Toward Understanding Spatial Dependence on Epidemic Thresholds in Networks

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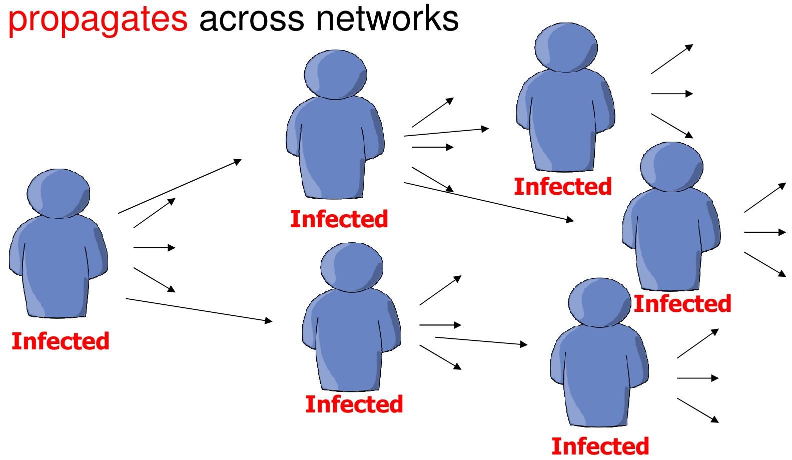
Social Influence in Online Social Networks

- Viral marketing ("word-of-mouth")
- Blog information cascading
- Rumor spreading
- ……

Bear a resemblance to epidemic process!

Epidemic Process

- Epidemic process is a process that information self-propagates across networks

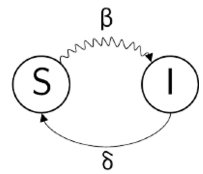


Epidemiological Models

- Susceptible-infected-recovered (SIR) model for independent cascading influence spread
- Susceptible-infected-susceptible (SIS) model for blog information cascading
- Susceptible-infected-cured (SIC) model for rumor and anti-rumor propagation

Epidemic Thresholds

- Fundamental metric used to evaluate epidemic spread
- Condition on which an information will either die out or become epidemic
- In SIS model,
  - Birth rate
  - Death rate
  - Ratio between birth rate and death rate
    - $\beta / \delta >$ epidemic threshold, become epidemic
    - $\beta / \delta <=$ epidemic threshold, die out



State of the Art

$$\tau = \frac{\mathbf{1}}{\lambda_{max}(\boldsymbol{A})}$$

- $\lambda_{max}(A)$ is the largest eigenvalue of the adjacency matrix A of the network

Assume that the status of nodes in the network are independent of each other!

Questions

- Can spatial dependence among nodes affect the epidemic threshold? If so, how significantly?
- How can we derive a more accurate epidemic threshold, taking into consideration a certain spatial dependence?
- Can the birth rate and the death rate affect the spatial dependence and thus the epidemic threshold? If so, how?

Outline

- Mathematical Framework
- Epidemic Thresholds in Regular Graphs
- Epidemic Thresholds in Arbitrary Networks

Mathematical Framework

- $X_i(t)$: status of node i at time t

$$X_i(t) = \begin{cases} 0, & \text{if susceptible} \\ 1, & \text{if infected} \end{cases}$$

$$\delta = P(X_i(t+1) = 0 \mid X_i(t) = 1)$$

$$I_i(t) = P(X_i(t+1) = 1 \mid X_i(t) = 0)$$

$$P(X_i(t+1) = 1) = P(X_i(t) = 1)(1-\delta) + P(X_i(t) = 0)\, I_i(t)$$

Mathematical Framework

$$\begin{aligned} I_i(t) &= \sum_{\mathbf{x}_{N_i}(t)} P(X_i(t+1) = 1, \mathbf{X}_{N_i}(t) = \mathbf{x}_{N_i}(t) | X_i(t) = 0) \\ &= \sum_{\mathbf{x}_{N_i}(t)} \boxed{P(\mathbf{X}_{N_i}(t) = \mathbf{x}_{N_i}(t) | X_i(t) = 0)} \left[1 - \prod_{j \in N_i} (1-\beta)^{x_j(t)}\right] \\ &= 1 - E\left[\prod_{j \in N_i} (1-\beta)^{X_j(t)} \middle| X_i(t) = 0\right], \qquad (6) \end{aligned}$$

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Epidemic Thresholds in Regular Graphs

- It has been shown that the epidemic threshold proposed in previous work does not work well in regular graphs.
- More importantly, due to the symmetric property of regular graphs, we can derive a closed-form expression for the epidemic threshold.

