Final Exam
Friday, 7-19-95 9:30-11:20
Math 131
Adam Coffman

Name ______________________ Student ID# ____________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 98 points out of a quarter total of 294.

Hypotheses for theorems 1–5.

k is a constant.

$\lim_{x \to c} f(x)$ exists.

$\lim_{x \to c} g(x)$ exists.

Conclusions:

1. $\lim_{x \to c} k = k$.
2. $\lim_{x \to c} x = c$.
3. $\lim_{x \to c} (k \cdot f(x)) = k \cdot \lim_{x \to c} f(x)$.
4. $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$.
5. $\lim_{x \to c} (f(x) \cdot g(x)) = (\lim_{x \to c} f(x)) \cdot (\lim_{x \to c} g(x))$.

6. **The Composite Limit Theorem.** If $\lim_{x \to c} g(x) = L$ AND f is continuous at L, then

$$\lim_{x \to c} (f \circ g)(x) = f(\lim_{x \to c} g(x))$$

which could also be written as

$$\lim_{x \to c} f(g(x)) = f(L)$$

1. State the definition of $|x|$.$^{(1\ point)}$

2. State the Triangle Inequality.$^{(1)}$

3. State the definition of "$f(x)$ is a decreasing function."$^{(1)}$

4. $^{(1)}$ State, in terms of absolute values, an inequality whose solution set is the set of numbers $x \neq c$ whose distance from c is less than δ.

5. $^{(1)}$ State the formula for the slope of the line through the points $(x, f(x))$ and $(c, f(c))$.

6. $^{(1)}$ State the definition of "b is a lower bound of the set $S \subseteq \mathbb{R}$."

7. $^{(1)}$ State the definition of "x is a rational number."

8. $^{(1)}$ State the definition of the composite function $h \circ g \circ f$.

9. $^{(1)}$ State the definition of

$$\lim_{x \to c} f(x) = L.$$

10. (1) State the "limit" (not the ϵ, δ) definition of "$f(x)$ is continuous at c."

11. (1) State the $\lim\limits_{x \to c}$ definition of the derivative of $f(x)$ at c:

$$f'(c) =$$

12. (1) State the $\lim\limits_{h \to 0}$ definition of the derivative of $f(x)$ at c:

$$f'(c) =$$

13. (1) State the ϵ, δ definition of "$f(x)$ is continuous at c."

14. (1) State an ϵ, δ definition of "$f(x)$ is differentiable at c."

15. (1) State the definition of "$f(c)$ is the minimum value of $f(x)$."

16. (1) State the Theorem on the Existence of Maximum and Minimum Values.

17. (1) The set of points (x, y) satisfying the equation $Ax + By + C = 0$ defines a line l in the plane. Find all the lines $Dx + Ey + F = 0$ that are parallel to l.

18. The following **three** parts (a, b, c) are "ϵ, δ" proofs. Don't use the limit theorems on page one, or any derivative theorems; use the ϵ, δ definition of limit. Do all of them; I will grade them, 6 points each with partial credit, but then I will only count your best two answers in the total score, for a maximum score of 12 points.

a. Prove, using the definition,

$$\lim_{x \to 1} 2x + 5 = 7.$$

scratchwork:

formal proof:

b. Prove, using the definitions of limit and continuity, that the function $f(x) = -3x + 20$ is continuous at $c = 4$.

scratchwork:

formal proof:

18c. Prove, using the definitions of limit and differentiability, that the derivative of the function $f(x) = x^2$ at $c = 3$ is equal to 6.

scratchwork:

formal proof:

19. Compute the following limits any way you want. You don't have to explain your reasoning. Write either a real number after the "=," or "**DNE**" if the limit Does Not Exist. Some of the formulas involve "c" — treat it as a constant and express your answer in terms of c. (1 pt each)

- $\lim_{x \to c}(x^3 + x^2c + xc^2 + c^3) =$
- $\lim_{x \to 1} \frac{|x-1|}{x-1} =$
- $\lim_{x \to 1^-} \frac{|x-1|}{x-1} =$
- $\lim_{x \to 1^+} \left(\frac{1}{x-1} - \frac{1}{|x-1|} \right) =$
- $\lim_{x \to 1^+} \sqrt{x - [\![x]\!]} =$
- $\lim_{h \to 0} \frac{(c+h)^{53} - c^{53}}{h} =$
- $\lim_{x \to c} \frac{x^{1/3} - c^{1/3}}{x - c} =$

  (Hint: this one exists for some c and not for others)

20. The following questions refer to the graph of the function $f(x)$ that I've sketched, with domain $(-1, 8)$. Some answers should be numbers, other answers will be sets of numbers: use "{}" or interval notation. 1 pt each.

- Maximum value of $f(x) =$
- $f(x)$ is not continuous at the points:
- $f(x)$ is not differentiable at the points:
- $|f(x)|$ is not differentiable at the points:
- the Minimum value of $|f(x)| =$
- $f'(x) = 0$ at the points:
- the points at which $\lim_{x \to c^-} f(x)$ exists but where $f(x)$ is not continuous:
- $f'(2)$ is approximately equal to:
- Give an example of a closed interval $[a, b] \subseteq (-1, 8)$ such that the average rate of change $\frac{f(b)-f(a)}{b-a}$ is positive.

