THE SCHWARZ-PICK LEMMA OF HIGH ORDER IN SEVERAL VARIABLES

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Abstract. We prove a high order Schwarz-Pick lemma for mappings between unit balls in complex spaces in terms of the Bergman metric. From this lemma, Schwarz-Pick estimates for partial derivatives of arbitrary order of mappings are deduced.

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1. Introduction

Let $\mathbb{B}_n$ be the unit ball in the complex space $\mathbb{C}^n$ of dimension n. The unit disk in the complex plane is denoted by $\mathbb{D}$. For $z = (z_1, \cdots, z_n)$ and $z' = (z'_1, \cdots, z'_n) \in \mathbb{C}^n$, denote $\langle z, z'\rangle = z_1\overline{z}'_1 + \cdots + z_n\overline{z}'_n$ and $|z| = \langle z, z\rangle^{1/2}$.

A multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ of dimension n consists of n non-negative integers α_j, $1 \leq j \leq n$, the degree of a multi-index α is the sum $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$, and we denote $\alpha! = \alpha_1! \cdots \alpha_n!$. For $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ and a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, let $z^\alpha = \prod\limits_{j=1}^{n} {z_j}^{\alpha_j}$. A holomorphic function f on $\mathbb{B}_n$ can be expressed by $f(z) = \sum\limits_{\alpha} a_\alpha z^\alpha$. For two multi-indexes $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $v = (v_1, \cdots, v_n)$, let $v^\alpha = v_1^{\alpha_1}, \cdots, v_n^{\alpha_n}$. Note that $v_j^{\alpha_j} = 1$ if $v_j = \alpha_j = 0$. Let $\Omega_{n,m}$ be the class of all holomorphic mappings f from $\mathbb{B}_n$ into $\mathbb{B}_m$. For $f \in \Omega_{n,m}$, if

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$f=(f_1,\cdots,f_m)$, $f_j(z)=\sum\limits_{\alpha} a_{j,\alpha}z^{\alpha}$ for $j=1,\cdots,m$, we denote $f(z)=\sum\limits_{\alpha} a_{\alpha}z^{\alpha}$, where $a_{\alpha}=(a_{1,\alpha},\cdots,a_{m,\alpha})$.

For $f\in\Omega_{1,1}$, the classical Schwarz-Pick lemma says that

$$\frac{|f'(z)|}{1-|f(z)|^2}\le\frac{1}{1-|z|^2}$$

holds for $z\in\mathbb{D}$. Recently, the above inequality has been generalized to the derivatives of arbitrary order by some authors [MSZ, Zh, DP]. The best result was proved in [DP]. It was proved that

$$\frac{|f^{(k)}(z)|}{1-|f(z)|^2}\le(1+|z|)^{k-1}\cdot\frac{k!}{(1-|z|^2)^k} \tag{1.1}$$

holds for $f\in\Omega_{1,1}$, $k\ge 1$ and $z\in\mathbb{D}$. The equality in (1.1) may be attained if $z=0$, and the equality statement has been established. If $k>1$ and $z\neq 0$, (1.1) is a strict inequality.

Chen and Liu [ChL] generalized (1.1) by proving the following Schwarz-Pick estimate for partial derivatives of arbitrary order of a function $f\in\Omega_{n,1}$:

$$\left|\frac{\partial^{|v|}f(z)}{\partial {z_1}^{v_1}\ldots\partial {z_n}^{v_n}}\right|\le n^{\frac{|v|}{2}}|v|!\binom{n+|v|-1}{n-1}^{n+2}\frac{1-|f(z)|^2}{(1-|z|^2)^{|v|}}(1+|z|)^{|v|-1} \tag{1.2}$$

holds for any $z\in\mathbb{B}_n$ and multi-index $v=(v_1,\cdots,v_n)\neq 0$.

On the unit ball $\mathbb{B}_n$, the Bergman metric $H_z(\beta,\beta)$ may be defined by

$$H_z(\beta,\beta)=\frac{(1-|z|^2)|\beta|^2+|\langle\beta,z\rangle|^2}{(1-|z|^2)^2}\quad\text{for}\quad z\in\mathbb{B}_n,\ \beta\in\mathbb{C}^n.$$

Commonly, there is a factor $(n+1)/2$ in the definition of the Bergman metric. In spite of ambiguity, we use the same notation for Bergman metrics in unit balls of different dimensions. This metric is invariant under the automorphism group of

$\mathbb{B}_n$. For $f \in \Omega_{n,m}$, the Schwarz-Pick lemma is formulated in terms of the Bergman metric (see [C]):

$$H_{f(z)}(f'(z)\beta, f'(z)\beta) \leq H_z(\beta, \beta) \quad \text{for} \quad z \in \mathbb{B}_n,\ \beta \in \mathbb{C}^n. \tag{1.3}$$

Here, $f'(z)$ is the Jacobian matrix of the mapping f at the point z, i.e., $f'(z) = (\partial f_j(z)/\partial z_k)_{1\leq j\leq m, 1\leq k\leq n}$, and we identify a point in complex space with a column matrix (column vector) so that $f'(z)\beta$ is the product of two matrixes. (1.3) is precise, the equality holds for mappings in the automorphism group of $\mathbb{B}_n$ if $m = n$.

The purpose of this paper is to generalize (1.3) to the high order Fréchet derivatives of mappings in $\Omega_{n,m}$ as was done in [DP] for the classical Schwarz-Pick lemma. For $f \in \Omega_{n,m}$, $k \geq 1$, $z \in \mathbb{B}_n$, the Fréchet derivative of f at z of order k is defined by

$$D_k(f, z, \beta) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial^k f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \beta^\alpha,$$

where $\beta \in \mathbb{C}_n$. $D_k(f, z, 1) = f^{(k)}(z)$ when $n = m = 1$. With this notation, our main result is expressed as follows:

Theorem: *Let $f \in \Omega_{n,m}$. Then, for $k \geq 1$, $z \in \mathbb{B}_n$ and $\beta \in \mathbb{C}^n$, we have*

$$H_{f(z)}(D_k(f, z, \beta),\ D_k(f, z, \beta)) \leq k!^2 \left(1 + \frac{|\langle \beta, z\rangle|}{((1-|z|^2)|\beta|^2 + |\langle \beta, z\rangle|^2)^{1/2}}\right)^{2(k-1)} (H_z(\beta, \beta))^k. \tag{1.4}$$

(1.4) coincides with (1.1) or (1.3) if $n = m = 1$ or $k = 1$ respectively. Note that the factor preceding $(H_z(\beta, \beta))^k$ is increasing with $|\langle \beta, z\rangle|$ from 0 to $|z|$.

As a consequence, we deduce from (1.4) a Schwarz-Pick estimate for partial derivatives of a mapping $f \in \Omega_{n,m}$:

$$\left|\left\langle \frac{\partial^{|v|} f(z)}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}},\ f(z) \right\rangle\right|^2 + (1 - |f(z)|^2) \left| \frac{\partial^{|v|} f(z)}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}} \right|^2$$

$$\leq \frac{|v|^{|v|}}{v^v} \left[v!(1+|z|)^{|v|-1} \cdot \frac{1-|f(z)|^2}{(1-|z|^2)^{|v|}} \right]^2 \tag{1.5}$$

holds for any multi-index $v = (v_1, \cdots, v_n) \neq 0$ and $z \in \mathbb{B}_n$. In particular, if $f \in \Omega_{n,1}$, we have

$$\left| \frac{\partial^{|v|} f(z)}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}} \right| \leq \sqrt{\frac{|v|^{|v|}}{v^v}} v!(1+|z|)^{|v|-1} \cdot \frac{1-|f(z)|^2}{(1-|z|^2)^{|v|}}. \tag{1.6}$$

The equalities in (1.5) and (1.6) may be attained if $z = 0$ and the equality statement is given. (1.6) is much better than (1.2) since the factor $\binom{n+|v|-1}{n-1}^{n+2}$ is canceled, $v! \leq |v|!$ and $\sqrt{|v|^{|v|}/v^v} \leq n^{|v|/2}$ (the equality holds if and only if $v_1 = \cdots = v_n$).

