Final Exam
Wednesday, 3-13-96 10:30-12:30
Math 152
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Name ________________________ Student ID# ____________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 150 points out of a quarter total of 450.

Hypotheses for theorems 1–7:
k is a constant.
$f(x)$ and $g(x)$ are differentiable functions, with derivatives $f'(x)$ and $g'(x)$.
The notation $\frac{d}{dx}$ is also used to mean "the derivative of".
Conclusions:

1. $\frac{d}{dx}k = 0$.
2. $\frac{d}{dx}x = 1$.
3. $\frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx}f(x)$.
4. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.
5. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + f'(x)g(x)$.
6. $g(x) \neq 0 \Rightarrow \frac{d}{dx}(\dfrac{f(x)}{g(x)}) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.
7. $\frac{d}{dx}((f \circ g)(x)) = f'(g(x)) \cdot g'(x)$.

Theorem 2.1 If $f(x)$ is continuous on $[a, b]$, then the function

$$F(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$, differentiable on (a, b), with derivative $F'(x) = f(x)$.

Theorem 2.2 The function $\log(x)$ is differentiable on $(0, \infty)$. $\frac{d}{dx}\log|x| = 1/x$ on $(-\infty, 0) \cup (0, \infty)$.

Theorem 2.3 (The Inverse Function Theorem) If $f(x)$ is differentiable and one-to-one on (a, b), and $f'(x) \neq 0$, then f^{-1} is differentiable, and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

In the following questions, give only one definition, version of the required statement, or example. In the proofs, you may cite any theorem from the cover page.

1. State the Fundamental Theorem of Calculus. (4 pts)

2. State the Mean Value Theorem. (4 pts)

3. State the "Second Derivative Test" theorem. (4 pts)

4. a. State the definition of "$f(x)$ is concave up on (a, b)." (2 pts)

b. Give an example of a polynomial which is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$. (2 pts)

5. a. Calculate the following derivatives any way you want. (3 pts each)

- $\frac{d}{dx}\log(\log(x)) =$
- $\frac{d}{dx}\cos^2(x^2) =$
- $\frac{d}{dx}\sin^{-1}(\sqrt{1-x^2}) =$
- $\frac{d}{dx}\int_0^x f(t-1)dt =$

b. Calculate the following integrals any way you want. (3 pts each)

- $\int \sin(x)\cos(x)dx =$
- $\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{1}{\sqrt{1-t^2}}dt =$
- $\int \frac{1}{f'(f^{-1}(x))}dx =$
- $\int x\exp(x^2)dx =$

6. a. Define $f(x) = \exp(-(x-\mu)^2)$, with domain $\mathbb{R}$. Sketch a graph of $f(x)$, and find all critical points, extreme values, and inflection points. ($\mu \in \mathbb{R}$ is a constant. 4 pts)

b. Find an antiderivative $F(x)$ of $f(x)$. (Hint: $f(x)$ is continuous, and it is a theorem that all continuous functions have antiderivatives, even if they cannot be computed in terms of elementary functions. 1 pt.)

c. Sketch a graph of an antiderivative $F(x)$ of $f(x)$ and indicate the critical points and inflection points of $F(x)$. (2 pts)

7. a. Sketch a graph of $f(x) = x^3 - 3x + 1$. (4 pts)

b. Find the exact integral $\int_{-2}^{2} f(t)dt$. (2 pts)

c. Find the Lower Sum for $f(x)$, on the interval $[-2, 2]$, using the partition $\{-2 < -1 < 0 < 1 < 2\}$. (Simplify at your own risk! 6 pts)

d. Find both points $c \in (-2, 2)$ such that the equality given by the Mean Value Theorem holds. (3 pts)

e. Give an example of a cubic polynomial $g(x)$ with exactly one real root c, such that c is not a critical point of $g(x)$. (Hint: modify the above $f(x)$ to get a function $g(x)$. 2 pts)

9. State the definition and domain of the following functions: (3 pts each)

- $\log(x) =$
- $e^x =$
- $\cosh(x) =$
- $x^x =$
- $x^{(x^x)} =$

10. This question is a max/min problem. (10 pts total)

a. Use the second derivative test to find the critical points and global minimum value of $f(x) = x - e \cdot \log(x)$ on the domain $(0, \infty)$. Hint: I'm using the number $e \in \mathbb{R}$ as a coefficient here: $\log(x)$ is not an exponent.

b. Conclude, from part a., that $x > 0 \implies x \geq e \cdot \log(x)$, and find all x where this is an equality.

c. On the same graph, sketch both functions x and $e \cdot \log(x)$.

11. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified and the functions are real-valued. $\mathbb{C}$ is the field of complex numbers. You need not explain your answer.$^{(1\ each)}$

______ If $f(x)$ has a local maximum value on $[a, b]$, then $f(x)$ is not one-to-one.

______ If $f(x)$ is continuous on $[a, b]$, then $\int_a^b f(x)dx = f(b) - f(a)$.

______ If $f(x)$ is continuous on (a, b) and one-to-one, then $f(x)$ is either increasing or decreasing.

______ $a,\ b \in \mathbb{R},\ a^2 + b^2 = 0 \implies a = 0$.

______ $a,\ b \in \mathbb{C},\ a^2 + b^2 = 0 \implies a = 0$.

______ If $f(x)$ is increasing on (a, b) then $f'(x) > 0$.

(for the next two questions, assume A and B are subsets of the plane and $\text{Area}(A)$ and $\text{Area}(B)$ both exist:)

______ $A \neq B$ AND $A \subseteq B \implies \text{Area}(A) < \text{Area}(B)$.

______ $\text{Area}(A \cap B) > 0 \implies \text{Area}(A \cup B) < \text{Area}(A) + \text{Area}(B)$.

______ If $h(x)$ is a solution of $y' = y$, then $h(x) = \exp(x + C)$ for some $C \in \mathbb{R}$.

______ If $p(x)$ is a polynomial function with domain $\mathbb{R}$ and $p(a) = 0$ and $p(b) = 0$ and $a < b$, then $\exists c \in (a, b)$ such that $p'(c) = 0$.

______ If $p(x)$ is a polynomial function with domain $\mathbb{R}$ and $p(a) = 0$ and $p(b) = 0$ and $a < b$, then $\exists$ a unique $c \in (a, b)$ such that $p'(c) = 0$.

______ If $f(x)$ is an integrable function on $[a, b]$, and $f(x) \geq 0$ on $[a, b]$, and $\int_a^b f(x)dx = 0$ then $f(x)$ is the constant function 0.

______ If $f(x)$ and $g(x)$ are continuous on $\mathbb{R}$ and $g'(x)$ is also continuous, then $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

______ If $\int_a^b f(x)dx = 0$ then $\left|\int_a^b f(x)dx\right| = 0$.

______ If $\int_a^b f(x)dx = 0$ then $\int_a^b (f(x)+1)dx = b-a$.

______ If $\int_a^b f(x)dx = 0$ then $\exists c \in [a,b]$ such that $f(c) = 0$.

______ If $f(x)$ is one-to-one, and $0 \notin \text{range}(f)$ then $g(x) = 1/f(x)$ is one-to-one.

______ $e = 2.7182818$

______ $\int \tan(x)dx = \log|\sec(x)| + C$.

______ $\int \tan(x)dx = -\log|\cos(x)| + C$.

______ If $f(t)$ is integrable on $[a,b]$ and $f(t) > 0$ on $[a,b]$, then $\int_a^x f(t)dt$ is an increasing function of x on $[a,b]$.

12. Verify the following trigonometric identities: you may assume the formulas $\sin^2(x) + \cos^2(x) = 1$ and $\sin(2x) = 2\sin(x)\cos(x)$. (2 pts each)

- $\sqrt{\frac{1+\sin(x)}{1-\sin(x)}} = \sec(x) + \tan(x)$
- $\cos(2x) = \cos^2(x) - \sin^2(x)$
- $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$.
- $x = \sin(\theta) \implies \cot(\theta) = \frac{\sqrt{1-x^2}}{x}$

13. $f(x)$ and $g(x)$ are one-to-one on $\mathbb{R}$. (2 pts each)
a. Find $(f \circ g)^{-1}(x)$.

b. If $h(x) = f(x + c)$, find $h^{-1}(x)$.

c. If $c \neq 0$ and $k(x) = g(1 + (x/c))$, find $k^{-1}(x)$.

d. Give an example to show that the product $f \cdot g$ is not necessarily one-to-one.

14. (8 pts) Suppose $f(x)$ is increasing and continuous on $[0, \infty)$, and $f(0) = 0$. By drawing a graph, and using the base·height formula for the area of a rectangle, demonstrate the inequality:

$$a \cdot b \leq \int_{x=0}^{x=a} f(x)dx + \int_{y=0}^{y=b} f^{-1}(y)dy.$$

15. a. Calculate the indefinite integral $\int \frac{\sqrt{1-x^2}}{x} dx$.

(Hints: Use the substitution $x = \sin(\theta)$. One of the terms that appeared when I tried this was $\int \csc(\theta) d\theta$, which equals $-\log|\csc(\theta) + \cot(\theta)|$. (5 pts)

b. Check your answer to b. (by differentiating it). (2 pts)