \bar{z}^2, z\bar{z})})^2, \\
p_3(z, \bar{z}^2, z\bar{z}) &= sz\overline{p_1(z, \bar{z}^2, z\bar{z})} + \bar{s}\bar{z}p_1(z, \bar{z}^2, z\bar{z}) + |p_1(z, \bar{z}^2, z\bar{z})|^2,
\end{aligned}$$

the RHS must be in $\mathcal{Q}$ (since the LHS is in $\mathcal{Q}$). In particular, $\bar{p}_1 \cdot (2\bar{s}\bar{z} + \bar{p}_1)$ and $p_1 \cdot (\bar{s}\bar{z} + \bar{p}_1)$ must both be in $\mathcal{Q}$ when p_1 is restricted to $\widetilde{M}$. Any p_1 satisfying this condition defines a formal stabilizing transformation Ψ, since then p_2, p_3 will be determined mod $\mathcal{J}$.

The normalization chosen for a transformation $\vec{p}$ in Theorem 3.4, which was intended to alter the defining functions, not preserve them, was that $\bar{z}\bar{p}_1$ and $\bar{z}p_1$ were not in $\mathcal{Q}$ when restricted to $\widetilde{M}$, and that p_2, p_3 could be arbitrary, but with $\mathrm{Q} = 0$ in the decomposition mod $\mathcal{J}$.

7 An example of an approximate solution

As an example of the approximate normalizing transformation $\vec{p}$, consider the following real algebraic surface with a non-degenerate complex tangent in $\mathbb{C}^3$, with $d = 2k + 1 \geq 3$, and $R > 0$:

$$z_2 = \bar{z}^2 + \frac{\bar{z}^d}{R - \bar{z}}, \tag{17}$$

$$z_3 = z\bar{z} + \frac{\bar{z}^d}{R - z}. \tag{18}$$

It is left as an exercise to show that the construction of Theorem 3.4 results in the following normalized $\vec{p}$, and as another exercise to verify $\vec{p}$ is an exact solution of equations (10) and (11). Note that the functions are holomorphic on Δ_R, with weights equal to $d-1$, d, d.

$$\begin{aligned}
p_1 &= \frac{Rz^{2k}}{2(R^2-z^2)}+\frac{z_2^k}{R},\\
p_2 &= \frac{2z_3z^{2k-1}}{R}-\frac{z_2^{k+1}}{R^2-z_2},\\
p_3 &= \frac{2z^{2k+3}z_2-R^2z^3z_2^k-2R^2z^{2k+3}+R^4zz_2^k+2R^4z^{2k+1}-2R^2z^{2k+1}z_2}{2R(R^2-z_2)(R^2-z^2)}\\
&\quad +z_3\frac{2zz_2^{k+1}-2R^2zz_2^k+2Rz_2^{k+1}-2R^3z_2^k+R^4z^{2k-1}-R^2z^{2k-1}z_2}{2R(R^2-z_2)(R^2-z^2)}.
\end{aligned}$$

8 Surfaces in higher dimensions

Since the expected real codimension of the complex tangent locus of a real m-manifold in $\mathbb{C}^n$ is $2(n-m+1)$ when $m \le n$, a real 2-manifold M with an isolated complex tangent in $\mathbb{C}^n$, $n>3$, is quite exceptional. However, for a sufficiently large family of surfaces, one might expect that while most are totally real, some of its members may have an isolated complex tangent. A naturally occuring example of such a family of surfaces in a complex 4-manifold is described by [C$_2$], where quadratic rational functions are used to map the real projective plane into $\mathbb{C}P^4$, and in some special cases, the rational functions define an embedding with exactly one complex tangent.

With coordinate system $(z, z_2, \ldots, z_n)$ for $\mathbb{C}^n$, the complex tangent plane of the surface M can be arranged to be the z-axis as in Section 2, and then the local real analytic defining equations are of the form $(z_2, \ldots, z_n) = (h_2(z,\bar{z}), \ldots, h_n(z,\bar{z}))$, with

$$\begin{aligned}
h_i(z,\zeta) &= \beta_i z\zeta+\gamma_i\zeta^2+\theta_i\zeta^3\\
&\quad +\alpha_i z^2+\kappa_i z^3+\lambda_i z^2\zeta+\mu_i z\zeta^2+\sum_{a+b\ge 4} h_i^{a,b}z^a\zeta^b,
\end{aligned}$$