For radial and normal partial derivatives, we have estimates more precise than (1.5) and (1.6). For $f \in \Omega_{n,1}$, we prove that

$$\left| \frac{\partial^{|v|} f(z)}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}} \right| \leq \sqrt{\frac{|v|^{|v|}}{v^v}} v! \mu(z) \cdot \frac{1-|f(z)|^2}{(1-|z|^2)^{(v_1+|v|)/2}} \tag{1.7}$$

holds for any multi-index $v = (v_1, \cdots, v_n) \neq 0$ and $z = (z_1, 0, \cdots, 0) \in \mathbb{B}_n$, where $\mu(z) = (1+|z|)^{|v|-1}$ if $v_1 = |v|$, and $\mu(z)$ is the sum of terms $c_j |z|^j$ with $j \leq v_1$ in $(1+|z|)^{|v|-1}$.

2. Some lemmas

The following results are known [R]. For a point a in a unit ball, let

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1-|a|^2} Q_a z}{1 - \langle z, a \rangle},$$

where $P_a z = \langle z, a\rangle a/\langle a, a\rangle$, $Q_a z = z - P_a z$. Note that $P_0(z) = 0$. Then, φ_a is injective and maps the unit ball onto itself,

$$\varphi_a(0) = a, \quad \varphi_a(a) = 0, \quad \varphi_a = \varphi_a^{-1},$$

and

$$\varphi_a'(0) = -(1 - |a|^2)P_a - (1 - |a|^2)^{1/2} Q_a,$$

$$\varphi_a'(a) = -\frac{1}{1 - |a|^2} P_a - \frac{1}{(1 - |a|^2)^{1/2}} Q_a.$$

Lemma 1. *If $f(z) = \sum_\alpha a_\alpha z^\alpha \in \Omega_{n,m}$, then*

$$\sum_\alpha |a_\alpha|^2 \left|\beta^{2\alpha}\right| \le 1 \tag{2.1}$$

holds for $\beta \in \partial\mathbb{B}_n$. Further,

$$\sum_\alpha |a_\alpha|^2 \cdot \frac{v^\alpha}{|v|^{|\alpha|}} \le 1 \tag{2.2}$$

holds for any multi-index $v = (v_1, \cdots, v_j) \neq 0$. As a consequence, we have

$$|a_v| \le \sqrt{\frac{|v|^{|v|}}{v^v}}. \tag{2.3}$$

Further, if $v_j \neq 0$ for $j = 1, \cdots, n$, then the equality in (2.3) holds only if $a_\alpha = 0$ for $\alpha \neq v$.

Proof. Let $\beta = (\beta_1, \cdots, \beta_n) \in \partial\mathbb{B}_n$ be fixed. For $0 < \sigma < 1$, we have

$$\begin{aligned} 1 &\ge \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} |f(\sigma\beta_1 e^{i\theta_1}, \cdots, \sigma\beta_n e^{i\theta_n})|^2 d\theta_1 \cdots d\theta_n \\ &= \frac{1}{(2\pi)^n} \sum_{j=1}^m \int_0^{2\pi} \cdots \int_0^{2\pi} |f_j(\sigma\beta_1 e^{i\theta_1}, \cdots, \sigma\beta_n e^{i\theta_n})|^2 d\theta_1 \cdots d\theta_n \\ &= \sum_{j=1}^m \sum_\alpha |a_{j,\alpha}|^2 \sigma^{2|\alpha|} |\beta_1|^{2\alpha_1} \cdots |\beta_n|^{2\alpha_n} = \sum_\alpha |a_\alpha|^2 \sigma^{2|\alpha|} |\beta_1|^{2\alpha_1} \cdots |\beta_n|^{2\alpha_n}. \end{aligned}$$

Letting $\sigma \to 1$ gives (2.1). Thus, for given $v = (v_1, \cdots, v_n) \neq 0$, letting $\beta_j = \sqrt{v_j/|v|}$ for $j = 1, \cdots, n$ in (2.1), we obtain (2.2). The lemma is proved. □

In the above proof, in order to get the best estimate (2.3) for a_v, we deduce (2.2) by choosing $\beta_j = \sqrt{v_j/|v|}$ in (2.1), since the maximum $\max\limits_{\beta \in \partial \mathbb{B}_n} |\beta^v| = \sqrt{\frac{v^v}{|v|^{|v|}}}$ is attained when $\beta_j = \sqrt{v_j/|v|}$ for $j = 1, \cdots, n$.

Lemma 2. *If $f(z) = \sum\limits_{\alpha} a_\alpha z^\alpha \in \Omega_{n,m}$, then*

$$\sum_{k=0}^{\infty} \left| \sum_{|\alpha|=k} a_\alpha \beta^a \right|^2 \leq 1 \tag{2.4}$$

holds for $\beta \in \partial \mathbb{B}_n$.

Proof. For $\beta \in \partial \mathbb{B}_n$, let

$$h(\lambda) = f(\beta\lambda) = \sum_{k=0}^{\infty} \left(\sum_{|\alpha|=k} a_\alpha \beta^\alpha \right) \lambda^k, \quad \lambda \in \mathbb{D}.$$

Then, $h(\mathbb{D}) \in \mathbb{B}_m$. Using (2.1), we obtain (2.4). The lemma is proved. □

Lemma 3. *Let $k \geq 2$ be a positive integer and $f(z) = \varphi_a(bz^k) + g(z)$ for $z \in \mathbb{D}$, where $a \in \mathbb{B}_m$, $b \in \partial \mathbb{B}_m$ and*

$$g(z) = \sum_{j=1}^{k-1} \sum_{n=0}^{\infty} a_{nk+j} z^{nk+j}$$

is a holomorphic mapping of $\mathbb{D}$ into $\mathbb{C}^m$. If $|f(z)| < 1$ for $z \in \mathbb{D}$, then $g(z) \equiv 0$.

Proof. Since $|g(z)| < 1 + |\varphi_a(bz^k)| < 2$, by Lemma 1, we have

$$\sum_{j=1}^{k-1} \sum_{n=0}^{\infty} |a_{nk+j}|^2 \leq 4.$$

Thus, for $j = 1, 2, \cdots, k-1$, every component of the mapping

$$g_j(z) = \sum_{n=0}^{\infty} a_{nk+j} z^{nk+j}$$

is in the Hardy class H^2 and, consequently, for almost every $\zeta \in \partial\mathbb{D}$, the radial limit $\lim\limits_{z\to\zeta} g_j(z)$ exists for all j. Let ζ be such a point and $\lambda = \varphi_a(b\zeta^k)$. Obviously, $\lambda \in \partial\mathbb{B}_m$. Denote $\omega = e^{2\pi i/k}$. For $l = 1, \cdots, k$, we have

$$\lim_{z\to\zeta} f(\omega^l z) = \varphi_a(b\zeta^k) + \sum_{j=1}^{k-1} \lim_{z\to\zeta} g_j(\omega^l z) = \lambda + \sum_{j=1}^{k-1} \omega^{lj} \lim_{z\to\zeta} g_j(z),$$

and, since $f(\mathbb{D}) \subset \mathbb{B}_m$,

$$\left| 1 + \sum_{j=1}^{k-1} \omega^{lj} \langle \lim_{z\to\zeta} g_j(z), \lambda\rangle \right| \le \left| \lim_{z\to\zeta} f(\omega^l z) \right| \le 1.$$

