Final Examination
Friday, 6-7-96 10:30-12:30
Math 153
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Name ________________________ Student ID# ____________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 150 points out of a quarter total of 450.

Definition 4.1 $\lim_{x \to c} f(x) = L$ means $\forall \epsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$.

Theorem 4.2 (Quotient Rule for Derivatives)
$g(x) \neq 0 \Rightarrow \frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.

Theorem 4.3 (Chain rule) $\frac{d}{dx}((f \circ g)(x)) = f'(g(x)) \cdot g'(x)$.

Theorem 4.4 $\sin(0) = 0$, $\cos(0) = 1$. $\frac{d}{dx}\sin(x) = \cos(x)$.

Theorem 4.5 If Σa_n is a convergent series, then $\lim_{n \to \infty} a_n = 0$.

Theorem 4.6 (Divergence Test) If $\lim_{n \to \infty} a_n \neq 0$, then Σa_n is a divergent series. (This is just the contrapositive of the previous theorem)

Theorem 4.7 If Σa_n and Σb_n are convergent series, then $\Sigma(a_n + b_n)$ converges to $(\Sigma a_n) + (\Sigma b_n)$.

Theorem 4.8 (Basic Comparison Test) Let Σa_k be a series with nonnegative terms.

- Σa_k converges if there exists a convergent series Σc_k with nonnegative terms such that $a_k \leq c_k$ for all k sufficiently large.
- Σa_k diverges if there exists a divergent series Σd_k with nonnegative terms such that $d_k \leq a_k$ for all k sufficiently large.

Theorem 4.9 (Term-by-term differentiation) If $\sum_{k=0}^{\infty} a_k x^k$ converges for $x \in (-c, c)$, then $\sum_{k=1}^{\infty} k a_k x^{k-1}$ also converges for $x \in (-c, c)$.

Theorem 4.10 (p-series) $\Sigma \frac{1}{k^p}$ converges if and only if $p > 1$.

(first four problems 3 pts each. Give only one answer.)
1. Define "a_n is a weakly increasing sequence."

2. State the ϵ, N definition of "the sequence c_k converges to C."

3. State the definition of "$\lim_{n\to\infty} a_n = +\infty$."

4. State the "∞/∞" form of L'Hôpital's Rule, for $\lim_{x\to c^+} \frac{f(x)}{g(x)}$.

5. (5 pts total) a. Which equations define a "conic" curve in the x, y-plane? (State the general form of the polynomial equation)

b. Give an example of a pair of conics which intersect each other at exactly one point.

c. Give an example of a pair of conics which intersect each other at exactly two points. (mention the two points at which they intersect)

(Problems 6-13 three points each)
6. Given a sequence a_i, define s_n, the sequence of partial sums of a_i.

7. State (in terms of $\lim\limits_{n\to\infty}$) the definition of "$\sum\limits_{i=1}^{\infty} a_i$ converges to L."

8. State (in terms of the ϵ, N definition of $\lim\limits_{n\to\infty}$) the definition of "$\sum\limits_{i=1}^{\infty} a_i$ converges to L."

9. State the definition of "$\sum\limits_{i=1}^{\infty} a_i$ is conditionally convergent."

10. Given a function $f(x)$ with n continuous derivatives on an open interval containing 0, define $P_n(x)$, its Taylor polynomial of degree n.

11. State the Theorem on the Differentiability of Power Series. (not the one on the cover page, but the one about functions $f(x)$.)

12. State the Theorem on the Integrability of Power Series.

13. Give an ϵ, N definition of the statement "$\int_a^\infty f(t)dt = L$."

14. (4 pts) Prove, by finding the limit of the sequence of partial sums, that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for $x \in (-1, 1)$.

15. (4 pts) Use the ϵ, N definition of limit and the triangle inequality to prove:

$$(\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n) = \lim_{n\to\infty} (a_n + b_n).$$

16. (3 pts each) Use the definition (problem #3) to show that the sequence $b_n = (n-1)^{1/4}$ diverges to $+\infty$ and also the sequence $c_n = n!$ diverges to $+\infty$.

17. (3 pts) Show that $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq \sqrt{n}$.

18. (3 pts) Show that

$$\sum_{i=0}^{n-1} \left(\sum_{k=i+1}^{n} \frac{1}{k} \right) = n.$$

19. (2 pts) Give an example of a sequence a_n such that a_n is *alternating* (either $a_{2n} > 0$ and $a_{2n+1} < 0$, or the signs are reversed), and such that $\Sigma a_n = +\infty$.

20. Use the Limit Comparison test or the Ratio Test to determine the convergence or divergence of the following series: (3 pts each)

- $\Sigma(\sin(1/n))^{(3/2)}$
- $\frac{1}{2} + \frac{1\cdot 3}{2\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 5\cdot 8} + \cdots + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 5\cdot 8\cdots(3n-1)} + \cdots$
- $\Sigma(n^3 - 2)^{-1/3}$

21. a. Consider $f(x) = 6e^{(-x^2)}$. Use Taylor's theorem directly (calculate derivatives, don't use the series for e^x you've memorized) to calculate the Taylor polynomials, centered at 0, of $f(x)$ of degree 0, 1, 2, 3, and 4. Hints: Some of the Taylor polynomials will be equal to each other.

b. Graph $f(x) = 6e^{(-x^2)}$ and all the Taylor polynomials you computed in part a. (8 pts total)

For questions 22 and 23, some of the limits may diverge. Indicate $+\infty$, $-\infty$, or "DNE" if neither applies.