Independent Model

- Assume spatial independence between nodes

$$
\begin{aligned}
I_i(t) &= 1 - \prod_{j \in N_i} E\left[(1-\beta)^{X_j(t)}\right] \\
&= 1 - \prod_{j \in N_i} [1 - \beta P(X_j(t) = 1)]
\end{aligned}
$$

$$\tau_{c,ind} = \frac{1}{k}$$

where k is the average nodal degree and the largest eigenvalue of adjacency matrix

Markov Model

- Assume spatial Markov dependence
- Inspired by the local Markov property of Markov Random Field

$$
\begin{aligned}
I_i(t) &= 1 - \prod_{j \in N_i} E\left[(1-\beta)^{X_j(t)} \middle| X_i(t) = 0\right] \\
&= 1 - \prod_{j \in N_i} [1 - \beta P(X_j(t) = 1 | X_i(t) = 0)]
\end{aligned}
$$

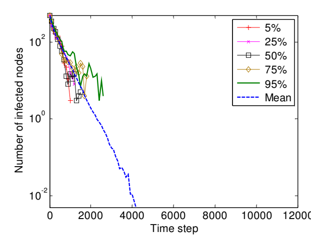
$$\tau_{c,mar} = \frac{1}{k(1-\rho)}, \qquad \rho = \frac{2\beta(1-\delta)}{(2\delta - \delta^2) + 2\beta(1-\delta)}$$

Simulation Setup

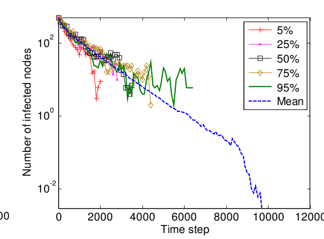
- Simulator is based on discrete time and random number generator
- Run 1000 times for each scenario
- Run long enough so that it reaches the steady state (e.g., 12000 time steps)
- Assign half of nodes to be infected initially

Sample Run of Epidemic Spread in a Ring Graph

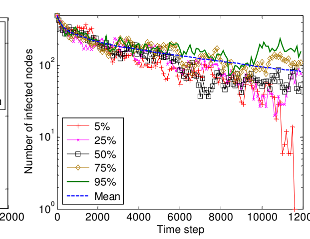
- δ = 0.1 and |V| = 1000



β = 0.13

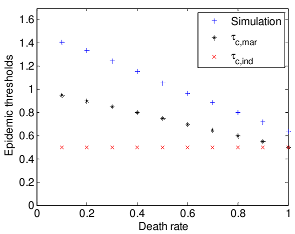


β = 0.14

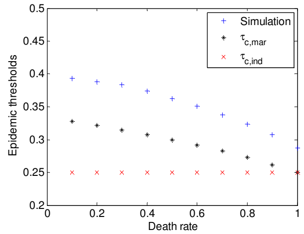


β = 0.15

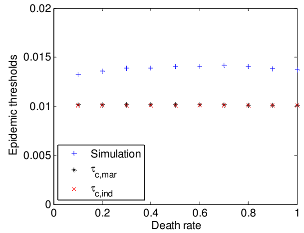
Performance Evaluation



Ring (|V| = 1000)



Lattice (|V| = 2500)



Complete (|V| = 100)

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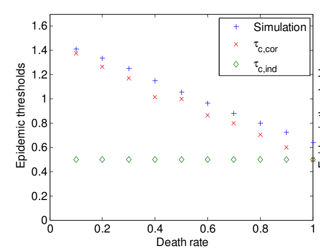
Conjecture Epidemic Thresholds of Arbitrary Network

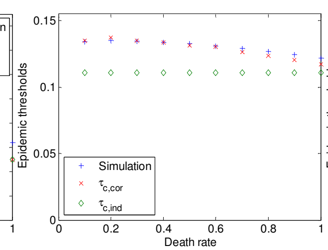
$$\tau_{c,ind} = \frac{1}{\lambda_{max}(A)} \qquad \tau_{c,mar} = \frac{1}{k(1-\rho)}$$

$$\tau_{c,mar} = \frac{1}{\lambda_{max}(A)(1-\rho_e)}$$

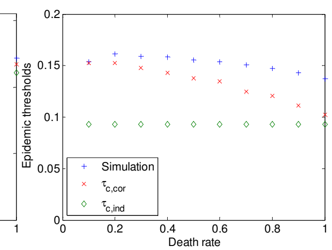
Performance Evaluation (|V| = 1000)