For $l = 1, \cdots, k$, let

$$A_l = \sum_{j=1}^{k-1} \omega^{lj} \langle \lim_{z\to\zeta} g_j(z), \lambda\rangle.$$

Then, $|1 + A_l| \le 1$ and, consequently, $\mathrm{Re}A_l \le 0$ for $l = 1, \cdots, k$. However,

$$\sum_{l=1}^{k-1} A_l = \sum_{l=1}^{k-1}\sum_{j=1}^{k-1} \omega^{lj} \langle \lim_{z\to\zeta} g_j(z), \lambda\rangle$$

$$= \sum_{j=1}^{k-1} \left(\langle \lim_{z\to\zeta} g_j(z), \lambda\rangle \sum_{l=1}^{k-1} \omega^{lj} \right) = -\sum_{j=1}^{k-1} \langle \lim_{z\to\zeta} g_j(z), \lambda\rangle = -A_k.$$

Thus, $\mathrm{Re}A_k = 0$. Noting that $|1 + A_k| \le 1$ we conclude that $A_k = 0$, i.e.,

$$\sum_{j=1}^{k-1} \langle \lim_{z\to\zeta} g_j(z), \lambda\rangle = 0.$$

Thus,

$$1 \ge \left| \lim_{z\to\zeta} f(z) \right|^2 = \left| \lambda + \sum_{j=1}^{k-1} \lim_{z\to\zeta} g_j(z) \right|^2 = 1 + \left| \sum_{j=1}^{k-1} \lim_{z\to\zeta} g_j(z) \right|^2 = 1 + \left| \lim_{z\to\zeta} g(z) \right|^2.$$

This shows that the radial limit of every component of $g(z)$ is equal to 0 at almost every $\zeta \in \partial\mathbb{D}$. According the general theory of H^p spaces, we conclude that $g(z) \equiv 0$. The lemma is proved. $\square$

3. The partial derivatives at the origin

Theorem 1. *Let* $f(z) = \sum_\alpha a_\alpha z^\alpha \in \Omega_{n,m}$*. Then,*

$$\left| \left\langle \sum_{|\alpha|=k} a_\alpha \beta^\alpha,\ a_0 \right\rangle \right|^2 + (1-|a_0|^2) \left| \sum_{|\alpha|=k} a_\alpha \beta^\alpha \right|^2 \le (1-|a_0|^2)^2. \tag{3.1}$$

holds for $k \ge 1$ *and* $\beta \in \partial \mathbb{B}_n$*.*

Proof. Let $k \ge 1$ and $\beta \in \partial\mathbb{B}_n$ be given. If $a_0 = 0$, (3.1) is a consequence of (2.4). Now, assume that $a_0 \neq 0$. Let

$$h(z) = \frac{1}{k} \sum_{l=1}^{k} f(e^{2l\pi i/k} z).$$

Then, $h(z) \in \Omega_{n,m}$, $h(0) = a_0$, and

$$h(z) = a_0 + \sum_{m=1}^{\infty} \sum_{|\alpha|=mk} a_\alpha z^\alpha.$$

Let $\phi = \varphi_{a_0} \circ h$. Obviously, $\phi \in \Omega_{n,m}$ and $\phi(0) = 0$. We have

$$\begin{aligned}
\phi(z) = {} & \frac{1}{1-\langle h(z), a_0\rangle} \left(-(a_0/|a_0|^2) \sum_{m=1}^{\infty} \sum_{|\alpha|=mk} \langle a_\alpha, a_0\rangle z^\alpha \right. \\
& \left. - \sqrt{1-|a_0|^2} \sum_{m=1}^{\infty} \sum_{|\alpha|=mk} a_\alpha z^\alpha + \sqrt{1-|a_0|^2}(a_0/|a_0|^2) \sum_{m=1}^{\infty} \sum_{|\alpha|=mk} \langle a_\alpha, a_0\rangle z^\alpha \right) \\
= {} & -\frac{1}{1-\langle h(z), a_0\rangle} \sum_{m=1}^{\infty} \sum_{|\alpha|=mk} \left(\frac{\langle a_\alpha, a_0\rangle a_0}{1+\sqrt{1-|a_0|^2}} + \sqrt{1-|a_0|^2} a_\alpha \right) z^\alpha \\
= {} & -\frac{1}{1-|a_0|^2} \sum_{|\alpha|=k} \left(\frac{\langle a_\alpha, a_0\rangle a_0}{1+\sqrt{1-|a_0|^2}} + \sqrt{1-|a_0|^2} a_\alpha \right) z^\alpha + \sum_{m=2}^{\infty} \sum_{|\alpha|=mk} c_\alpha z^\alpha.
\end{aligned}$$

Thus, using (2.4), we obtain

$$\frac{1}{(1-|a_0|^2)^2} \left| \sum_{|\alpha|=k} \left(\frac{\langle a_\alpha \beta^\alpha, a_0\rangle a_0}{1+\sqrt{1-|a_0|^2}} + \sqrt{1-|a_0|^2} a_\alpha \beta^\alpha \right) \right|^2 \le 1.$$

A simple calculation gives

$$\left|\sum_{|\alpha|=k}\left(\frac{\langle a_\alpha\beta^\alpha, a_0\rangle a_0}{1+\sqrt{1-|a_0|^2}}+\sqrt{1-|a_0|^2}a_\alpha\beta^\alpha\right)\right|^2$$

$$=\left|\frac{\left\langle\sum_{|\alpha|=k}a_\alpha\beta^\alpha,\ a_0\right\rangle a_0}{1+\sqrt{1-|a_0|^2}}+\sqrt{1-|a_0|^2}\sum_{|\alpha|=k}a_\alpha\beta^\alpha\right|^2$$

$$=\frac{\left|\left\langle\sum_{|\alpha|=k}a_\alpha\beta^\alpha,\ a_0\right\rangle\right|^2|a_0|^2}{\left(1+\sqrt{1-|a_0|^2}\right)^2}+(1-|a_0|^2)\left|\sum_{|\alpha|=k}a_\alpha\beta^\alpha\right|^2$$

$$+\frac{2\sqrt{1-|a_0|^2}\left|\left\langle\sum_{|\alpha|=k}a_\alpha\beta^\alpha, a_0\right\rangle\right|^2}{1+\sqrt{1-|a_0|^2}}$$

$$=\left|\left\langle\sum_{|\alpha|=k}a_\alpha\beta^\alpha,\ a_0\right\rangle\right|^2+(1-|a_0|^2)\left|\sum_{|\alpha|=k}a_\alpha\beta^\alpha\right|^2.$$

This shows (3.1). The theorem is proved. □

Theorem 2. *Let* $f(z)=\sum_\alpha a_\alpha z^\alpha\in\Omega_{n,m}$. *Then,*

$$|\langle a_v,\ a_0\rangle|^2+(1-|a_0|^2)\,|a_v|^2\le\frac{|v|^{|v|}}{v^v}(1-|a_0|^2)^2. \tag{3.2}$$

holds for any multi-index $v\neq 0$. *Further, if the equality holds for some* $v=(v_1,\cdots,v_n)$ *with* $v_j\neq 0$ *for* $j=1,\cdots,n$, *then*

$$f(z)=a_0+\frac{a_vz^v}{1+\frac{\langle a_v,a_0\rangle z^v}{1-|a_0|^2}}=a_0+a_vz^v+\cdots. \tag{3.3}$$

Conversely, if $v\neq 0$, $a_0\in\mathbb{B}_m$ *and* $a_v\in\mathbb{C}^m$ *satisfy the equality in (3.2), then the mapping* f *expressed by (3.3) belongs to* $\Omega_{n,m}$.