for $i=2,\ldots,n$. The non-degeneracy condition is that the $(n-1)\times 3$ matrix of the β_i, γ_i, θ_i complex coefficients has full rank. In this case, a complex linear transformation of $(z_2, \ldots, z_n)$ can diagonalize the first three rows of the matrix, and it is clear that $\alpha_i z^2$ and the remaining cubic terms can be eliminated by weight 2 and 3 transformations of the form $z_i \mapsto z_i + p_i(z, z_2, z_3)$, leaving the following cubic normal form:

$$\begin{aligned} z_2 &= \bar{z}^2 + e_2(z, \bar{z}) \\ z_3 &= z\bar{z} + e_3(z, \bar{z}) \\ z_4 &= \bar{z}^3 + e_4(z, \bar{z}) \\ z_s &= e_s(z, \bar{z}), \end{aligned}$$

with degree ≥ 4 functions e_2, e_3, e_4, and e_s, $5 \leq s \leq n$.

Theorem 8.1. *Given $n \geq 4$, and assuming M is a non-degenerate surface in $\mathbb{C}^n$, (in the sense that it has the above defining equations), there exists a local biholomorphism Ψ taking M to the real variety $\widetilde{M}$ defined by:*

$$z_2 = \bar{z}^2, \ z_3 = z\bar{z}, \ z_4 = \bar{z}^3, \ z_s = 0.$$

Proof. As in Theorem 3.4, the main idea is that any series $e(z, \bar{z})$ can be decomposed as a sum of three series, but the $\bar{z}^{2k+1}$ terms now can be normalized using $z_4 = \bar{z}^3$. Ψ need not involve a $z \mapsto z + p_1$ component, and this is such a significant simplification that the rapid convergence argument will not be needed. Rather than constructing an approximate Ψ as in Theorem 3.4, the exact inverse map ψ, defined by

$$\psi(z, z_2, \ldots, z_n) = (z, z_2 + P_2(z, z_2, z_3, z_4), \ldots, z_n + P_n(z, z_2, z_3, z_4)),$$

can be directly computed in terms of $e_2, \ldots, e_n$. The condition that defines a normalizing ψ is that if $\vec{z} \in \widetilde{M}$, then $\psi(\vec{z}) \in M$, so that the following equations must hold when $\vec{z} = (z, z_2, \ldots, z_n)$ is restricted to $(z, \bar{z}^2, z\bar{z}, \bar{z}^3, 0, \ldots, 0)$:

$$\begin{aligned} z_2 + P_2(\vec{z}) &= \bar{z}^2 + e_2(z, \bar{z}) \\ z_3 + P_3(\vec{z}) &= z\bar{z} + e_3(z, \bar{z}) \\ z_4 + P_4(\vec{z}) &= \bar{z}^3 + e_4(z, \bar{z}) \\ z_s + P_s(\vec{z}) &= e_s(z, \bar{z}). \end{aligned}$$

Evidently these equations can be solved exactly, by showing that any real analytic $e(z, \bar{z})$ is agrees in some neighborhood with the restriction to $\widetilde{M}$ of some function $P(z, z_2, z_3, z_4)$ that is complex analytic on a polydisc in $\mathbb{C}^n$. To solve for P_i in terms of e_i, $i = 2, \ldots, n$, normalize P_i so that

$$P_i(z, z_2, z_3, z_4) = P_i^E(z, z_2) + z_3 P_i^O(z, z_2) + z_4 P_i^C(z_2).$$

(As in Theorem 3.4, the superscripts E and O abbreviate even and odd, and C denotes the use of a "cubic" term.) The function

$$\frac{1}{2}(e_i(0,\zeta) - e_i(0,-\zeta)) = \sum e_i^{0,2k+1}\zeta^{2k+1}$$

begins with at least a fifth degree term; let $P_i^C(z_2) = \sum e_i^{0,2k+1} z_2^{k-1}$, so that $\bar{z}^3 P_i^C(\bar{z}^2) = \sum e_i^{0,2k+1}\bar{z}^{2k+1}$. If $e_i(z,\zeta)$ is complex analytic on D_r, and $|e_i|_r < \infty$, then the series P_i^C is convergent for $z_2 < r^2$, and it and its derivatives are bounded (on a smaller disc) by some multiple of $|e_i|_r$, as in Corollary 3.5. The rest of the details are omitted, but straightforward modifications of the calculations in Section 3 will find P_i^E and P_i^O in terms of e_i and P_i^C, and Lemma 4.3 will show that ψ is invertible on some neighborhood of the origin. ∎

The complexification of $\widetilde{M}$ in $\mathbb{C}^4$ is the complex surface defined by the ideal $\langle z_2z_3 - z_1z_4, z_2^3 - z_4^2, z_1z_2^2 - z_3z_4, z_1^2z_2 - z_3^2\rangle$.

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