Ring ($\lambda_{max}(A)$ = 2)



ER random ($\lambda_{max}(A)$ = 9.03)



Power law ($\lambda_{max}(A)$ = 10.77)

Conclusions and Future Work

$$\tau_{c,ind} = \frac{1}{\lambda_{max}(A)} \longrightarrow \tau_{c,mar} = \frac{1}{\lambda_{max}(A)(1-\rho_e)}$$

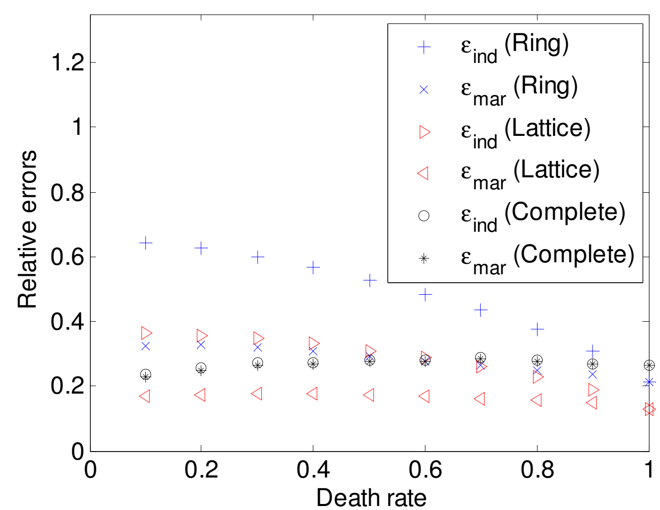
- Future work
  - Derive the equation for epidemic thresholds in irregular graphs
  - Apply our observations for predicting and controlling the dynamics of the epidemic spreading process

Thanks for your attention

Algorithm 1 Finding epidemic threshold τ_c

Input: δ, β_{low}, β_{high}
Output: τ_c
while $\beta_{high} - \beta_{low} > \epsilon$ **do**
  $\beta = (\beta_{high} + \beta_{low})/2$
  Simulate epidemic spread using β and δ
  Average the number of final infections over 1000 runs and get avg_inf_num
  if $avg\_inf\_num > 0$ **then**
    $\beta_{high} = \beta$
  else
    $\beta_{low} = \beta$
  end if
end while
$\tau_c = \beta_{low}/\delta$

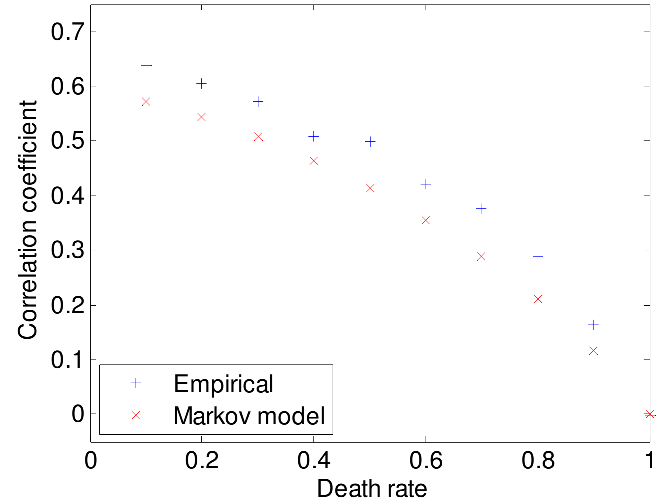
Relative Errors



Algorithm 2 Finding spatial correlation coefficient ρ_e

Input: δ, τ_c, β_s
Output: ρ_e
Set $\beta = \tau_c \times \delta$ and $found = 0$
while $found = 0$ **do**
  Simulate epidemic spread using β and δ
  Average the number of final infections over 1000 runs and get avg_inf_num
  Average the correlation coefficient over 1000 runs and get ρ
  if $avg\_inf\_num \geq 1$ **then**
    $\rho_e = \rho$
    $found = 1$
  end if
  $\beta = \beta + \beta_s$
end while

Correlation Coefficients at Epidemic Thresholds in Ring Graph (|V| = 1000)



Relative Errors