Proof. Let $v=(v_1,\cdots,v_n)\neq 0$ be given and $k=|v|$. As in the proof of the above theorem, consider h and ϕ. Let

$$b_v=-\frac{1}{1-|a_0|^2}\left(\frac{\langle a_v,a_0\rangle a_0}{1+\sqrt{1-|a_0|^2}}+\sqrt{1-|a_0|^2}a_v\right).$$

Using Lemma 1 to the function ϕ, by (2.3), we have $|b_v|^2 v^v/|v|^{|v|} \le 1$ and

$$\left|\left(\frac{\langle a_v, a_0\rangle a_0}{1+\sqrt{1-|a_0|^2}} + \sqrt{1-|a_0|^2}a_v\right)\right|^2 \le \frac{|v|^{|v|}}{v^v}(1-|a_0|^2)^2.$$

The same calculation as in the proof of the above theorem gives

$$\left|\left(\frac{\langle a_v, a_0\rangle a_0}{1+\sqrt{1-|a_0|^2}} + \sqrt{1-|a_0|^2}a_v\right)\right|^2 = |\langle a_v,\ a_0\rangle|^2 + (1-|a_0|^2)\,|a_v|^2\,.$$

This shows (3.2).

Now, let the equality in (3.2) holds for some $v = (v_1, \cdots, v_n)$ with $v_j \neq 0$ for $j = 1, \cdots, n$. If $a_0 = 0$, then $|a_v|^2 v^v/|v|^{|v|} = 1$, the equality in (2.3) holds and, consequently, $f(z) = a_v z^v$. This shows (3.3). In the case $a_0 \neq 0$, we have $|b_v|^2 v^v/|v|^{|v|} = 1$. Then, the same reasoning shows $\phi(z) = b_v z^v$ and, consequently,

$$h(z) = \varphi_{a_0}(b_v z^v) = \frac{a_0 - \frac{\langle b_v, a_0\rangle}{|a_0|^2} a_0 z^v - \sqrt{1-|a_0|^2}\left(b_v z^v - \frac{\langle b_v, a_0\rangle}{|a_0|^2} a_0 z^v\right)}{1-\langle b_v, a_0\rangle z^v}. \tag{3.4}$$

Note that

$$\langle b_v, a_0\rangle = -\frac{\langle a_v, a_0\rangle}{1-|a_0|^2}. \tag{3.5}$$

Replacing $\langle b_v, a_0\rangle$ in (3.4) by (3.5), by a straightforward calculation, we obtain

$$h(z) = a_0 + \frac{a_v z^v}{1 + \frac{\langle a_v, a_0\rangle z^v}{1-|a_0|^2}}.$$

If $k = 1$, $f(z) = h(z)$ and (3.3) is true. In the case $k \geq 2$, we have $f(z) = \varphi_{a_0}(b_v z^v) + g(z)$ with

$$g(z) = \sum_{j=1}^{k-1} \sum_{m=0}^{\infty} \sum_{|\alpha|=mk+j} a_\alpha z^\alpha.$$

Let $0 \le \theta_1, \cdots, \theta_n \le 2\pi$ be fixed. For $\lambda \in \mathbb{D}$, define

$$\begin{aligned}\psi(\lambda) &= f\left(e^{i\theta_1}\sqrt{v_1}\lambda/\sqrt{|v|}, \cdots, e^{i\theta_n}\sqrt{v_n}\lambda/\sqrt{|v|}\right) \\ &= \varphi_{a_0}\left(be^{i(v_1\theta_1+\cdots+v_n\theta_n)}\sqrt{\frac{v^v}{|v|^{|v|}}}\lambda^k\right) \\ &\qquad + \sum_{j=1}^{k-1}\sum_{m=0}^{\infty}\left(\sum_{|\alpha|=mk+j} a_\alpha e^{i(\alpha_1\theta_1+\cdots+\alpha_n\theta_n)} \cdot \frac{\sqrt{v^\alpha}}{\sqrt{k^{mk+j}}}\right)\lambda^{mk+j}.\end{aligned}$$

Using Lemma 3 to ψ, we have

$$p_{m,j}(\theta_1, \cdots, \theta_n) = \sum_{|\alpha|=mk+j} a_\alpha \sqrt{v^\alpha} e^{i(\alpha_1\theta_1+\cdots+\alpha_n\theta_n)} = 0$$

for $j = 1, \cdots, k-1$ and $m = 0, 1, \cdots$. Note that the above equality holds for arbitrary $\theta_1, \cdots, \theta_n$. Thus, for any multi-index α' with $|\alpha'| = mk+j$, $1 \le j \le k-1$, $m \ge 0$, we have

$$a_{\alpha'}\sqrt{v^{\alpha'}} = \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} e^{-i(\alpha'_1\theta_1+\cdots+\alpha'_n\theta_n)} p_{m,j}(\theta_1, \cdots, \theta_n)d\theta_1\cdots d\theta_n = 0.$$

It is proved that $a_\alpha = 0$ for any multi-index α with $|\alpha| = mk+j$, $1 \le j \le k-1$, $m \ge 0$. Then we obtain $f(z) = h(z)$ and (3.3) is proved again. The last conclusion of the theorem is easy to verify. The theorem is proved. □

Remark 1. Define

$$f(z) = a_{1,0}z_1 + a_{0,2}z_2^2 = z_1 + \frac{1}{3}z_2^2, \quad \text{for} \quad z = (z_1, z_2) \in \mathbb{B}_2.$$

It is easy to verify that $f \in \Omega_{2,1}$. Let $v = (1, 0)$. We have $a_0 = 0, a_v = 1$. v, a_0, and a_v satisfy the equality in (3.2), but $f(z)$ is not expressed by (3.3). This example shows that the condition $v_j \neq 0$ for $j = 1, \cdots, n$ in the second part of the above theorem cannot be omitted.

Corollary 1. *Let* $f(z) = \sum_{\alpha} a_\alpha z^\alpha \in \Omega_{n,m}$. *Then, for any multi-index* $v \neq 0$,

$$|a_v| \leq \sqrt{\frac{|v|^{|v|}}{v^v}} \sqrt{1 - |a_0|^2}$$

if $m \geq 2$; *and*

$$|a_v| \leq \sqrt{\frac{|v|^{|v|}}{v^v}} (1 - |a_0|^2)$$

if $m = 1$ *or, more general,* $\lambda_1 a_0 + \lambda_2 a_v = 0$ *with* $\lambda_1, \lambda_2 \in \mathbb{C}$.

4. The Schwarz-Pick lemma of high order

First we consider mappings from the unit disk into a unit ball $\mathbb{B}_m$. The following theorem is the special case that $n = 1$ of our general Schwarz-Pick lemma of high order.

Theorem 3. *Let* $f \in \Omega_{1,m}$. *Then,*

$$|\langle f^{(k)}(z), f(z)\rangle|^2 + (1 - |f(z)|^2)|f^{(k)}(z)|^2 \leq \left[\frac{k!(1 - |f(z)|^2)}{(1 - |z|^2)^k}(1 + |z|)^{k-1}\right]^2 \quad (4.1)$$

holds for $k \geq 1$ *and* $z \in \mathbb{D}$.