21. Use the Limit Theorems #1–6 from the cover page **one at a time** to carefully calculate the following limits: (Justify each step with the numbered theorem; for example, I've already done "a.")$^{(2\ each)}$

a. $\lim_{x\to 2} 2x + 2 = \lim_{x\to 2} 2x + \lim_{x\to 2} 2$ (using #4)
$= 2\lim_{x\to 2} x + \lim_{x\to 2} 2$ (#3) $= 2\lim_{x\to 2} x + 2$ (#1)
$= 2 \cdot 2 + 2$ (#2) $= 6.$

b. $\lim_{x\to 2} x^2(2x-1) =$

c. $f(x) = 4x^2 + 1$. Use either limit definition of derivative and the limit theorems to calculate $f'(c)$.

d. As in part c., use the limit theorems to show that if $f(x) = \sqrt{x}$ and $c > 0$, then $f'(c) = \frac{1}{2\sqrt{c}}$. You may use (and mention) the fact that the functions $\sqrt{x}$ and $\frac{1}{x}$ are continuous on $(0, \infty)$.

22. Hypotheses for theorems 1–7: k is a constant, and $f(x)$ and $g(x)$ are differentiable functions, with derivatives $f'(x)$ and $g'(x)$. The notation $\frac{d}{dx}$ is also used to mean "the derivative of".

1. $\frac{d}{dx}k = 0.$
2. $\frac{d}{dx}x = 1.$
3. $\frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx}f(x).$
4. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$
5. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + f'(x)g(x).$
6. $g(x) \neq 0 \Rightarrow \frac{d}{dx}(\dfrac{f(x)}{g(x)}) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$
7. $\frac{d}{dx}((f \circ g)(x)) = f'(g(x)) \cdot g'(x).$

Use these Derivative Theorems #1–7 **one at a time** to carefully calculate the following derivatives: (Justify each step with the numbered theorem; here's an example of what to do:)$^{(3\ each)}$

a. $\frac{d}{dx}2x + 2 = \frac{d}{dx}2x + \frac{d}{dx}2$ (using #4) $= 2\frac{d}{dx}x + \frac{d}{dx}2$ (#3) $= 2 \cdot 1 + \frac{d}{dx}2$ (#2) $= 2 + 0$ (#1) $= 2$.

b. $\frac{d}{dx}(3x^2 + 1) =$

c. $\frac{d}{du}\frac{1}{u^2} =$

(do not use the "power rule" on b. or c.)

d. $\frac{d}{dx}(x^2 + 1)^{10} =$

(use the fact that $\frac{d}{du}u^{10} = 10u^9$.)

e. $\frac{d}{dx}f(1/x) =$

(assume $f(u)$ is differentiable for $u \neq 0$.)

23. The following functions have domain $\mathbb{R}$. Find their derivatives any way you want. Also specify the **domain** of the derivative function, which will not necessarily be all of $\mathbb{R}$. (3 pts each).

- $f(x) = 4x^3 + 6x^2 + 7x + 1$.
- $b(x) = \frac{1}{1+x^2}$.
- $d(x) = x^{1/5}$.
- $g(x) = |x|$.
- $c(x) = \sqrt{|x|}$.
- $h(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.
- $j(x) = [\![x]\!]$.

24. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. You need not explain your answer.$^{(1\ each)}$

____________ $c \in (a, b) \iff \{c\} \subseteq (a, b)$.

____________ If a line intersects the graph of $f(x)$ only at the point $(c, f(c))$, then the line is tangent to the graph.

____________ If $f(x)$ is positive for all x, then $f'(x)$ is positive for all x where $f(x)$ is differentiable.

____________ If $f(x)$ is increasing on (a, b), then $f(x)$ is differentiable on (a, b).

____________ -6 is a solution of the inequality $|x^2 - 5| < 10$.

____________ $\lim_{x \to c} f(x) = L \implies \lim_{x \to c} f(x^2) = L^2$.

____________ $|x| - |y| \leq |x - y|$

____________ $||x| - |y|| \leq |x - y|$

____________ $f(x) = x^3 \implies \frac{df}{dx}(x) = 3x^2$.

____________ $\frac{df}{dx}(x) = 3x^2 \implies f(x) = x^3$.

____________ If $g(c) = 0$ and $g(x)$ is differentiable at c, then $\lim_{x \to c} \dfrac{g(x)}{x - c}$ exists.

25. Sketch the ellipse $x^2/25 + y^2/16 = 1$. Check, for example, that the point $(4, \frac{12}{5})$ satisfies the equation, so it's on the ellipse. Find the equation of the line through $(4, \frac{12}{5})$ that's tangent to the ellipse. (If you have time, calculate the line's slope exactly. For lots of partial credit, approximate the slope by looking at your sketch.) (3 pts)