22. Consider the function $f(x) = \frac{\log(x)}{1+\log(x)}$. Use the fact that $\log(1/e) = -1$ in the following calculations: (2 pts each)

- $\lim_{x \to 0^+} f(x) =$
- $\lim_{x \to 1/e^-} f(x) =$
- $\lim_{x \to 1/e^+} f(x) =$
- $\lim_{x \to \infty} f(x) =$
- $\lim_{x \to 0^+} f'(x) =$

23. Do the following calculations any way you want. (3 each, best 5 of six.)

- $\displaystyle \lim_{x \to 0} \frac{\sin(kx)}{\sin(x)} =$
- $\displaystyle \lim_{x \to 0} \frac{1 + x + \frac{x^2}{2} - e^x}{x^3} =$
- $\int_0^\infty e^{t^2}\,dt =$
- $\lim_{x \to 0^+} (x^x)^x =$
- Taylor series for $\cos(x)$=
- For $p > 0$, $\lim_{x \to 0^+} \log(p^{(1/x)}) =$

  Hint: There are three answers, depending on what p is.

24. True or False. Write **T** or **F** next to each sentence. The numbers i, k, and n refer to natural numbers in the domain of a sequence. You need not explain your answer. $^{(1\ each)}$

______ If $\Sigma a_k x^k$ converges on $(-c, c)$, and $0 < d < c$, then $\Sigma a_k x^k$ converges on $(0, d)$.

______ If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)g(x) = 0$.

______ If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)g(x) = \infty$.

______ If $\lim_{x\to a} f(x) = -\infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x) + g(x) = 0$.

______ If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} (\frac{1}{2} + f(x))^{g(x)} = \frac{1}{2}$.

______ If $0 \le b_n \le c_n$ and the sequence c_n is decreasing and convergent, then b_n is convergent.

______ If $0 \le b_n \le c_n$ and the sequence c_n is decreasing and convergent, then b_n is decreasing.

______ $\sum(-\frac{1}{2})^i$ diverges.

______ If $\int_0^\infty f(x)dx$ exists then $\lim_{x\to\infty} f(x) = 0$.

______ if $f(x) \ge 0$ and $\lim_{x\to\infty} f(x) = 0$ then $\int_0^\infty f(x)dx$ exists.

______ If $\int_0^\infty f(x)dx$ exists and $f(x)$ is continuous on $[0, \infty)$ then $f(x)$ is bounded on $[0, \infty)$.

______ If the sequence d_n is defined recursively by $d_1 = 2$ and $d_n = 1/(d_{n-1})$ for $n > 1$, then d_n is divergent.

______ If the sequence d_n is defined recursively by $d_1 = 2$ and $d_n = (d_{n-1})^{-1/2}$ for $n > 1$, then d_n is divergent.

______ If $a_i \ge 0$ and $\sum a_i$ converges, then so does $\sum a_{(i^2)}$.

______ If A and B are linear operators with the same domain, then $A - B$ (defined by $(A-B)(\vec{v}) = A(\vec{v}) - B(\vec{v})$) is also a linear operator.

______ If Σa_i is conditionally convergent, then $\Sigma(-a_i)$ is convergent.

______ If the sequence a_n is alternating, then $\Sigma \frac{a_n}{n}$ is convergent

25. Prove: If $0 < a_i \leq b_i$, and $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} b_i$, then a_i and b_i are the same sequence. (4 pts)

26. Give an example of a sequence a_n such that the power series $\Sigma a_k x^k$ converges for all $x \in \mathbb{R}$, and prove that your example is a convergent series. (5 pts)

27. (3 pts) Give an example of a function $f(x)$ such that $f^{(n)}(0)$ exists but $f^{(n+1)}(0)$ does not. (f must depend on the unknown natural number n)

28. (2 pts) Give an example of a sequence d_n such that the series Σd_n diverges, but the sum of the even terms Σd_{2n} converges.

29. (1 pt) Write any six terms in order from the Fibonacci sequence.

30. (4 pts) Use the Basic Comparison Test to show that the sum of two absolutely convergent series is also absolutely convergent.

31. (1 pt) Is the set of absolutely convergent series, $\{(a_1, a_2, \ldots)$ such that $\Sigma|a_n|$ converges$\}$, a vector space?

32. On the back of this page, or elsewhere, draw a pattern of circles in $\mathbb{R}^2$, the infinite (x, y)-plane, so that the total area of the disks $(\Sigma\pi r_n^2)$ is finite, but no line in the plane misses all the circles. (4 pts)

33. (4 pts) Adam has infinitely many checkers (or cards, or billiard balls, or cats), indexed by natural numbers 1, 2, 3, 4, 5, etc. He lines up the first ten in order $1, \ldots, 10$, and then removes the first (#1). Then the next ten, $11, \ldots, 20$ are lined up after the remaining nine, $2, \ldots, 10$, and the second (#2) is removed. Then, checkers $21, \ldots, 30$ are placed after $3, \ldots, 20$, and the third (#3) is removed. Describe the paradox that arises if one assumes that Adam, after infinitely many steps, uses all the checkers.