Proof. Let $\xi \in \mathbb{D}$ and a positive integer k be fixed. We consider $g = f \circ \varphi_\xi \in \Omega_{1,m}$, where

$$\varphi_\xi(z) = \frac{\xi - z}{1 - \bar{\xi} z}.$$

Let $g(z) = \sum_{l=0}^{\infty} c_l z^l$ with $c_l = (c_{1,l}, \cdots, c_{m,l})$ for $l = 1, 2, \cdots$. Then $c_0 = f(\xi)$ and, by (3.2) for $n = 1$,

$$|\langle c_l, c_0\rangle|^2 + (1 - |c_0|^2)|c_l|^2 \leq (1 - |c_0|^2)^2 \quad (4.2)$$

holds for $l \geq 1$.

It is easy to verify that

$$\frac{d^l(\varphi_\xi(z)^j)}{dz^l}\bigg|_{z=\xi} = \begin{cases} 0, & l < j; \\ \frac{(-1)^j(\bar{\xi})^{l-j}}{(1-|\xi|^2)^l}\frac{l!(l-1)!}{(l-j)!(j-1)!}, & l \geq j. \end{cases}$$

Let

$$A_j = \frac{(-1)^j\overline{\xi}^{k-j}}{(1-|\xi|^2)^k}\frac{k!(k-1)!}{(k-j)!(j-1)!}.$$

Since $f = g\circ\varphi_\xi$, we have

$$f^{(k)}(\xi) = \sum_{j=1}^{k} c_j A_j,$$

and, using (4.2) and the Schwarz inequality,

$$\begin{aligned}
&|\langle f^{(k)}(\xi), f(\xi)\rangle|^2 + (1-|f(\xi)|^2)|f^{(k)}(\xi)|^2 \\
&\qquad = \left|\sum_{j=1}^{k} A_j\langle c_j, c_0\rangle\right|^2 + (1-|c_0|^2)\left|\sum_{j=1}^{k} c_j A_j\right|^2 \\
&\qquad \leq \sum_{j=1}^{k}|A_j|\sum_{j=1}^{k}|A_j||\langle c_j, c_0\rangle|^2 + (1-|c_0|^2)\sum_{j=1}^{k}|A_j|\sum_{j=1}^{k}|A_j||c_j|^2 \\
&\qquad = \sum_{j=1}^{k}|A_j|\sum_{j=1}^{k}|A_j|\left(|\langle c_j, c_0\rangle|^2 + (1-|c_0|^2)|c_j|^2\right) \\
&\qquad \leq (1-|c_0|^2)^2\left(\sum_{j=1}^{k}|A_j|\right)^2.
\end{aligned}$$

On the other hand,

$$\sum_{j=1}^{k}|A_j| = \frac{k!}{(1-|\xi|^2)^k}\sum_{j=1}^{k}\frac{(k-1)!|\xi|^{k-j}}{(k-j)!(j-1)!} = \frac{k!}{(1-|\xi|^2)^k}(1+|\xi|)^{k-1}.$$

This shows (4.1). The theorem is proved. □

Corollary 2. *Let $f \in \Omega_{1,m}$. Then, for $k \geq 1$ and $z \in \mathbb{D}$,*

$$|f^{(k)}(z)| \leq \frac{k!(1-|f(z)|^2)^{1/2}}{(1-|z|^2)^k}(1+|z|)^{k-1};$$

and

$$|f^{(k)}(z)| \leq \frac{k!(1-|f(z)|^2)}{(1-|z|^2)^k}(1+|z|)^{k-1}$$

if $\lambda_1 f(z) + \lambda_2 f^{(k)}(z) = 0$ *with* $\lambda_1, \lambda_2 \in \mathbb{C}$.

Remark 2. In [DP], the authors proved that (1.1) is asymptotically sharp in the sense that for any two points $z, w \in \mathbb{D}$, there exists a holomorphic function $f_{z,w}$ on $\mathbb{D}$, such that $f_{z,w}(z) = w$, $f_{z,w}(\mathbb{D}) \subset \mathbb{D}$, and

$$\lim_{w \to \partial\mathbb{D}} \frac{|f_{z,w}^{(k)}(z)|}{(1-|f_{z,w}(z)|^2)} = \frac{k!(1+|z|)^{k-1}}{(1-|z|^2)^k}$$

holds for any positive integer k. In the same way, we can construct examples of mappings to show (4.1) is also asymptotically sharp. For fixed points $\xi \in \mathbb{D} \setminus \{0\}$, $\arg \xi = \theta$, and $w \in \mathbb{B}_m \setminus \{0\}$, let $b = -(1-|w|^2)/|w|$, and define

$$g_w(z) = \frac{w}{|w|} \frac{|w|-z}{1-|w|z} = w(1 + bz + b|w|z^2 + b|w|^2 z^3 + \cdots),$$

and

$$f_w(z) = g_w\left(-e^{-i\theta}\frac{\xi - z}{1-\bar{\xi}z}\right).$$

Then, $f_w(\xi) = w$, and

$$f_w^{(k)}(\xi) = \frac{-e^{-ki\theta}k!(1-|w|^2)w}{|w|(1-|\xi|^2)^k} \sum_{v=1}^{k} \frac{|w|^{v-1}(k-1)!|\xi|^{k-v}}{(v-1)!(k-v)!},$$

$$\begin{aligned}\frac{|\langle f_w^{(k)}(\xi), f_w(\xi)\rangle|^2 + (1-|f_w(\xi)|^2)|f_w^{(k)}(\xi)|^2}{(1-|f_w(\xi)|^2)^2} &= \frac{|f_w^{(k)}(\xi)|^2}{(1-|f_w(\xi)|^2)^2} \\ &= \left(\frac{k!}{(1-|\xi|^2)^k} \sum_{v=1}^{k} \frac{|w|^{v-1}(k-1)!|\xi|^{k-v}}{(v-1)!(k-v)!}\right)^2.\end{aligned}$$

Thus,

$$\lim_{w \to \partial\mathbb{B}_m} \frac{|\langle f_w^{(k)}(\xi), f_w(\xi)\rangle|^2 + (1-|f_w(\xi)|^2)|f_w^{(k)}(\xi)|^2}{(1-|f_w(\xi)|^2)^2}$$

$$= \left(\frac{k!}{(1-|\xi|^2)^k} \sum_{v=1}^{k} \frac{(k-1)!|\xi|^{k-v}}{(v-1)!(k-v)!}\right)^2 = \left(\frac{k!}{(1-|\xi|^2)^k}(1+|\xi|)^{k-1}\right)^2.$$

Now, we are ready to prove our main result.

Theorem 4. *Let $f \in \Omega_{n,m}$. Then,*

$$H_{f(z)}(D_k(f,z,\beta),\ D_k(f,z,\beta)) \\ \leq k!^2 \left(1 + \frac{|\langle \beta, z\rangle|}{((1-|z|^2)|\beta|^2 + |\langle \beta, z\rangle|^2)^{1/2}}\right)^{2(k-1)} (H_z(\beta,\beta))^k \qquad (4.3)$$

holds for $k \geq 1$, $\beta \in \mathbb{C}^n \setminus \{0\}$ and $z \in \mathbb{B}_n$. Further, in the case $n \leq m$, the equality in (4.3) *holds for $k = 1$, some $z = \xi \in \mathbb{B}_n$, and any $\beta \in \mathbb{C}^n$, i.e.,*

$$H_{f(\xi)}(f'(\xi)\beta, f'(\xi)\beta) = H_\xi(\beta,\beta) \qquad (4.4)$$

holds for any $\beta \in \mathbb{C}^n$, if and if $F'(0) = \varphi'_{f(\xi)}(f(\xi))f'(\xi)\varphi'_\xi(0)$ satisfies $\overline{F'(0)}^T F'(0) = I$, where I is the identity matrix of $n \times n$, and

$$f(z) = f(\xi) + \left(\frac{1-\langle z,\xi\rangle}{1-|\xi|^2} + \frac{\overline{f(\xi)}^T f'(\xi)(z-\xi)}{1-|f(\xi)|^2}\right)^{-1} f'(\xi)(z-\xi). \qquad (4.5)$$

Proof. Let $k \geq 1$, $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{C}^n \setminus \{0\}$ and $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{B}_n$ be given. First assume that $\beta \in \partial\mathbb{B}_n$. We consider the disk

$$\Delta = \{\zeta \in \mathbb{C} : |\xi + \zeta\beta|^2 = |\xi_1 + \beta_1\zeta|^2 + \cdots + |\xi_n + \beta_n\zeta|^2 < 1\}.$$

To make the equation of Δ clearer, let U be a unitary matrix such that $U\beta = (1, 0, \cdots, 0)^T$. Denote $U\xi = \eta = (\eta_1, \cdots, \eta_n)^T$. Here we identify a point in $\mathbb{C}^n$ with a column matrix of $n \times 1$. Since

$$|\xi + \zeta\beta|^2 = |U(\xi + \zeta\beta)|^2 = |\eta_1 + \zeta|^2 + |\eta_2|^2 + \cdots + |\eta_n|^2,$$

we have

$$\Delta = \{\zeta \in \mathbb{C} : |\eta_1 + \zeta|^2 < 1 - |\eta_2|^2 - \cdots - |\eta_n|^2\}.$$

Thus, if we set $\sigma = (1 - |\eta_2|^2 - \cdots - |\eta_n|^2)^{1/2}$, $\gamma = \sigma\beta$, and

$$\zeta = \sigma\omega - \eta_1, \quad z = L(\omega) = \xi + \omega\gamma - \eta_1\beta,$$

$g(\omega) = f(L(\omega))$ is a holomorphic mapping from $\mathbb{D}$ into $\mathbb{B}_m$.

Using (4.1) to the mapping g and the point $\omega = \omega' = \eta_1/\sigma$, we have

$$|\langle g^{(k)}(\omega'), g(\omega')\rangle|^2 + (1 - |g(\omega')|^2)|g^{(k)}(\omega')|^2 \le \left[\frac{k!(1 - |g(\omega')|^2)}{(1 - |\omega'|^2)^k}(1 + |\omega'|)^{k-1}\right]^2.$$

Note that $g(\omega') = f(\xi)$, $|\eta| = |\xi|$, $\eta_1 = \langle \xi, \beta\rangle$ and

$$\sigma^2 = 1 - |\eta|^2 + |\eta_1|^2 = 1 - |\xi|^2 + |\langle \beta, \xi\rangle|^2,$$

$$|\omega'| = \frac{|\langle \beta, \xi\rangle|}{(1 - |\xi|^2 + |\langle \beta, \xi\rangle|^2)^{1/2}}, \quad 1 - |\omega'|^2 = \frac{1 - |\xi|^2}{\sigma^2}.$$

By the chain rule,

$$g^{(k)}(\omega') = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial^k f(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \gamma^\alpha = \sigma^k \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial^k f(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \beta^\alpha.$$

Thus,

$$\left|\left\langle \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial^k f(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \beta^\alpha,\ f(\xi)\right\rangle\right|^2 + (1 - |f(\xi)|^2)\left|\sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial^k f(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \beta^\alpha\right|^2$$

$$\le k!^2 (1 - |f(\xi)|^2)^2 \left[\frac{1 - |\xi|^2 + |\langle \beta, \xi\rangle|^2}{(1 - |\xi|^2)^2}\right]^k \left(1 + \frac{|\langle \beta, \xi\rangle|}{(1 - |\xi|^2 + |\langle \beta, \xi\rangle|^2)^{1/2}}\right)^{2(k-1)}.$$

(4.3) is proved for $z = \xi$ and any $\beta \in \partial\mathbb{B}_n$. For a general β, we may consider $\beta/|\beta|$, since (4.3) is homogeneous for β. (4.3) is proved completely.

Now assume that $n \le m$ and (4.4) holds for any $\beta \in \mathbb{C}^n$. Consider $F = \varphi_{f(\xi)} \circ f \circ \varphi_\xi$. By the invariance of the Bergman metric, $H_0(F'(0)\beta, F'(0)\beta) = H_0(\beta, \beta)$, i.e., $|F'(0)\beta| = |\beta|$, holds for any $\beta \in \mathbb{C}^n$. This shows that the $m \times n$-matrix

$F'(0)$ satisfies $\overline{F'(0)}^T F'(0) = I$, where I is the identity matrix of $n \times n$. Note that $F'(0) = \varphi'_{f(\xi)}(f(\xi))f'(\xi)\varphi'_\xi(0)$. Thus, for $z \in \mathbb{B}_n$, $F(z) = F'(0)z$ and

$$f(z) = \varphi_{f(\xi)}\left(\varphi'_{f(\xi)}(f(\xi))f'(\xi)\varphi'_\xi(0)\varphi_\xi(z)\right). \tag{4.6}$$

Using the formulas for φ_a at the beginning of Section 2, we have

$$f'(\xi)\varphi'_\xi(0)\varphi_\xi(z) = \frac{(1-|\xi|^2)f'(\xi)(z-\xi)}{1-\langle z,\xi\rangle},$$

$$\varphi'_{f(\xi)}(f(\xi))f'(\xi)\varphi'_\xi(0)\varphi_\xi(z)$$

$$= -\frac{1-|\xi|^2}{1-\langle z,\xi\rangle}\left(\frac{(1-(1-|f(\xi)|^2)^{1/2})\overline{f(\xi)}^T f'(\xi)(z-\xi)}{|f(\xi)|^2(1-|f(\xi)|^2)}f(\xi) + \frac{f'(\xi)(z-\xi)}{(1-|f(\xi)|^2)^{1/2}}\right),$$

$$\langle\varphi'_{f(\xi)}(f(\xi))f'(\xi)\varphi'_\xi(0)\varphi_\xi(z), f(\xi)\rangle = -\frac{(1-|\xi|^2)\overline{f(\xi)}^T f'(\xi)(z-\xi)}{(1-\langle z,\xi\rangle)(1-|f(\xi)|^2)}, \tag{4.7}$$

$$P_{f(\xi)}(\varphi'_{f(\xi)}(f(\xi))f'(\xi)\varphi'_\xi(0)\varphi_\xi(z)) = -\frac{(1-|\xi|^2)\overline{f(\xi)}^T f'(\xi)(z-\xi)}{|f(\xi)|^2(1-\langle z,\xi\rangle)(1-|f(\xi)|^2)}f(\xi). \tag{4.8}$$

(4.5) follows from (4.6), (4.7) and (4.8). Conversely, if $A = \varphi'_{f(\xi)}(f(\xi))f'(\xi)\varphi'_\xi(0)$ satisfies $\overline{A}^T A = I$ and (4.5) holds, then

$$f(z) = \varphi_{f(\xi)}\left(\varphi'_{f(\xi)}(f(\xi))f'(\xi)\varphi'_\xi(0)\varphi_\xi(z)\right)$$

and, by the invariance of the Bergman metric,

$$H_{f(\xi)}(f'(\xi)\beta, f'(\xi)\beta) = H_{f(\xi)}(\varphi'_{f(\xi)}(0)A\varphi'_\xi(\xi)\beta, \varphi'_{f(\xi)}(0)A\varphi'_\xi(\xi)\beta)$$

$$= H_0(A\varphi'_\xi(\xi)\beta, A\varphi'_\xi(\xi)\beta) = |A\varphi'_\xi(\xi)\beta|^2 = |\varphi'_\xi(\xi)\beta|^2$$

$$= H_0(\varphi'_\xi(\xi)\beta, \varphi'_\xi(\xi)\beta) = H_\xi(\beta,\beta)$$

holds for any $\beta \in \mathbb{C}^n$. The theorem is proved. □

5. Schwarz-Pick estimates for derivatives of any order

On the basis of Theorem 4, we can deduce an estimate for partial derivatives of arbitrary order of mappings in $\Omega_{n,m}$.

Theorem 5. *Let $f \in \Omega_{n,m}$. Then,*

$$\left|\left\langle \frac{\partial^{|v|} f(z)}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}},\ f(z)\right\rangle\right|^2 + (1-|f(z)|^2)\left|\frac{\partial^{|v|} f(z)}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}}\right|^2$$
$$\leq \frac{|v|^{|v|}}{v^v}\left[v!(1+|z|)^{|v|-1} \cdot \frac{1-|f(z)|^2}{(1-|z|^2)^{|v|}}\right]^2. \tag{5.1}$$

holds for any multi-index $v = (v_1, \cdots, v_n) \neq 0$ and $z \in \mathbb{B}_n$. In particular, if $f \in \Omega_{n,1}$, then (5.1) becomes

$$\left|\frac{\partial^{|v|} f(z)}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}}\right| \leq \sqrt{\frac{|v|^{|v|}}{v^v}}\, v!(1+|z|)^{|v|-1} \cdot \frac{1-|f(z)|^2}{(1-|z|^2)^{|v|}}. \tag{5.2}$$

Proof. Let $v = (v_1, \cdots, v_n) \neq 0$ and $\xi \in \mathbb{B}_n$ be given, and $k = |v|$. By (4.3),

$$\left|\left\langle \sum_{|\alpha|=k} \frac{k}{\alpha!}\frac{\partial^k f(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}\frac{v^\alpha}{|v|},\ f(\xi)\right\rangle\right|^2 + (1-|f(\xi)|^2)\left|\sum_{|\alpha|=k} \frac{k}{\alpha!}\frac{\partial^k f(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}\frac{v^\alpha}{|v|}\right|^2 \leq A^2,$$

where

$$A = k!(1+|\xi|)^{k-1} \cdot \frac{1-|f(\xi)|^2}{(1-|\xi|^2)^k}.$$

Define

$$g(z) = \frac{1}{A}\left\langle \sum_{|\alpha|=k} \frac{k!}{\alpha!}\frac{\partial^k f(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} z^\alpha,\ f(\xi)\right\rangle = \frac{1}{A}\sum_{|\alpha|=k}\left\langle \frac{k!}{\alpha!}\frac{\partial^k f(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}},\ f(\xi)\right\rangle z^\alpha,$$

$$h(z) = \frac{1}{A}(1-|f(\xi)|^2)^{1/2}\sum_{|\alpha|=k} \frac{k!}{\alpha!}\frac{\partial^k f(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} z^\alpha,$$

and $\phi = (g, h)$. Using (2.2) to ϕ, which is a holomorphic mapping from $\mathbb{B}_n$ into $\mathbb{B}_{2m}$ and satisfies $|\phi(z)|^2 < 1$ for $z \in \mathbb{B}_n$, we have

$$\sum_{|\alpha|=k}\left(\left|\left\langle \frac{k!}{\alpha!}\frac{\partial^k f(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}},\ f(\xi)\right\rangle\right|^2 + (1-|f(\xi)|^2)\left|\frac{k!}{\alpha!}\frac{\partial^k f(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}\right|^2\right)\cdot\frac{v^\alpha}{|v|^{|\alpha|}} \leq A^2.$$

In particular,

$$\left|\left\langle \frac{k!}{v!}\frac{\partial^k f(\xi)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}},\ f(\xi)\right\rangle\right|^2+(1-|f(\xi)|^2)\left|\frac{k!}{v!}\frac{\partial^k f(\xi)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}}\right|^2\le\frac{|v|^{|v|}}{v^v}A^2.$$

This shows (5.1) and the theorem is proved. □

Theorem 6. *Let $f\in\Omega_{n,m}$. Then,*

$$\left|\left\langle \frac{\partial^{|v|} f(z)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}},\ f(z)\right\rangle\right|^2+(1-|f(z)|^2)\left|\frac{\partial^{|v|} f(z)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}}\right|^2$$
$$\le\frac{|v|^{|v|}}{v^v}\left[v!\mu(z)\cdot\frac{1-|f(z)|^2}{(1-|z|^2)^{(v_1+|v|)/2}}\right]^2 \tag{5.3}$$

holds for any multi-index $v=(v_1,\cdots,v_n)\neq 0$ and $z=(z_1,0,\cdots,0)\in\mathbb{B}_n$, where $\mu(z)=(1+|z|)^{|v|-1}$ if $v_1=|v|$, and $\mu(z)$ is the sum of terms $c_j|z|^j$ with $j\le v_1$ in $(1+|z|)^{|v|-1}$. In particular, if $f\in\Omega_{n,1}$, then (5.3) becomes

$$\left|\frac{\partial^{|v|} f(z)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}}\right|\le\sqrt{\frac{|v|^{|v|}}{v^v}}v!\mu(z)\cdot\frac{1-|f(z)|^2}{(1-|z|^2)^{(v_1+|v|)/2}}. \tag{5.4}$$

Proof. Let $v=(v_1,\cdots,v_n)\neq 0$ and $\xi=(\xi_1,\cdots,\xi_n)\in\mathbb{B}_m$ be given. $v_1=|v|$, (5.3) follows from (5.1). Now assume that $v_1<|v|$. Let $k=|v|$ and

$$g(z)=f(\varphi_\xi(z))=\sum_\alpha c_\alpha z^\alpha.$$

Then, $c_0=f(\xi)$ and, by (3.2),

$$|\langle c_\alpha,\ c_0\rangle|^2+(1-|c_0|^2)\,|c_\alpha|^2\le\frac{|\alpha|^{|\alpha|}}{\alpha^a}(1-|c_0|^2)^2 \tag{5.5}$$

holds for any multi-index $\alpha\neq 0$. Thus, we have

$$f(z)=g(\varphi_\xi(z))=\sum_\alpha c_\alpha\varphi_\xi(z)^\alpha,$$

where

$$\varphi_\xi(z)=\left(\frac{\xi_1-z_1}{1-\bar{\xi}_1 z_1},-\frac{(1-|\xi|^2)^{1/2}z_2}{1-\bar{\xi}_1 z_1},\cdots,-\frac{(1-|\xi|^2)^{1/2}z_n}{1-\bar{\xi}_1 z_1}\right).$$

For a multi-index α, denote $\alpha = (\alpha_1, \alpha')$ with $\alpha' = (\alpha_2, \cdots, \alpha_n)$. Then, it is easy to see that

$$\left.\frac{\partial^k(\varphi_\xi(z)^\alpha)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}}\right|_{z=\xi} = \begin{cases} 0, & \alpha' \neq v'; \\ 0, & \alpha' \neq v',\ \alpha_1 > v_1; \\ \frac{(-1)^{\alpha_1+|v'|}(\bar{\xi})^{v_1-\alpha_1}}{(1-|\xi|^2)^{(v_1+|v|)/2}}\frac{v!(k-1)!}{(v_1-\alpha_1)!(\alpha_1-1+|v'|)!}, & \alpha' = v',\ 0 \le \alpha_1 \le v_1. \end{cases}$$

Thus, letting

$$A_j = \frac{(-1)^{j+|v'|}(\bar{\xi})^{v_1-j}}{(1-|\xi|^2)^{(v_1+|v|)/2}}\frac{v!(k-1)!}{(v_1-j)!(j-1+|v'|)!},$$

we have

$$\left.\frac{\partial^k f(z)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}}\right|_{z=\xi} = \sum_{j=0}^{v_1} A_j c_{j,v'},$$

and, by the Schwarz inequality and (5.5),

$$\left|\left\langle \left.\frac{\partial^{|v|}f(z)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}}\right|_{z=\xi},\ f(\xi)\right\rangle\right|^2 + (1-|f(\xi)|^2)\left|\left.\frac{\partial^{|v|}f(z)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}}\right|_{z=\xi}\right|^2$$

$$= \left|\sum_{j=0}^{v_1} A_j\langle c_{j,v'}, c_0\rangle\right|^2 + (1-|c_0|^2)\left|\sum_{j=0}^{v_1} A_j c_{j,v'}\right|^2$$

$$\le \sum_{j=0}^{v_1}|A_j|\sum_{j=0}^{v_1}|A_j||\langle c_{j,v'}, c_0\rangle|^2 + (1-|c_0|^2)\sum_{j=0}^{v_1}|A_j|\sum_{j=0}^{v_1}|A_j||c_{j,v'}|^2$$

$$\le \frac{|v|^{|v|}}{v^v}(1-|c_0|^2)^2\left(\sum_{j=0}^{v_1}|A_j|\right)^2.$$

Here, we use the obvious inequality $\frac{|\alpha|^{|\alpha|}}{\alpha^\alpha} \le \frac{|v|^{|v|}}{v^v}$ if $\alpha_j \le v_j$ for $j = 1, \cdots, n$. Note that

$$\sum_{j=0}^{v_1}|A_j| = \frac{v!}{(1-|\xi|^2)^{(v_1+|v|)/2}}\sum_{j=0}^{v_1}\frac{(k-1)!|\xi|^{v_1-j}}{(v_1-j)!(j-1+|v'|)!}$$

$$= \frac{v!}{(1-|\xi|^2)^{(v_1+|v|)/2}}\sum_{l=0}^{v_1}\frac{(k-1)!|\xi|^l}{l!(k-1-l)!}$$

(5.3) is proved. (5.4) follows from (5.3) directly and the proof is complete. □

Remark 3. Let a multi-index $v = (v_1, \cdots, v_n)$, $\xi = (\xi_1, 0, \cdots, 0) \in \mathbb{B}_n$ and $w \in \mathbb{C}$ be fixed. Define

$$g(z) = \frac{w - \sqrt{|v|^{|v|}/v^v}\, z^v}{1 - \overline{w}\sqrt{|v|^{|v|}/v^v}\, z^v} \quad \text{for} \quad z \in \mathbb{B}_n,$$

and $f = g \circ \varphi_\xi$. Then, $g \in \Omega_{n,1}$, $f \in \Omega_{n,1}$, $f(\xi) = w$, and

$$g(z) = w - (1 - |w|^2)\sqrt{|v|^{|v|}/v^v}\, z^v - \overline{w}(1 - |w|^2)(|v|^{|v|}/v^v) z^{2v} - \cdots,$$

$$\begin{aligned} \left.\frac{\partial^{|v|} f(z)}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}}\right|_{z=\xi} &= -\sqrt{\frac{|v|^{|v|}}{v^v}}(1 - |w|^2) \cdot \left.\frac{\partial^{|v|} (\varphi_\xi(z))^v}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}}\right|_{z=\xi} \\ &= (-1)^{|v|+1}\sqrt{\frac{|v|^{|v|}}{v^v}}\, v! \cdot \frac{1 - |f(\xi)|^2}{(1 - |\xi|^2)^{(v_1+|v|)/2}}. \end{aligned}$$

This shows that the estimate (5.4) is precise up to a constant less than $2^{|v|-1}$.

Remark 4. If $v_1 = |v| = k$, (5.4) becomes

$$\left|\frac{\partial^k f(z)}{\partial z_1^k}\right| \leq k!(1 + |z|)^{k-1} \cdot \frac{1 - |f(z)|^2}{(1 - |z|^2)^k}. \tag{5.5}$$

(5.5) is also a consequence of (5.2). For given $\xi = (\xi_1, 0, \cdots, 0) \in \mathbb{B}_n \setminus \{0\}$ and $w \in \mathbb{C} \setminus \{0\}$, let $\theta = \arg \xi - \arg w$ and defined

$$g(z) = \frac{w + e^{-i\theta} z_1}{1 + \overline{w} e^{-i\theta} z_1} \quad \text{for} \quad z = (z_1, \cdots, z_n) \in \mathbb{B}_n,$$

and $f = g \circ \varphi_\xi$. Then, $g \in \Omega_{n,1}$, $f \in \Omega_{n,1}$, $f(\xi) = w$, and

$$g(z) = w + (1 - |w|^2)e^{-i\theta} z_1 - \overline{w}(1 - |w|^2)e^{-2i\theta} z_1^2 + \overline{w}^2(1 - |w|^2)e^{-3i\theta} z_1^3 + \cdots.$$

Thus, for any positive integer k, we have

$$\begin{aligned} \left.\frac{\partial^k f(z)}{\partial z_1^k}\right|_{z=\xi} &= (1 - |w|^2)\sum_{j=1}^{k} \overline{w}^{j-1} e^{-ji\theta} \left.\frac{d^k}{dz_1^k}\left(\frac{(\xi_1 - z_1)^j}{(1 - \overline{\xi}_1 z_1)^j}\right)\right|_{z_1=\xi_1} \\ &= -\frac{k!(1 - |w|^2)}{(1 - |\xi|^2)^k} \sum_{j=1}^{k} \frac{(k-1)! e^{-ji\theta} \overline{w}^{j-1} \overline{\xi}^{k-j}}{(j-1)!(k-j)!} \end{aligned}$$

$$= -\frac{k!(1-|w|^2)|\xi|^k w}{(1-|\xi|^2)^k |w| \xi^k} \sum_{j=1}^{k} \frac{(k-1)|w|^{j-1}|\xi|^{k-j}}{(j-1)!(k-j)!}$$

$$= -\frac{k!(1-|w|^2)|\xi|^k w}{(1-|\xi|^2)^k |w| \xi^k} (|w|+|\xi|)^{k-1}$$

and, consequently,

$$\lim_{w\to\partial\mathbb{D}} \frac{\left|\partial^k f(z)/\partial z_1^k\big|_{z=\xi}\right|}{1-|f(\xi)|^2} = \frac{k!(1+|\xi|)^{k-1}}{(1-|\xi|^2)^k}.$$

This shows that (5.5) is asymptotically sharp.

References

[C] H. H. Chen, Boundedness from below of composition operators on Bloch spaces, Science in China (Chinese), Series A, (4) 33 (2003), 289-296.

[ChL] Z. H. Chen and Y. Liu, Schwarz-Pick estimates for bounded holomorphic functions in the unit ball of $\mathbb{C}^n$, Preprint.

[DP] S. Y. Dai and Y. F. Pan, Note on Schwarz-Pick estimates for bounded and positive real part analytic functions, Proceedings of the American Mathematical Society, (2) 136 (2008), 635-640.

[MSZ] B. Maccluer, K. Stroethoff, and R. H. Zhao, Generalized Schwarz-Pick estimates, Proceedings of the American Mathematical Society, 131 (2003), 593-599.

[R] W. Rudin, Function theory in the unit ball of $\mathbb{C}^n$, Spring-Verlag New York Inc., 1980.

[Zh] M. Z. Zhang, Generalized Schwarz-Pick Lemma, Acta Mathematica Sinica, Chinese Series, (3) 49 (2006), 613-616.

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